Bayesian Estimation of DSGE Models: Lessons from Second-order Approximations

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Motivation

- Dynamic Stochastic General Equilibrium (DSGE) model
- Likelihood-based estimation with log-linearization: DYNARE!!
- For welfare analysis, first-order accurate approximation is not sufficient: (Kim and Kim, 2003)

Example: Welfare Reversal

- Two countries, $i = 1, 2$
- No dynamics
- Preference: CRRA

$$u(C_i) = \frac{C_i^{1-\tau} - 1}{1 - \tau}$$

- Technology: stochastic endowment

$$Z_i = \exp(u_i)$$

where u_i is iid with

$$u_i = \begin{cases} 1 & \text{with prob. } 1/2 \\ -1 & \text{with prob. } 1/2 \end{cases}$$

Example: Welfare Reversal

- Autarchy: $C_i^A = Z_i$
- Complete market, risk-sharing: $C_i^M = \frac{Z_1 + Z_2}{2}$
- Deterministic steady-state: $\bar{Z} = \bar{C} = 1$
- Log-linearization:

$$\hat{c}_i^{M1} = \frac{1}{2}(\hat{z}_1 + \hat{z}_2)$$

$$C_i^{M1} = \sqrt{Z_1 Z_2}$$

- Second-order approximation:

$$\hat{c}_i^{M2} = \frac{1}{2}(\hat{z}_1 + \hat{z}_2) + \frac{1}{8}(\hat{z}_1 - \hat{z}_2)^2$$

$$C_i^{M2} = C_i^{M1} \exp\left(\frac{1}{8}(\log Z_1 - \log Z_2)^2\right)$$

Example: Welfare Reversal

- Welfare: Expected utility

k	$\mathbf{E}\left[u\left(C_i^k\right)\right]$	$\tau = 0.5$	$\tau = 2$
$A:$	$\frac{1}{2}u(e) + \frac{1}{2}u(e^{-1})$	.2553	$-.5431$
$M:$	$\frac{1}{4}u(e) + \frac{1}{4}u(e^{-1}) + \frac{1}{2}u\left(\frac{e+e^{-1}}{2}\right)$	.3698	$-.0956$
$M1:$	$\frac{1}{4}u(e) + \frac{1}{4}u(e^{-1}) + \frac{1}{2}u\left(1\right)$	.1276	$-.2715$
$M2:$	$\frac{1}{4}u(e) + \frac{1}{4}u(e^{-1}) + \frac{1}{2}u\left(e^{1/2}\right)$	.4117	$-.0748$

Motivation

- Second-order approximation: DYNARE!! Again!!
- Likelihood-based estimation of second-order approximated models is not trivial.
- Short cut:

  (1) Estimation based on linearized model

  (2) Plugging the linear estimates into the second-order approximated model to do policy analysis
- Can this practice be misleading?

New Keynesian DSGE Model

- Representative household
- Monopolistically competitive firms

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}$$

- Nominal rigidities: quadratic adjustment cost

$$AC_t(j) = \frac{\varphi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \pi \right)^2 Y_t(j)$$

- No capital accumulation
- Linear production with perfectly competitive factor market

New Keynesian DSGE Model

- Monetary policy:
  Interest feedback rule with nominal interest rate target

$$R_t = {R_t^*}^{1-\rho_R} R_{t-1}^{\rho_R} \exp(\epsilon_{R,t})$$

$$R_t^* = r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^*}\right)^{\psi_2}$$

- Fiscal policy:
  Passive with exogenously driven government spending
- Three shocks:
  Technology, government spending, and monetary policy

Symmetric Equilibrium

After detrending, in log-deviation (from steady-state) form:

$$
\begin{aligned}
1 &= \mathbf{E}_t \left[e^{-\tau(\hat{c}_{t+1}-\hat{c}_t)+\hat{R}_t-\hat{\pi}_{t+1}-\hat{z}_{t+1}} \right] \\
0 &= \frac{1-\nu}{\nu\varphi\pi^2}\left(1-e^{\tau\hat{c}_t}\right)+\left(e^{\hat{\pi}_t}-1\right)\left[\left(1-\frac{1}{2\nu}\right)e^{\hat{\pi}_t}+\frac{1}{2\nu}\right] \\
&\quad - \beta\,\mathbf{E}_t\left[\left(e^{\hat{\pi}_{t+1}}-1\right)e^{-\tau(\hat{c}_{t+1}-\hat{c}_t)+(\hat{y}_{t+1}-\hat{y}_t)+\hat{\pi}_{t+1}}\right] \\
e^{\hat{c}_t-\hat{y}_t} &= e^{-\hat{g}_t}-\frac{\varphi\pi^2 g}{2}\left(e^{\hat{\pi}_t}-1\right)^2 \\
\hat{R}_t &= \rho_R\hat{R}_{t-1}+(1-\rho_R)\psi_1\hat{\pi}_t+(1-\rho_R)\psi_2\left(\hat{y}_t-\hat{g}_t\right)+\epsilon_{R,t} \\
\hat{g}_t &= \rho_g\hat{g}_{t-1}+\epsilon_{g,t} \\
\hat{z}_t &= \rho_z\hat{z}_{t-1}+\epsilon_{z,t}
\end{aligned}
$$

Representation

The solution of the rational expectations system takes the form

$$s_t = \Phi(s_{t-1}, \epsilon_t; \theta)$$

where

$$\begin{aligned} s_t &= [\hat{y}_t, \hat{c}_t, \hat{\pi}_t, \hat{R}_t, \epsilon_{R,t}, \hat{g}_t, \hat{z}_t]' \\ \epsilon_t &= [\epsilon_{R,t}, \epsilon_{g,t}, \epsilon_{z,t}]' \end{aligned}$$

Log-linearization and Rational Expectation

- Log-linearized economy

$$\begin{aligned}
\hat{c}_t &= \mathbf{E}_t\left[\hat{c}_{t+1}\right] - \frac{1}{\tau}\left(\hat{R}_t - \mathbf{E}_t\left[\hat{\pi}_{t+1}\right] - \mathbf{E}_t\left[\hat{z}_{t+1}\right]\right) \\
\hat{\pi}_t &= \beta \mathbf{E}_t[\hat{\pi}_{t+1}] + \kappa \hat{c}_t \\
\hat{y}_t &= \hat{c}_t + \hat{g}_t
\end{aligned}$$

where $\kappa = \tau \frac{1-\nu}{\nu \pi^2 \varphi}$.

Without rigidities where $\nu = 0$ and $\varphi = 0$, κ tends to infinity.

- The solution of the log-linearized rational expectations system

$$s_t = \Phi^{(s)} s_{t-1} + \Phi^{(\epsilon)} \epsilon_t$$

Second-Order Approximation

- Judd (1998), Collard and Juillard (2001), Jin and Judd (2002), Schmitt-Grohe and Uribe (2004), Kim, Kim, Schaumburg, and Sims (2005), Swanson, Anderson, and Levin (2005), Klein (2005)

- Partition s_t into endogenous and exogenous variables, $[x_t', z_t']'$

- The solution of the second-order approximated rational expectations system

$$\begin{aligned} z_t &= \Psi^{(z)} z_{t-1} + \epsilon_t \\ x_{j,t} &= \Psi_j^{(1)} w_t + \left(\Psi_j^{(0)} + w_t' \Psi_j^{(2)} w_t \right) \end{aligned}$$

where $w_t = [x_{t-1}', z_t']'$.

State-Space Representation: Transition

- Linear transition from log-linearization:

$$s_t = \Phi^{(s)} s_{t-1} + \Phi^{(\epsilon)} \epsilon_t$$

where $s_t = [x_t', z_t']'$

- Quadratic transition from second-order approximation:

$$\begin{aligned} z_t &= \Psi^{(z)} z_{t-1} + \epsilon_t \\ x_{j,t} &= \Psi_j^{(1)} w_t + \left(\Psi_j^{(0)} + w_t' \Psi_j^{(2)} w_t \right) \end{aligned}$$

where $w_t = [x_{t-1}', z_t']'$

State-Space Representation: Measurement

- Output growth rate, inflation, and nominal interest rate

$$
\begin{aligned}
\text{YGR}_t &= \gamma^{(Q)} + 100\,(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) + u_{y,t} \\
\text{INFL}_t &= \pi^{(A)} + 400\hat{\pi}_t + u_{\pi,t} \\
\text{INT}_t &= \pi^{(A)} + r^{(A)} + 4\gamma^{(Q)} + 400\hat{R}_t + u_{r,t}
\end{aligned}
$$

- Linear measurement:

$$
y_t = \Theta^{(0)} + \Theta^{(s)} s_t + u_t
$$

Filter and Likelihood

- *Initialization at the beginning of t:* $p\left(s_{t-1}|Y^{t-1}\right)$
- *Prediction/Forecasting:*

$$p\left(s_t|Y^{t-1}\right) = \int p\left(s_t|s_{t-1}\right) p\left(s_{t-1}|Y^{t-1}\right) ds_{t-1}$$

- *Filtering/Updating by Bayes' Theorem:*

$$p\left(s_t|Y^t\right) = \frac{p\left(y_t|s_t\right) p\left(s_t|Y^{t-1}\right)}{p\left(y_t|Y^{t-1}\right)}$$

- *Likelihood evaluation:*

$$\ln \mathcal{L}(\theta|Y) = \ln p\left(y_1|\theta\right) + \sum_{t=2}^{T} \ln p\left(y_t|Y^{t-1}, \theta\right)$$

Particle Filter

- Use Monte Carlo method to integrate out the latent variables

- Pioneering works:
  - Gordon, Salmon, and Smith (1993)
  - Kitagawa (1996)

- Economics literature:
  - Stochastic volatility:
    Pitt and Shephard (1999), Kim, Shephard, and Chib (1998), Duan and Fulop (2007)

  - Neoclassical growth model:
    Fernandez-Villaverde and Rubio-Ramirez (2004,2005,2006)

Prior Specification

Parameter	Density	Mean	Std.	Description
τ	Gamma	2	0.5	inverse EIS
ν	Beta	0.1	0.05	degree of imperfect competition
κ	Gamma	0.3	0.1	Phillips curve, $\kappa = \frac{\tau}{\pi^2 \varphi}\left(\frac{1-\nu}{\nu}\right)$
c/y	Beta	0.85	0.05	consumption-output ratio
ψ_1	Gamma	1.5	0.25	responsiveness to inflation
ψ_2	Gamma	0.5	0.25	responsiveness to output gap
ρ_R	Beta	0.5	0.2	policy inertia
$r^{(A)}$	Gamma	0.55	0.25	steady state real interest rate
$\pi^{(A)}$	Gamma	3	2	target inflation
$\gamma^{(Q)}$	Normal	0.8	0.25	steady state output growth rate

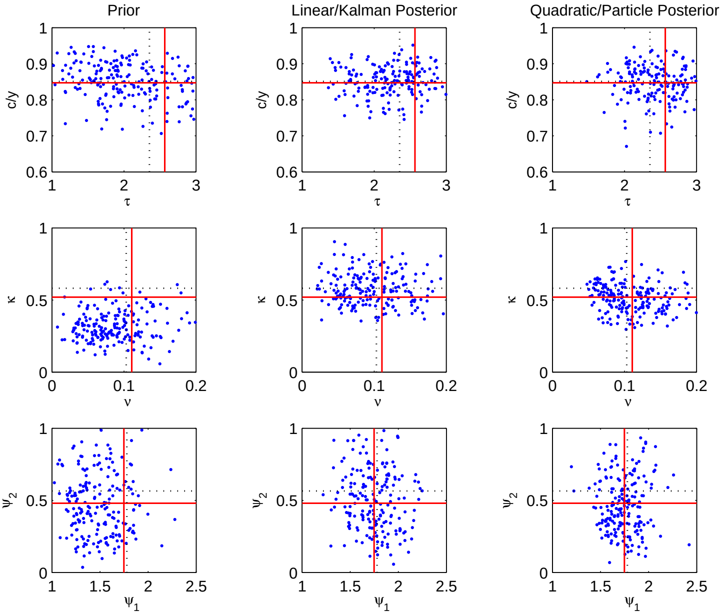
ρ_g, ρ_z, σ_R, σ_g, and σ_z

Posterior Distribution

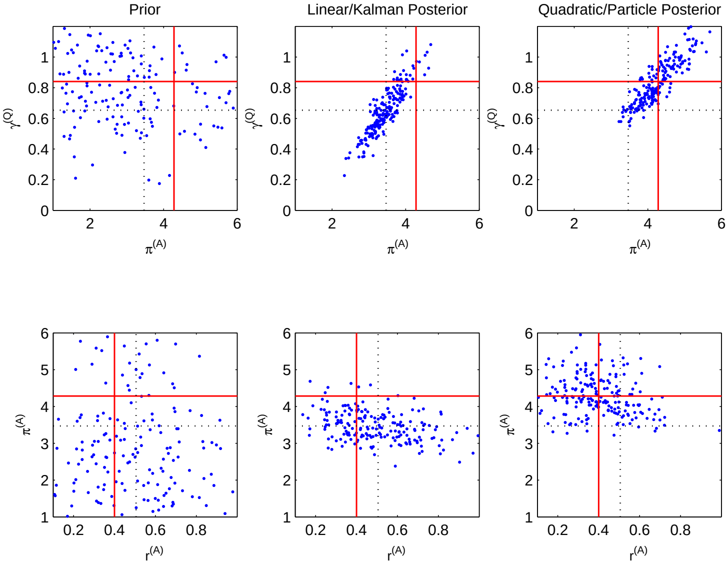
- Log marginal data densities:
  Modified harmonic mean estimator by Geweke (1999)

Linear/Kalman	Quadratic/Particle
−323.984	−322.200

- Information on ν and c/y
- Steady state parameters, $r^{(A)}$, $\pi^{(A)}$, and $\gamma^{(Q)}$
- Policy parameters, ψ_1 and ψ_2



Prior
Linear/Kalman Posterior
Quadratic/Particle Posterior
c/y
τ
κ
ν
ψ_2
ψ_1



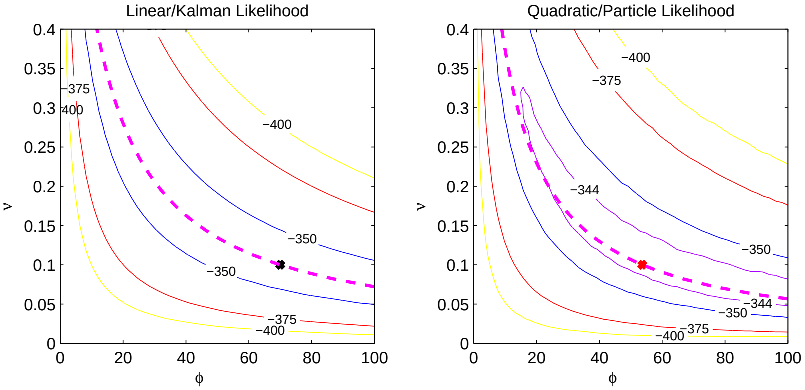
Prior
Linear/Kalman Posterior
Quadratic/Particle Posterior
γ(Q)
π(A)
r(A)

Posterior Distribution

	Prior		Posterior			
			Linear/Kalman		Quadratic/Particle	
	Mean	90% Interval	Mean	90% Interval	Mean	90% Interval
τ	2.004	[1.175, 2.793]	2.352	[1.486, 3.128]	2.565	[1.757, 3.269]
ν	0.100	[0.022, 0.175]	0.103	[0.025, 0.179]	0.111	[0.052, 0.171]
κ	0.300	[0.138, 0.452]	0.582	[0.389, 0.763]	0.520	[0.361, 0.678]
c/y	0.850	[0.771, 0.931]	0.851	[0.773, 0.930]	0.847	[0.776, 0.994]
ψ_1	1.508	[1.091, 1.907]	1.779	[1.417, 2.147]	1.749	[1.456, 2.027]
ψ_2	0.500	[0.110, 0.866]	0.565	[0.123, 0.978]	0.480	[0.140, 0.816]
ρ_R	0.499	[0.148, 0.823]	0.808	[0.760, 0.863]	0.806	[0.755, 0.853]
$r^{(A)}$	0.548	[0.161, 0.918]	0.506	[0.186, 0.797]	0.400	[0.159, 0.632]
$\pi^{(A)}$	2.992	[0.225, 5.765]	3.470	[2.772, 4.205]	4.282	[3.430, 5.212]
$\gamma^{(Q)}$	0.801	[0.380, 1.197]	0.653	[0.405, 0.913]	0.840	[0.630, 1.081]
ρ_g	0.801	[0.651, 0.959]	0.956	[0.927, 0.987]	0.959	[0.932, 0.985]
ρ_z	0.661	[0.429, 0.909]	0.937	[0.905, 0.969]	0.943	[0.917, 0.976]
$100\sigma_R$	0.500	[0.210, 0.790]	0.195	[0.165, 0.229]	0.205	[0.169, 0.242]
$100\sigma_g$	1.261	[0.537, 1.996]	0.682	[0.589, 0.784]	0.658	[0.596, 0.719]
$100\sigma_z$	0.501	[0.218, 0.796]	0.180	[0.146, 0.213]	0.176	[0.144, 0.206]

Identification

- Likelihood contour from simulated data
- $\kappa = \frac{\tau}{\pi^2 \varphi}\left(\frac{1-\nu}{\nu}\right)$



Expected Utility as Welfare Measures

- Infinite sum of the discount utilities

$$\begin{aligned} W_0 &= \mathbf{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\tau}-1}{1-\tau}-\chi_H y_t\right)\right] \\ &= \mathbf{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\tau}-1}{1-\tau}\right)\right]-\chi_H\mathbf{E}_0\left[\sum_{t=0}^{\infty}\beta^t y_t\right] \\ &= U_0-\chi_H V_0 \end{aligned}$$

- Approximations in recursive form

$$U_t=\frac{c_t^{1-\tau}-1}{1-\tau}+\beta\mathbf{E}_t U_{t+1}, \qquad V_t=y_t+\beta\mathbf{E}_t V_{t+1}$$

Welfare Cost

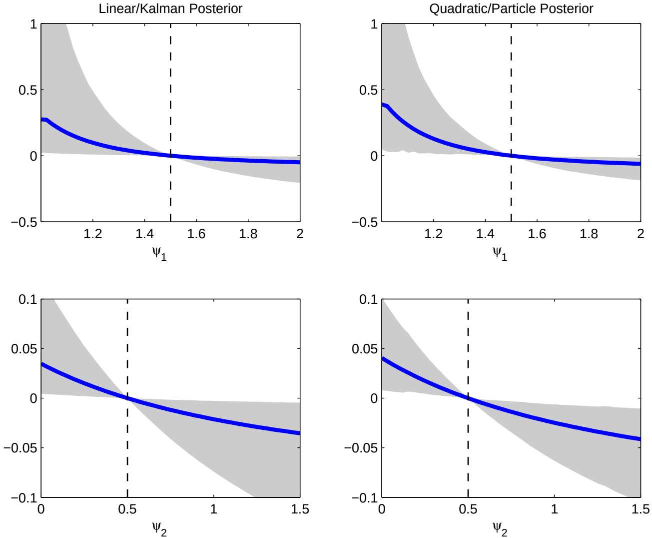
- Two sets of parameters, reference and alternative, implies different levels of welfare.

- The fraction of consumption in the alternative to be increased (or to be given up) to achieve the welfare level of the reference.

$$W_t^R = \mathbf{E}_t \left(\sum_{s=0}^{\infty} \left(\beta^A\right)^s \left(\frac{\left[(1+\lambda) C_{t+s}^A / A_{t+s}^A\right]^{1-\tau^A} - 1}{1-\tau^A} - \chi_H^A H_{t+s}^A \right) \right]$$

- Welfare differentials
  - Fix nonpolicy parameters
  - Differentials relative to baseline policy parameter

Welfare Cost



Generalized Impulse Responses

- Impulse responses in a linear model
  - Symmetry
  - Scalability
  - Path-independency: past and future

- Koop, Pesaran, and Potter (1996): Generalized Impulse Response

$$GI(n) \quad = \quad \mathbf{E}\left[x_{t+n}|\epsilon_t, \Omega_{t-1}\right] - \mathbf{E}\left[x_{t+n}|\Omega_{t-1}\right]$$

Generalized Impulse Responses

- Consider a nonlinear process

$$X_t = f(X_{t-1}) + \epsilon_t$$

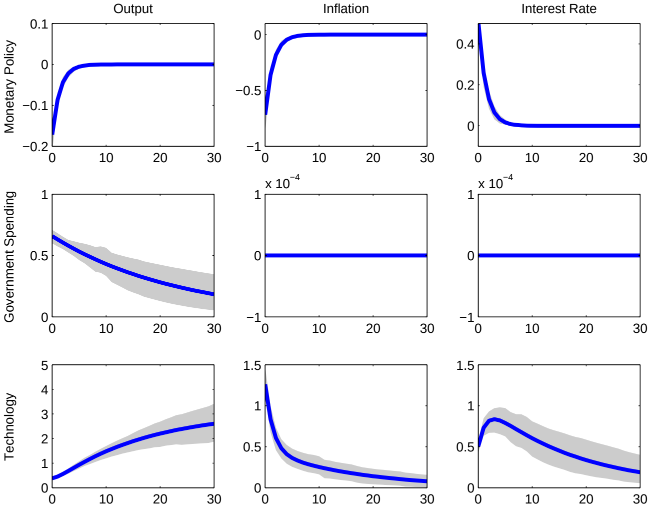
and two alternative shock processes which differ only in time t, that is, $\epsilon_k = \epsilon_k'$ for all $k \neq t$.

- Then conventional impulse response at time $t+2$ can be written as

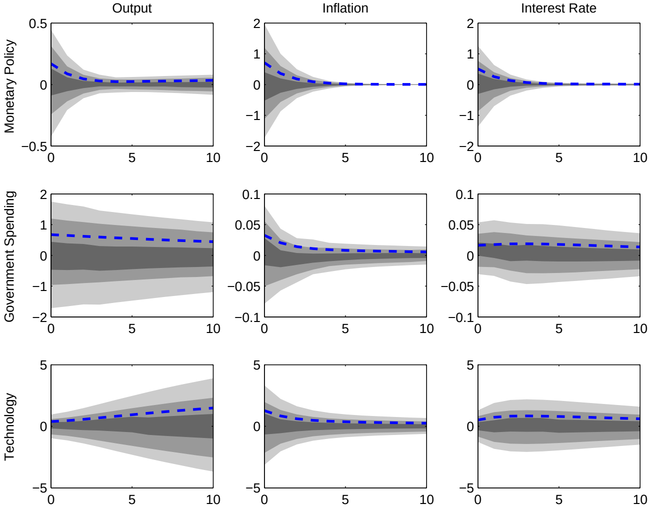
$$\begin{aligned} X_{t+2}' - X_{t+2} &= f[f\{f(X_{t-1}) + \epsilon_t\} + \epsilon_{t+1}] \\ &\quad - f[f\{f(X_{t-1}) + \epsilon_t'\} + \epsilon_{t+1}] \end{aligned}$$

which implies the path-dependence.

Impulse Responses



Impulse Responses For Nonlinear Processes



Conclusion

- Particle filter for second-order approximated DSGE model
- Stylized monetary model in line with Woodford (2003)
- U.S. quarterly data

- More identification on structural parameters
- Better fit as measured by the marginal data density
- Bias corrections on parameter estimates
- Aggressive estimates on welfare cost
- Unveils neglected impulse response dynamics