International Risk Sharing and Wealth Allocation with Higher Order Cumulants

Online Appendix

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A Summary of the three-country model

In this section we present the model in a way that lends itself to recursive solution, as implemented in the Fortran codes (see `core_model.f90`). In particular, the solution strategy consists of guessing B's and C's prices in country A's consumption units, recursively solving for the other variables and thus verifying the guess. Hence we guess $p_{B,t}$ and $p_{C,t}$. Using the aggregate price equations we obtain

$$p_{A,t} = \left(\frac{1 - (1-\nu_A)\big(\varsigma_A p_{B,t}^{1-\theta} + (1-\varsigma_A) p_{C,t}^{1-\theta}\big)}{\nu_A} \right)^{\frac{1}{1-\theta}} \tag{A.1}$$

$$Q_{A,B,t} = \left(\nu_B p_{B,t}^{1-\theta} + (1-\nu_B)\left(\varsigma_B p_{A,t}^{1-\theta} + (1-\varsigma_B) p_{C,t}^{1-\theta}\right)\right)^{\frac{1}{1-\theta}} \tag{A.2}$$

$$Q_{A,C,t} = \left(\nu_C p_{C,t}^{1-\theta} + (1-\nu_C)\left(\varsigma_C p_{A,t}^{1-\theta} + (1-\varsigma_C) p_{B,t}^{1-\theta}\right)\right)^{\frac{1}{1-\theta}} \tag{A.3}$$

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Using the risk-sharing constants

$$C_{B,t} = Q_{A,B,t}^{-\frac{1}{\rho}} \kappa_{A,B} C_{A,t} \qquad\qquad C_{C,t} = Q_{A,C,t}^{-\frac{1}{\rho}} \kappa_{A,C} C_{A,t}. \tag{A.4}$$

The goods market equilibrium for the three countries is given by

$$n_A p_{A,t} Y_{A,t} = (p_{A,t})^{1-\theta} \left(n_A \nu_A + n_B \varsigma_B (1-\nu_B) Q_{A,B,t}^{\theta-\frac{1}{\rho}} \kappa_{A,B} + n_C \varsigma_C (1-\nu_C) Q_{A,C,t}^{\theta-\frac{1}{\rho}} \kappa_{A,C} \right) C_{A,t} \tag{A.5}$$

$$n_B p_{B,t} Y_{B,t} = (p_{B,t})^{1-\theta} \left(n_B \nu_B Q_{A,B,t}^{\theta-\frac{1}{\rho}} \kappa_{A,B} + n_A \varsigma_A (1-\nu_A) + n_C (1-\varsigma_C)(1-\nu_C) Q_{A,C,t}^{\theta-\frac{1}{\rho}} \kappa_{A,C} \right) C_{A,t} \tag{A.6}$$

$$n_C p_{C,t} Y_{C,t} = (p_{C,t})^{1-\theta} \left(n_C \nu_C Q_{A,C,t}^{\theta-\frac{1}{\rho}} \kappa_{A,C} + n_A (1-\varsigma_A)(1-\nu_A) + n_B (1-\varsigma_B)(1-\nu_B) Q_{A,B,t}^{\theta-\frac{1}{\rho}} \kappa_{A,B} \right) C_{A,t}. \tag{A.7}$$

Summing up these three equations gives

$$C_{A,t} = \mu_{A,t} Y_{W,t} \tag{A.8}$$

where

$$\mu_{A,t} = \left[n_A + n_B Q_{A,B,t}^{1-\frac{1}{\rho}} \kappa_{A,B} + n_C Q_{A,C,t}^{1-\frac{1}{\rho}} \kappa_{A,C} \right]^{-1} \tag{A.9}$$

Note that $\mu_{A,t} > 0$ can be larger or smaller than 1. Likewise, by applying equations (A.4) we have

$$C_{B,t} = \mu_{B,t} Y_{W,t}, \text{ and } C_{C,t} = \mu_{C,t} Y_{W,t} \tag{A.10}$$

where per equations (A.4)

$$\mu_{B,t} = \mu_{A,t} \kappa_{A,B} Q_{A,B,t}^{-\frac{1}{\rho}}, \text{ and } \mu_{C,t} = \mu_{A,t} \kappa_{A,C} Q_{A,C,t}^{-\frac{1}{\rho}}. \tag{A.11}$$

Note also that $n_A \mu_A + n_B \mu_B Q_{A,B,t} + n_C \mu_C Q_{A,C,t} = 1$.

Then use the labor supply equations to generate global output, i.e., from

$$L_{A,t}=\left(C_{A,t}^{-\rho}\frac{1-\alpha}{\chi}p_{A,t}D_{A,t}\right)^{\frac{1}{\varphi+\alpha}} \tag{A.12}$$

$$L_{B,t}=\left({C_{B,t}^{*}}^{-\rho}\frac{1-\alpha}{\chi}p_{B,t}Q_{A,B,t}^{-1}D_{B,t}\right)^{\frac{1}{\varphi+\alpha}} \tag{A.13}$$

$$L_{C,t}=\left({C_{C,t}^{*}}^{-\rho}\frac{1-\alpha}{\chi}p_{C,t}Q_{A,C,t}^{-1}D_{C,t}\right)^{\frac{1}{\varphi+\alpha}} \tag{A.14}$$

Or, using the results so far

$$n_{A}p_{A,t}D_{A,t}L_{A,t}^{1-\alpha}=n_{A}p_{A,t}D_{A,t}\left(\mu_{A}^{-\rho}\frac{1-\alpha}{\chi}p_{A,t}D_{A,t}\right)^{\frac{(1-\alpha)}{\varphi+\alpha}}Y_{W,t}^{-\frac{\rho(1-\alpha)}{\varphi+\alpha}} \tag{A.15}$$

$$n_{B}p_{B,t}D_{B,t}L_{B,t}^{1-\alpha}=n_{B}p_{B,t}D_{B,t}\left(\mu_{B}^{-\rho}\frac{1-\alpha}{\chi}p_{B,t}Q_{A,B,t}^{-1}D_{B,t}\right)^{\frac{(1-\alpha)}{\varphi+\alpha}}Y_{W,t}^{-\frac{\rho(1-\alpha)}{\varphi+\alpha}} \tag{A.16}$$

$$n_{C}p_{C,t}D_{C,t}L_{C,t}^{1-\alpha}=n_{C}p_{C,t}D_{C,t}\left(\mu_{C}^{-\rho}\frac{1-\alpha}{\chi}p_{C,t}Q_{A,C,t}^{-1}D_{C,t}\right)^{\frac{(1-\alpha)}{\varphi+\alpha}}Y_{W,t}^{-\frac{\rho(1-\alpha)}{\varphi+\alpha}} \tag{A.17}$$

Adding them up yields

$$Y_{W,t}=\left[\begin{array}{c} n_{A}p_{A,t}D_{A,t}\left(\mu_{A}^{-\rho}\frac{1-\alpha}{\chi}p_{A,t}D_{A,t}\right)^{\frac{(1-\alpha)}{\varphi+\alpha}}+\\ n_{B}p_{B,t}D_{B,t}\left(\mu_{B}^{-\rho}\frac{1-\alpha}{\chi}p_{B,t}Q_{A,B,t}^{-1}D_{B,t}\right)^{\frac{(1-\alpha)}{\varphi+\alpha}}+\\ n_{C}p_{C,t}D_{C,t}\left(\mu_{C}^{-\rho}\frac{1-\alpha}{\chi}p_{C,t}Q_{A,C,t}^{-1}D_{C,t}\right)^{\frac{(1-\alpha)}{\varphi+\alpha}} \end{array}\right]^{\frac{\varphi+\alpha}{\rho(1-\alpha)+\varphi+\alpha}} \tag{A.18}$$

Finally use

$$n_{B}Y_{B,t}=\left(p_{B,t}\right)^{-\theta}\left(n_{B}\nu_{B}Q_{A,B,t}^{\theta-\frac{1}{\rho}}\kappa_{A,B}+n_{A}\varsigma_{A}\left(1-\nu_{A}\right)+n_{C}\left(1-\varsigma_{C}\right)\left(1-\nu_{C}\right)Q_{A,C,t}^{\theta-\frac{1}{\rho}}\kappa_{A,C}\right)C_{A,t} \tag{A.19}$$

$$n_{C}Y_{C,t}=\left(p_{C,t}\right)^{-\theta}\left(n_{C}\nu_{C}Q_{A,C,t}^{\theta-\frac{1}{\rho}}\kappa_{A,C}+n_{A}\left(1-\varsigma_{A}\right)\left(1-\nu_{A}\right)+n_{B}\left(1-\varsigma_{B}\right)\left(1-\nu_{B}\right)Q_{A,B,t}^{\theta-\frac{1}{\rho}}\kappa_{A,B}\right)C_{A,t} \tag{A.20}$$

to verify the guess for $p_{B,t}$ and $p_{C,t}$.

A.1 What changes under GHH preferences

The derivation above uses the separable CRRA first-order conditions; here we record what changes under the non-separable GHH preferences. Both specifications share the same constant-relative-risk-aversion curvature ρ; under GHH the felicity is

$$u(C_{j,t}, L_{j,t}) = \frac{X_{j,t}^{1-\rho}}{1-\rho}, \qquad X_{j,t} = C_{j,t} - \chi \frac{L_{j,t}^{1+\varphi}}{1+\varphi},$$

i.e., the same CRRA aggregator applied to the composite X, which bundles consumption and the disutility of labour. The marginal utilities are $u_C = X^{-\rho}$ and $u_L = -X^{-\rho}\chi L^{\varphi}$.

Labour supply has no wealth effect. The intratemporal condition $u_L + w_{j,t}\, u_C = 0$ reduces to $\chi L_{j,t}^{\varphi} = w_{j,t}$, because the common factor $X^{-\rho}$ cancels. With the competitive wage $w_{j,t} = (1-\alpha)\, p_{j,t} Q_{A,j,t}^{-1} D_{j,t} L_{j,t}^{-\alpha}$ (with $Q_{A,A,t} \equiv 1$), labour supply becomes

$$L_{j,t} = \left(\frac{1-\alpha}{\chi}\, p_{j,t}\, Q_{A,j,t}^{-1}\, D_{j,t} \right)^{\frac{1}{\varphi+\alpha}},$$

i.e., the consumption term $C_{j,t}^{-\rho}$ that appears in the CRRA labour-supply equations above drops out. Hours, and hence output $Y_{j,t} = D_{j,t} L_{j,t}^{1-\alpha}$, are pinned down by prices alone: the world output value $Y_{W,t} = \sum_j n_j p_{j,t} Y_{j,t}$ follows directly, without the fixed point in $Y_{W,t}$ (and the $\mu_j^{-\rho}$ terms) that the CRRA wealth effect produces in the world-output equation above.

Risk sharing is over the composite. The stochastic discount factor and the risk-sharing conditions are built from $u_C = X^{-\rho}$, so it is the composite X—not consumption—that is shared in Negishi-weighted proportions. The constants in (A.4) become

$$X_{B,t} = Q_{A,B,t}^{-1/\rho}\, \kappa_{A,B}\, X_{A,t}, \qquad X_{C,t} = Q_{A,C,t}^{-1/\rho}\, \kappa_{A,C}\, X_{A,t},$$

and the whole block from (A.4) to (A.9) carries over verbatim with consumption replaced by the composite ($C \to X$, same ρ): $X_{j,t} = \mu_{j,t} X_{W,t}$ with the same weights (A.9), where the world composite

$$X_{W,t} = Y_{W,t} - \sum_j n_j \, Q_{A,j,t} \, \chi \, \frac{L_{j,t}^{1+\varphi}}{1+\varphi}$$

is world output value net of the value of labour disutility, each country's disutility valued at its real exchange rate $Q_{A,j,t}$ (with $Q_{A,A,t} \equiv 1$).

Recovering consumption and prices. Given $X_{j,t}$ and $L_{j,t}$, consumption is recovered as $C_{j,t} = X_{j,t} + \chi L_{j,t}^{1+\varphi}/(1+\varphi)$. The goods-market-clearing and price-index equations are otherwise unchanged, since the consumption basket—the CES aggregator and the price indices p and Q—is the same. Consequently the level effect κ is determined exactly as in the CRRA case with consumption replaced by the composite X.

B A useful efficient procedure

We detail a practical step which is particularly useful for numerical solutions of DSGE models of any size, e.g., using Dynare (Juillard, 1996).[1] We then offer a higher-order accurate solution of the Negishi weights. For this illustration we focus on the two-country version of the model, *wlog*.

We illustrate the procedure focusing on Negishi weights, which also happen to feature differently than the other variables in the model.[2] We begin by observing that the risk-sharing condition can be solved backward to yield:

$$\ln \zeta_{A,t} - \ln \zeta_{B,t} + \ln Q_{A,B,t} = \ln \zeta_{A,0} - \ln \zeta_{B,0} + \ln Q_{A,B,0} := \rho \ln \kappa_{A,B}. \tag{B.1}$$

[1] We solve our model using a specific code in Wolfram's Mathematica for which we don't need to use this practical suggestion.

[2] For non-separable preferences or recursive preferences, e.g., à la Bansal and Yaron (2004), a similar decomposition can be obtained. Details are available from the authors on request.

where the subscript 0 indicates the time zero in which the risk-sharing agreement is entered for the first time.

The initial distribution of wealth under a full set of period-by-period state contingent (Arrow) securities is pinned down by a condition on the initial distribution of Arrow securities across borders. Take the m-order series-expansion of $\kappa_{A,B}$ around $\omega = 0$

$$\ln \kappa_{A,B}(\omega) \approx \kappa_{A,B}^{(0)} + \kappa_{A,B}^{(1)}\omega + \ldots + \kappa_{A,B}^{(m)}\frac{1}{m!}\omega^m \tag{B.2}$$

where $\kappa_{A,B}^{(m)} := \left.\dfrac{\partial^m \ln \kappa_{A,B}}{\partial \omega^m}\right|_{\omega=0}$. At each order m, solve for $\kappa_{A,B}^{(m)}$ and proceed recursively, starting from $\kappa_{A,B}^{(0)} = \bar{\kappa}_{A,B}$ and $\kappa_{A,B}^{(1)} = 0$ (as certainty equivalence holds at first order). By way of example, a second order expansion implies that

$$\widetilde{\kappa}_{A,B} := \ln \kappa_{A,B}(\omega) - \ln \kappa_{A,B}^{(0)} \approx \frac{1}{2}\kappa_{A,B}^{(2)} \tag{B.3}$$

where *wlog* we set $\omega = 0$.[3]

Our solution algorithm (whether applied analytically or numerically) proceeds as follows

1. Expand to the order of interest the system of equations constituting the model;
2. Find the RE solution for all variables as a function of $\widetilde{\kappa}_{A,B}$;
3. Use the appropriate series expansion of budget constraint under the condition $S_{A,0} = 0$ to solve for $\widetilde{\kappa}_{A,B}$.

For higher orders of approximation this algorithm can be used recursively starting from lower orders to build the solution for higher orders, i.e., to construct a solution for each of the variables of the model with the same structure as in equation (B.2).

[3] The accuracy of the approximation clearly depends on the size of ω. Nevertheless, we can normalize this to 0 and scale appropriately the standard deviation of the underlying shocks, *wlog*.

It is worth stressing that this solution technique does not amount to finding the "risky steady state" à la Coeurdacier et al. (2011). Despite some similarities in the two methods, our technique is specific to the derivation of $\kappa_{A,B}$. It relies on the existence of an explicit condition that can be imposed to solve for $\kappa_{A,B}$—a given initial distribution of assets. The intuition is simple. Under complete markets we can solve the model economy (i.e., find the state-space representation of the endogenous dynamic variables) conditionally on the initial, time-invariant distribution of assets. Consider the representation of a DSGE model under a second order perturbation[4]

$$AE_t\widetilde{X}_{t+1}^{(2)} + B\widetilde{X}_t^{(2)} + CE_t\left(\widetilde{X}_{t+1}^{(1)} \otimes \widetilde{X}_{t+1}^{(1)}\right) + D\left(\widetilde{X}_t^{(1)} \otimes \widetilde{X}_t^{(1)}\right) + F\left(\widetilde{X}_t^{(1)} \otimes \varepsilon_t\right) + G\widetilde{\kappa}_{A,B} = 0 \tag{B.4}$$

where X_t is a vector of variables, $\tilde{X}^{(i)} := X^{(i)} - X^{(0)}$, ε_t is an i.i.d. vector of innovations, A, B, C, D, F, G and H (below) are conformable matrices of coefficients. Importantly, $\kappa_{A,B}$ does not appear in the coefficient matrices A, B, C, D, F, G which are only reflecting deterministic steady state information. As argued above, the non-linear terms are of lower order (here first order). In this example these terms are solved separately from the system

$$E_tA\widetilde{X}_{t+1}^{(1)} + B\widetilde{X}_t^{(1)} + H\varepsilon_t = 0. \tag{B.5}$$

Notably, $\widetilde{\kappa}_{A,B}^{(1)}$ is missing from the first order, as it is zero under certainty equivalence.

Since $\widetilde{\kappa}_{A,B}^{(2)}$ is time invariant, we can re-write equation (B.4) as

$$AE_t\check{X}_{t+1}^{(2)} + B\check{X}_t^{(2)} + CE_t\left(\widetilde{X}_{t+1}^{(1)} \otimes \widetilde{X}_{t+1}^{(1)}\right) + D\left(\widetilde{X}_t^{(1)} \otimes \widetilde{X}_t^{(1)}\right) + F\left(\widetilde{X}_t^{(1)} \otimes \varepsilon_t\right) = 0 \tag{B.6}$$

where $\check{X}_t^{(2)} = \widetilde{X}_t^{(2)} + (A+B)^{-1} G\widetilde{\kappa}_{A,B}^{(2)}$. Note that under complete markets the system (B.6) does not need to include the households' budget constraint, which will instead be used in a second step to solve for $\widetilde{\kappa}_{A,B}^{(2)}$.

[4]To simplify the illustration we assume that the model has at most one period ahead expectations and that the system is represented in "companion form" with all lags subsumed in the vector X_t.

Summing up, in the class of models where $\kappa_{A,B}$ enters log linearly,[5] the solution procedure consists naturally of two steps: i) solve for allocations and prices using system (B.6); ii) use the (approximated) budget constraint at time 0 to solve for $\widetilde{\kappa}^{(2)}_{A,B}$. These two steps will then allow us to recover $\widetilde{X}^{(2)}_t$. The same procedure holds for any order of approximation.

To avoid misinterpretations of the algorithm, it is important to stress that there is only one (standard) perturbation taking place, i.e., along the "risk" loading parameter ω. The solution of the resulting system of series expansions is recursive with respect to $\kappa_{A,B}$ at each order of approximation – at least in this class of models. The solution could as well be obtained all at once (less efficiently). The key is that we go from the original non-linear model to a system of series expansions of all the "risk-sensitive" variables, including $\kappa_{A,B}$.[6]

The second-order solution of a DSGE model can be written in a second-order VAR form as (e.g., following Dynare notation)

$$\check{X}_t = \check{A}\check{X}_{t-1} + \check{B}\varepsilon_t + \frac{1}{2}\left[C\left(\check{X}_{t-1} \otimes \check{X}_{t-1}\right) + \check{D}\left(\varepsilon_t \otimes \varepsilon_t\right) + 2\check{F}\left(\check{X}_{t-1} \otimes \varepsilon_t\right)\right] + \frac{1}{2}\check{G}\vec{\Sigma^2} \tag{B.7}$$

$\check{A}$, $\check{B}$, $\check{C}$, $\check{D}$, $\check{F}$, $\check{G}$ are conformable matrices, and for any column vectors x and z, $(x \otimes z)$ is the vectorized outer product of these vectors. $\Sigma^2 := E_t\left(\varepsilon_{t+1}\varepsilon'_{t+1}\right)$, and $\vec{\cdot}$ is the vectorization operator.[7]

The key term in equation (B.7) is the last one, which shifts the mean of variables

[5] This is the case of not only CRRA preferences, but also Epstein-Zin preferences.

[6] A similar recursivity between dynamic allocations and time-invariant financial allocations emerges in long-run portfolio decisions, as discussed by Devereux and Sutherland (2011). There, the "zero order" asymptotic portfolio composition can be solved recursively from the dynamic allocation of the model's variables. This is possible since the "zero order" portfolio shares are time-invariant (like our $\kappa_{A,B}$) and the dynamics of the model can be defined conditionally on these shares. Like for the portfolio case, no ad-hoc assumptions or heuristic techniques are needed. The solution emerges from the mechanical application of series expansion techniques and solution techniques for dynamic rational expectation models.

[7] To date, Dynare returns only the product $\check{G}\vec{\Sigma^2}$ in the variable "oo_.dr.ghs2". In order to implement our algorithm this product must be factorized into the two components. This can be easily done by modifying Dynare function `dyn_second_order_solver.m` at about line 173, by adding a new variable e.g., `dr.G=LHS\(-RHS);`, where LHS and RHS are variables defined in the function.

in proportion to the exogenous risk, captured by the variance matrix Σ^2 (also referred to as the stochastic steady state in the literature). Using regular perturbations (see e.g. Lombardo and Uhlig, 2018), none of the matrices in (B.7) depends on exogenous risk. This means that the only place where σ_κ appears is in Σ^2.

The vector $\check{X}_{A,t}$ contains the variable measuring Arrow-Debreu securities. Assume the latter are in position i_{AD}, and that σ_κ occupies position j_{σ_κ} in the vector $\vec{\Sigma^2}$. Then we have that

$$\begin{aligned} \check{X}_{A,t}[i_{AD}] = \check{A}[i_{AD},:]\check{X}_{t-1} + \check{B}[i_{AD},:]\varepsilon_t + \frac{1}{2}\left[\check{C}[i_{AD},:]\left(\check{X}_{t-1}\otimes\check{X}_{t-1}\right)\right. \\ \left. +\check{D}[i_{AD},:]\left(\varepsilon_t\otimes\varepsilon_t\right) + 2\check{F}[i_{AD},:]\left(\check{X}_{t-1}\otimes\varepsilon_t\right)\right] + \frac{1}{2}\check{G}[i_{AD},:]\vec{\Sigma^2} \end{aligned} \tag{B.8}$$

where for a matrix $\check{X}$, $\check{X}[i,j]$ denotes the element in row i and column j, and where $X[i,:]$ denotes the row i of matrix X; for a vector z, $z[j_{\sigma_\kappa}]$ is the j_{σ_κ}-th element in z. In particular, $\vec{\Sigma^2}[j_{\sigma_\kappa}] = \sigma_\kappa^2$.

Note that if we set $\bar{\kappa}_{A,B} = 1$, we can solve for σ_κ^2 that satisfies some restriction on $\check{X}_{A,t}[i_{AD}]$. In particular we know that under complete markets it must be that $\check{X}_0[i_{AD}] = 0$ (Ljungqvist and Sargent, 2012). One way to implement this condition is to assume that at time 0 and -1 the economy was at the stochastic steady state, i.e., all elements of equation (B.8) are zero except the last one, i.e.[8]

$$\check{X}_0[i_{AD}] = 0 = \frac{1}{2}\check{G}[i_{AD},:]\vec{\Sigma^2} \tag{B.9}$$

Then we can solve for σ_κ^2 as

$$\sigma_\kappa^2 = -\frac{\check{G}[i_{AD}, j_{\sigma_\kappa}^\perp]\vec{\Sigma^2}[j_{\sigma_\kappa}^\perp]}{\check{G}[i_{AD}, j_{\sigma_\kappa}]} \tag{B.10}$$

where $j_{\sigma_\kappa}^\perp$ denotes all the elements excluding j_{σ_κ}. Now we simply need to swap values,

[8]Equally easily implementable is any other condition, e.g., $E\check{X}_0[i_{AD}] = 0$.

i.e., $\bar{\kappa}_{A,B} \leftarrow \sigma_\kappa^2 \; \sigma_\kappa^2 \leftarrow 1$. With this assignment of values, $\kappa_{A,B}$ is the second-order accurate risk-sharing constant that implements complete markets.

Our proposed algorithm correctly implements complete markets up to second order accuracy. It should be noted also that our approach does not affect the first-order solution. This solution correctly describes growth rates of variables, since the risk-sharing constant is invariant to time (Ljungqvist and Sargent, 2012).

This approach is reminiscent of the solution algorithm proposed by Devereux and Sutherland (2011) (DS) to solve for portfolio shares up to second order. DS introduce an auxiliary iid shock in the budget constraint of investors as a placeholder for portfolio shares. By knowing the position of this auxiliary shock DS can then use simple linear algebra to derive the shares. Although we solve a different problem, our algorithm shares with DS the idea of using auxiliary iid shocks as placeholders for parameters that would otherwise drop out of the perturbed solution.

C Global solution method

We solve the model globally using standard time-iteration methods. With CRRA preferences (separable or of the GHH kind), finding allocations and prices for given κ does not require solving for expectations. In practice we first solve the model (once) for all possible values of the states. Then we solve for the value function and Arrow securities by iteration. Alternatively these two variables could be found by simulation. With Epstein-Zin preferences (which we consider in codes available on request, but that we omit to discuss for conciseness) the model and the value function must be solved simultaneously. In particular, (a) we iterate on the Bellman equations to determine the policy function relating welfare to the state variables of our model; and (b) we iterate on the budget constraint to determine Arrow securities as a function of the states. In the simple CRRA case, we only have exogenous states (TFP); we considered both possibilities: including κ as a pseudo state variable and solving for κ directly: for each candidate κ we solve the model.[9] We can thus describe the policy functions for welfare and Arrow securities as $\mathcal{W}_{i,t} = \mathcal{W}_i\left(\kappa_{A,B}, \kappa_{A,C}, D_{A,t}, D_{B,t}, D_{C,t}\right)$ and $\mathcal{S}_{i,t} = \mathcal{S}_i\left(\kappa_{A,B}, \kappa_{A,C}, D_{A,t}, D_{B,t}, D_{C,t}\right)$, $i = \{A, B, C\}$. Once we find a solution (i.e., a fixed point such that the residual of the underlying three equations is 10^{-14}), we can determine the optimal κ_s, i.e., $\kappa^{\dagger}_{A,B}$ and $\kappa^{\dagger}_{A,C}$ such that

$$\mathcal{S}_i\left(\kappa^{\dagger}_{A,B}, \kappa^{\dagger}_{A,C}, 1, 1, 1\right) = 0. \tag{C.1}$$

To determine $\kappa^{\dagger}_s$ and evaluate welfare at that value, we approximate the policy functions using cubic splines.[10]

The solution for κ consists of finding Negishi weights that satisfy the symmetric asset allocation at time zero (see main text), the relative share of consumption consistent with a symmetric distribution of Arrow securities across countries at time zero, assuming that the economy is not hit by any shock at that time. The same concept we use under perturbation.[11]

[9] Technically speaking κ is a state variable, since it depends on time-zero moments and allocations. Nevertheless κ is time invariant and could be treated as a deep parameter, that ensures that Arrow securities satisfy the "time zero" asset allocation condition. We thank Felix Kubler for suggesting this approach.

[10] The Negishi weights $(\kappa_{A,B}, \kappa_{A,C})$ are discretised on an evenly-spaced grid of $N_\kappa + 1$ nodes spanning $[0.45, 1.35]$, and the optimum $\kappa^{\dagger}_s$ is located by bicubic-spline interpolation of the asset-market conditions $\mathcal{S}_i(\kappa_{A,B}, \kappa_{A,C}, 1, 1, 1) = 0$, so its resolution is finer than the nodal spacing. We set $N_\kappa = 18$. This choice places the no-transfer reference $\kappa = 1$—against which the welfare gains are normalised—*exactly* on a node: a node coincides with $\kappa = 1$ if and only if N_κ is a multiple of 18, since $(1 - 0.45)/(1.35 - 0.45) = 11/18$. It also keeps the state space, of size $(N_\kappa + 1)^2 N_S^3$, tractable at $N_S = 13$.

[11] The global solution of our three country model is implemented in Fortran—a useful reference for the methods we adopt is Fehr and Kindermann (2018). The codes are available from the authors upon request. Solving the three-country model takes several hours on a standard PC. In recent literature, important advancements in the solution of non-linear DSGE models make use of Machine Learning, e.g., Scheidegger and Bilionis (2019), Maliar et al. (2021) and Azinovic et al. (2022). We make a Python implementation of these techniques to solve a two-country version of our model available online. We leave extensions to more than two countries to future research.

D Analytical results

In the text, we have referred to $a_\bullet$ and $b_\bullet$ as complex convolutions of deep parameters (ρ, θ, δ and n_A). Below we show the analytical expression for $a_\bullet$ and $b_\bullet$ for the special case of $n_A = \frac{1}{2}$:

$$a_{\kappa,2} = \frac{1}{4(1-\delta)} \tag{D.1}$$

$$a_{\kappa,2^2} = \frac{\rho}{32(1-\delta)} \tag{D.2}$$

$$a_{\kappa,3} = \frac{1}{12(1-\delta)} \tag{D.3}$$

$$a_{\kappa,4} = \frac{1}{48(1-\delta)} \tag{D.4}$$

$$a_{\kappa,\Gamma} = \frac{\delta(\rho-1)(\theta(\rho-2)-1)}{32(1-\delta)\theta} \tag{D.5}$$

$$a_{\gamma,A} = \frac{\delta(2\theta - 2\theta^2 + \rho - 4\theta\rho + 3\theta^2\rho)}{8(1-\delta)\theta^2} \tag{D.6}$$

$$a_{\gamma,B} = \frac{\delta(\rho - \theta^2 - 2\theta)}{8(1-\delta)\theta^2} \tag{D.7}$$

$$a_{\phi,A} = \frac{\delta(\theta-1)(\rho-1)(-2-\rho+\theta(1+5\rho)-\theta^2(7\rho-4))}{48(1-\delta)\theta^3} \tag{D.8}$$

$$a_{\phi,B} = \frac{\delta(\theta-1)(\rho-1)(-2-\rho-\theta(-1+\rho)-\rho-\theta^2(\rho-4))}{48(1-\delta)\theta^3} \tag{D.9}$$

$$a_{\eta,A} = \frac{\delta(\theta-1)}{384(1-\delta)\theta^4}(\rho(-6+3\rho+\rho^2)-\theta^2(-4+27\rho^2-17\rho^3)- \tag{D.10}$$

$$\theta(8-24\rho+3\rho^2+7\rho^3)-\theta^3(-8+30\rho-39\rho^2+15\rho^3)) \tag{D.11}$$

$$a_{\eta,B} = \frac{\delta(\theta-1)}{384(1-\delta)\theta^4}(\rho(-6+3\rho+\rho^2)-\theta^2(-4+3\rho^2-\rho^3)- \tag{D.12}$$

$$\theta(8-3\rho+\rho^2)-\theta^3(-8+18\rho-9\rho^2+\rho^3)) \tag{D.13}$$

$$b_{\gamma} = \frac{\delta(\theta-1)(\rho-1)}{\theta} \tag{D.14}$$

$$b_{\phi} = \frac{\delta(\theta-1)(2-\theta-3\theta\rho-\theta^2(4-9\rho+3\rho^2))}{4\theta^3} \tag{D.15}$$

$$b_{\gamma_A^2} = \frac{\delta(\theta-1)(\rho-1)(-3\delta\theta(1+\theta(\rho-2))(\rho-1))}{2\theta^3} \tag{D.16}$$

$$b_{\gamma_B^2} = -\frac{\delta(\theta-1)(\rho-1)(-3\delta\theta(1+\theta(\rho-2))(\rho-1))}{2\theta^3} \tag{D.17}$$

$$b_{\gamma_B\gamma_A} = 0 \tag{D.18}$$

$$b_{\eta} = \frac{\delta(\theta-1)(\rho-1)(-2+\theta+3\theta\rho+\theta^2(2-5\rho+\rho^2))}{2\theta^3} \tag{D.19}$$

Table 1: Values of parameters a and b under our calibration and $n_A = 0.5$

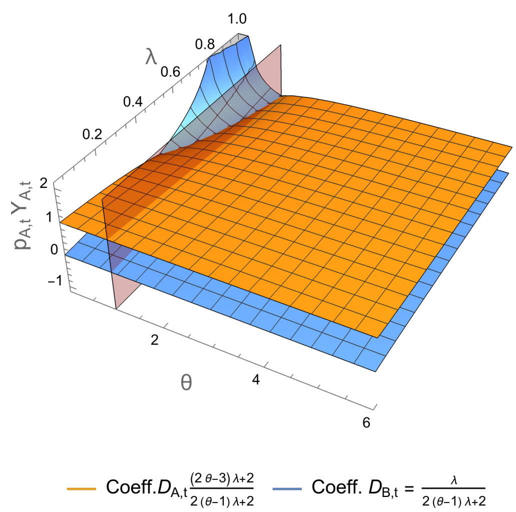
$a_{\kappa,2}$	$a_{\kappa,2^2}$	$a_{\kappa,3}$	$a_{\kappa,4}$	$a_{\kappa,\Gamma}$	$a_{\gamma,A}$	$a_{\gamma,B}$	$a_{\phi,A}$
12.5	6.25	4.17	1.04	12.25	14.97	9.52	12.93

$a_{\phi,B}$	$a_{\eta,A}$	$a_{\eta,B}$	b_{γ}	b_{ϕ}	$b_{\gamma_A^2}$	$b_{\gamma_B^2}$	$b_{\gamma_B\gamma_A}$	b_{η}
4.73	6.94	1.22	0.98	1.94	11.52	11.52	0	2.83

E Home bias and trade elasticity

In the main text, we have argued that trade elasticity regulates the relative "exposure" of a country to own and foreign production's shocks. We also argue that relative exposure is affected by the degree of home bias in consumption. We now offer a close-up analysis of the interaction between these two parameters. Figure 1 plots the response coefficient of a country's income to domestic and foreign output shocks. The plot shows that home bias can shift the threshold value of θ at which the relative exposure to own vs. foreign shocks falls from above to below 1. With sufficiently strong home bias (e.g., $\lambda \leq 0.3$), nonetheless, a country's income remains relatively more exposed to domestic shocks than to foreign shocks.

Figure 1: Response coefficients of country A's income to domestic and foreign shocks



Note: Coefficients of country A's and country B's TFP in the first-order approximation of country A's income. The vertical plane crosses the θ axis at 1.

F Correlation of TFP across countries and gains from risk sharing

In the main text, for the sake of simplicity, we have assumed that the stochastic processes for TFP are independent across countries. It is well understood that these processes may nonetheless reflect some degree of correlation of the underlying sources of risk. The traditional IRBC literature, e.g., Backus et al. (1992) indeed assumes that the TFP processes are correlated across countries. In this appendix we briefly discuss the implication for risk sharing. We do so analytically focusing on the instructive limit case in which shocks are perfectly correlated, yet they hit countries with different intensities. Without loss of generality, we develop our argument using the two-country version of the model.

In particular, we assume that

$$D_{B,t} = \zeta D_{A,t};\ \zeta \geq 0. \tag{F.1}$$

Equation (F.1) implies that $corr\left(D_{A,t}, D_{B,t}\right) = 1$. Yet, the intensity of the shocks differs across countries. In a stylized yet compelling way, this case captures one of the core premises of our analysis—strong evidence that shocks, even when global in nature, affect countries in largely asymmetric ways.

To show that perfect correlation of shocks does not rule out gains from risk sharing, we solve the model imposing equation (F.1). Focusing on skewness, up to fourth order of accuracy, we will have:

$$\frac{\partial RGRS}{\partial \phi_A} = \frac{\beta(\zeta-1)^3(\theta-1)^2 n^2 \rho(\theta\rho+\theta+\rho-3)}{2(\beta-1)\theta^3} + \frac{\beta(\zeta-1)^2(\theta-1)^2\rho(\zeta(2\theta\rho-\theta+\rho-2)+(\theta-1)(\rho-2)+3n(-\zeta(2\theta\rho+\rho-3)+2\theta+\rho-3))}{6(\beta-1)\theta^3} \tag{F.2}$$

Similar expressions, albeit more involved, hold for variance and kurtosis. The main takeaway is straightforward: as long as global shocks hit countries with different intensities (or transmit across borders asymmetrically), the gains from risk sharing are not zero, and will generally differ across borders.

We conclude with a comment on the maintained view that the benefits from international insurance are small if shocks are positively correlated across borders or have global nature. From our analysis above, it follows that a higher degree of correlation of GDP across countries may reduce or increase the gains from risk sharing, depending on the underlying heterogeneity of shock intensity across borders. In this sense, the high degree of dispersion in the distribution of moments that we document in the main text is likely to generate gains from insurance that dominate the mitigating effect of positive correlations.

G Generating skewed-leptokurtic distributions

We generate skewed and leptokurtic distributions by adopting the mixed-Gaussian distribution approach discussed in Farmer and Toda (2017, FT henceforth).[12] In particular we proceed in two steps. First, we calibrate the parameters of a three-element Gaussian mixture (i.e., weights, means and standard deviations of each element) by minimizing the relative distance between the observed moments and those generated by this distribution. Second, we implement the algorithm suggested by FT. This consists of matching low-order moments of the conditional distribution using relative entropy as the objective function (i.e., the Kullback-Leibler information).[13]

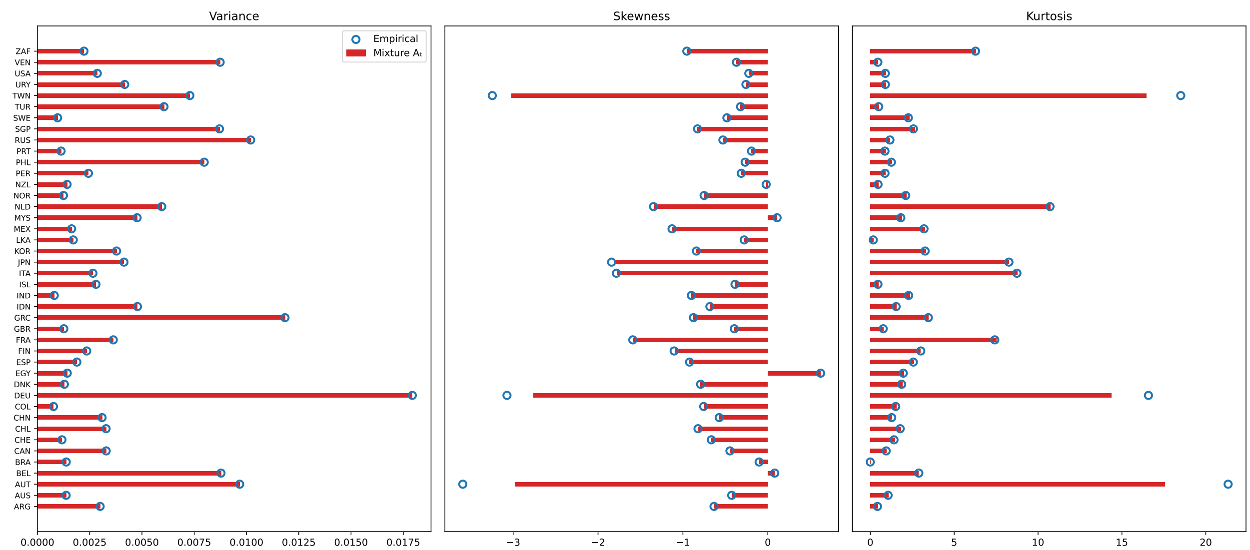
We proxy country risk (the TFP distribution in our model) by the distribution of detrended log real GDP per capita. Because rare disasters are central to our mechanism, we use the long historical sample obtained by splicing the Barro–Ursúa macroeconomic dataset with PWT 11.0, spanning 1901–2023. This sample includes the pre-1960 catastrophes (the two World Wars and the Great Depression) that generate the fat left tails of interest. It comprises 42 countries with continuous annual series over the period. Unlike standard practice, we do not trim extreme observations: retaining the rare disasters is precisely the object of interest.

On the basis of these data we proceed as follows. First, we detrend log real GDP per capita with the Hodrick–Prescott filter (Hodrick and Prescott, 1997) with smoothing parameter $\lambda = 100$ (the standard value for annual data), computing the trend directly as $(I + \lambda D'D)^{-1}y$, where D is the second-difference operator. Second, we fit an AR(1) to the cyclical component. Third, we compute the first four moments of the AR(1) innovation. Fourth, we calibrate a three-element Gaussian mixture to these moments. Fifth, we discretize the resulting process with the FT method on an evenly-spaced grid of $N_S = 13$ points; the

[12] See also Civale et al. (2017). The empirically relevant case is of leptokurtic distributions (fatter tails than the Gaussian). We thus refer for simplicity to leptokurtosis as shorthand for non-Gaussian kurtosis.

[13] We refer to Farmer and Toda (2017) for details. We implemented their method converting and adapting their Matlab codes into Python.

Figure 2: Moment matching: empirical versus discretized mixed-Gaussian



Note: Variance, skewness and excess kurtosis, country by country. Circles: empirical moments, from the estimated AR(1) for the cyclical component of log real GDP per capita (HP filter, $\lambda = 100$). Bars: moments of the ergodic distribution of the FT-discretized (Farmer and Toda, 2017) mixed-Gaussian process calibrated to the empirical innovation moments.

number of nodes is odd so that one node coincides with the (zero) conditional mean, which is needed to represent the sharp peak of a leptokurtic distribution. Figure 2 compares, country by country, the empirical moments (circles, obtained by simulating the estimated AR process) with those of the ergodic distribution of the discretized process (bars).[14]

H Generation of an artificial sample of economies

A mixed-Gaussian distribution (with the underlying constraints on its parameters) can map into a limited set of values for its moments. Only within this set is it possible to draw

[14]The FT step matches the conditional moments by minimizing the Kullback–Leibler divergence to the calibrated mixture on the finite grid. Matching the fourth moment requires the target to lie in the set of moments attainable by a distribution supported on the grid (its convex hull). When it does not, FT matches fewer moments at the affected nodes. This occurs only for near-Gaussian, thin-tailed countries, whose high moments are immaterial for welfare; the leptokurtic countries that drive our results are matched closely (excess kurtosis within roughly 10% of target). The first step yields negligible residuals for all countries.

moments that are independent of each other. Along the border of this set, changing one moment requires adjusting other moments, thus generating a correlation among them. Algorithm 1 is designed to draw from the set of admissible moments (second to fourth) using a three-term mixed-Gaussian distribution.

Algorithm 1 Construction of the artificial sample of countries

1: Express the TFP process of each country in terms of a three-term mixed-Gaussian distribution, e.g., for country A (analogously for country B)

$$\sigma_A \varepsilon_{A,t} \approx p_1 \mathcal{N}(m_1, v_1) + p_2 \mathcal{N}(m_2, v_2) + (1 - p_1 - p_2)\mathcal{N}(m_3, v_3)$$

where $\mathcal{N}(m_i, v_i)$ is the Gaussian PDF with mean m_i and variance v_i and p_i is the weight of the i term in the mixed-Gaussian distribution;

2: Express the parameters v_1, v_2 and v_3 in terms of $\gamma_A = E\left(\sigma_A^2 \varepsilon_{A,t}^2\right)$, $\phi_A = E\left(\sigma_A^3 \varepsilon_{A,t}^3\right)$ and $\eta_A = E\left(\sigma_A^4 \varepsilon_{A,t}^4\right)$ and the other parameters of the distributions.

3: Draw random values (from uniform distributions) for p_1, p_2,γ_A, ϕ_A, η_A, m_2 and m_3. Where m_1 is pinned down by the assumption that $E(\sigma_A \varepsilon_{A,t}) = 0$;

4: Discretize the resulting mixed-Gaussian distributions. Denote by N_D the number of these distributions;

5: Use the N_D distributions to parameterize the TFP process of country A in N_D economies, where sizes are set randomly and country B is the same in all of the N_D economies.

6: Find the solution for all the N_D economies.

To search for the admissible set we drew 100,000 values for the second-to-fourth uncentered moments from uniform distributions. The limits of these distributions were set so that: standard deviation $\in (0.001, 0.15)$, skewness $\in (-3, 3)$ and kurtosis $\in (3, 20)$. Moreover, we drew the weights of the first two terms of the mixed-Gaussian distribution from the $(0, 1)$ interval (imposing that the sum of all three weights should be 1). Finally the mean of the second and third Gaussian terms were drawn from the $(-1, 1)$ interval; the mean of the first term being set so to obtain a zero-mean for the whole mixed-Gaussian distribution.

The relative size n_A of each artificial economy is drawn at random, uniformly in log population over the range spanned by the data, while the partner economy B is held fixed across all draws; the resulting sizes cover the empirical range but do not correspond

Table 2: Distribution of generated sample

	5%	25%	50%	75%	95%
size	0.0007	0.0034	0.0259	0.1884	0.5646
standard deviation	0.0557	0.0939	0.1199	0.1357	0.1470
skewness	-1.8391	-0.7158	0.0571	0.8448	2.0182
kurtosis	4.8536	9.0786	13.3176	16.8538	19.1461

to any actual country (the sample consists of artificial units). Openness is governed by a common home-bias parameter, $\lambda_A = \lambda = 0.3$ for every economy, so that the import share $1 - \nu_A = (1 - n_A)\lambda$ defined in Section 2 is pinned down by size alone; openness is therefore not varied independently of size in this sample.

The distribution of the key parameters of this generated sample is summarized in Table 2.

Figure 3 shows a sub-set of variables of the artificial sample generated by following Algorithm 1. Particularly noteworthy are the scatter plots relating the three moments of interest. For example, the third and fourth columns of the second-last row show the set of admissible values for the fourth uncentered moment mapped against the second and third moments. Only within the set is it possible to pick independent moments.

H.1 Size and higher moments under GHH preferences

Table 3 in the main text reports, for CRRA preferences, the dependence of relative welfare, the level effect and the terms of trade on size and the cumulants of the artificial sample. Table 3 reproduces the exercise under GHH preferences (welfare in permanent-consumption units). The signs are unchanged: smaller and riskier economies, and those with more negative skewness, gain more, while the level effect moves in the opposite direction. The magnitudes, however, are uniformly larger under GHH—by a factor of roughly two to three—and the amplification is most pronounced for kurtosis, so that fat tails are an even stronger driver of

Figure 3: Distributions of the key variables in our ad-hoc sample.

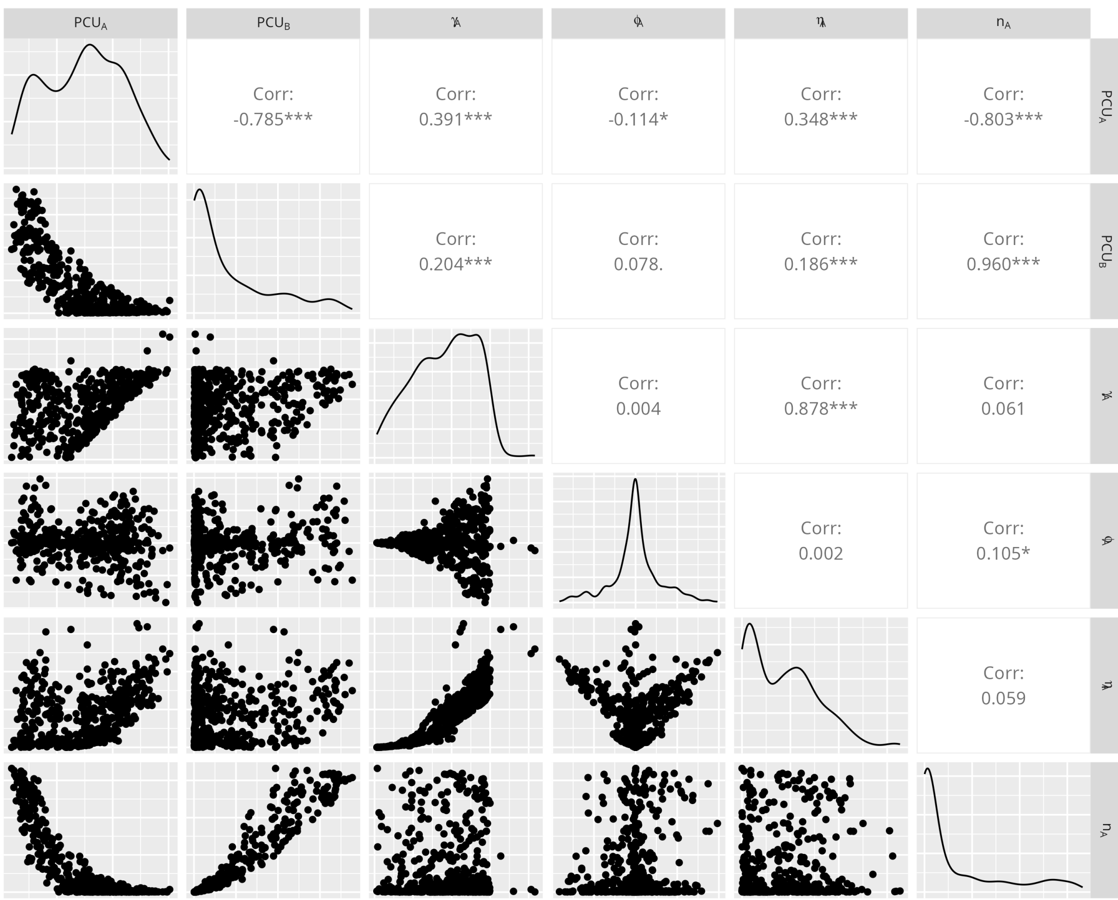
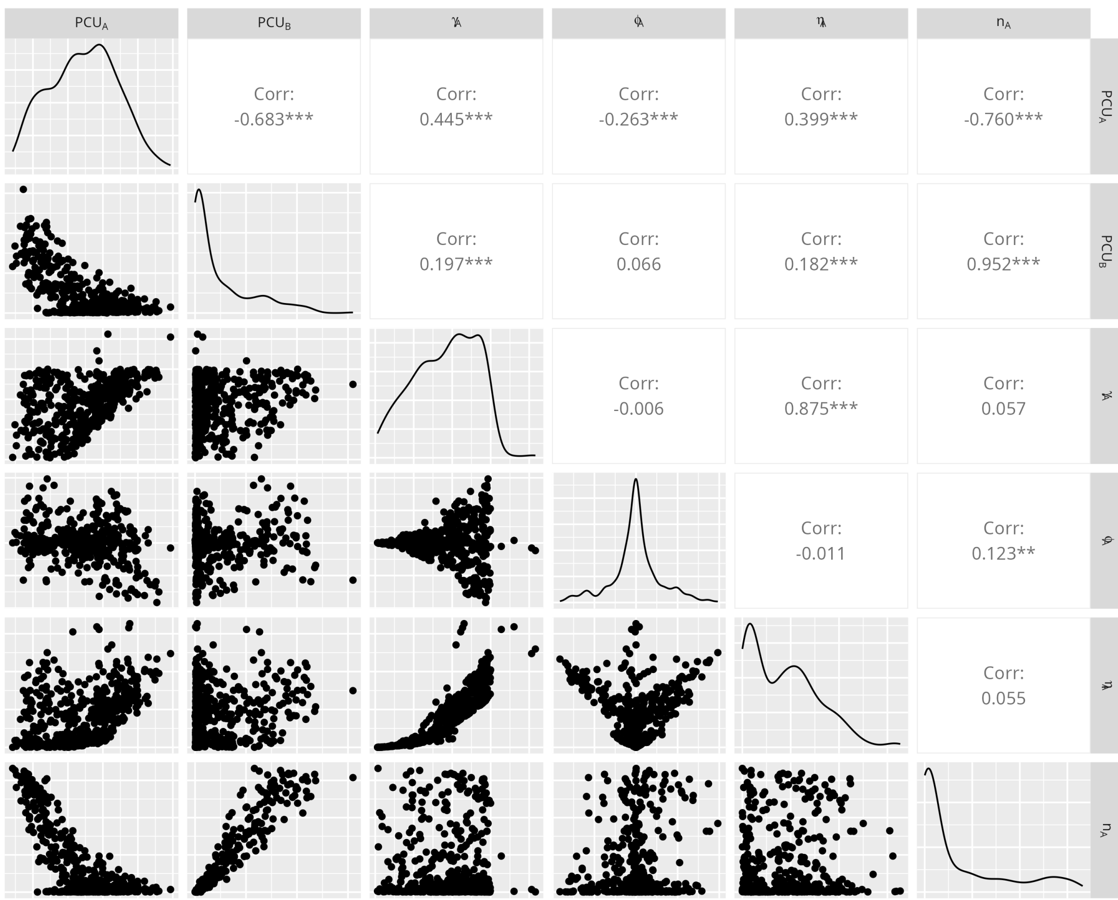


Figure 4: Distributions of the key variables in our ad-hoc sample — GHH preferences.



the cross-country distribution of gains than under CRRA.

Table 3: Relative welfare (PCU), κ, and the terms of trade as a function of cumulants and country size — GHH preferences

	$PCU_A - PCU_B$	$\ln \frac{\kappa^{cm}_{A,B}}{\kappa^{au}_{A,B}} \times 100$	$\ln \frac{ToT_{cm}}{ToT_{au}} \times 100$
Constant	0.013 [0.009, 0.017]	−0.019 [−0.022, −0.015]	−0.012 [−0.029, 0.006]
$n_A - n_B$	−4.321 [−4.333, −4.310]	6.884 [6.875, 6.892]	4.284 [4.240, 4.331]
$\gamma_A - \gamma_B$	67.796 [66.720, 68.887]	15.843 [15.052, 16.645]	34.720 [30.174, 39.272]
$\phi_A - \phi_B$	−98.800 [−101.300, −96.328]	−33.189 [−35.058, −31.308]	−36.464 [−46.793, −26.586]
$\eta_A - \eta_B$	71.400 [65.058, 77.644]	15.151 [10.330, 19.798]	19.448 [−7.156, 45.148]
$(n_A - n_B) \times (\gamma_A - \gamma_B)$	−97.919 [−101.153, −94.745]	155.486 [152.911, 157.866]	89.984 [76.452, 102.299]
$(n_A - n_B) \times (\phi_A - \phi_B)$	93.073 [86.799, 99.361]	−192.429 [−197.509, −187.524]	−107.730 [−133.541, −82.188]
$(n_A - n_B) \times (\eta_A - \eta_B)$	−32.552 [−50.959, −13.730]	96.285 [81.757, 110.950]	63.148 [−11.449, 140.307]
R^2	0.999	1.000	0.987

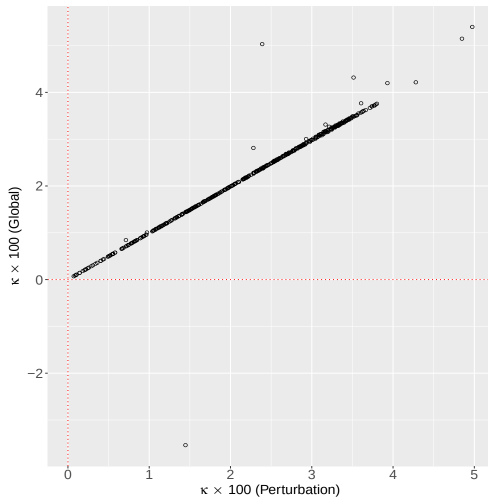
Note: Bayesian estimation of a linear model with interaction terms (GHH preferences; welfare in permanent-consumption units, PCU). Posterior means with 95% Highest Posterior Density intervals in brackets. Number of observations: 477.

I Comparison of the perturbation solution with the global solution

In the main text we have argued that the perturbation-based solution of the model delivers qualitatively reliable results. This section offers an example of the gap between the perturbation and global solution for a key variable in our analysis, κ.[15] For this comparison both solutions impose the same configuration: a two-country endowment economy ($\alpha = 1$) with equal sizes ($n = \frac{1}{2}$), no home bias ($\lambda = 1$, so that PPP holds), a non-stochastic endowment in country B, trade elasticity $\theta = 6$, risk aversion $\rho = 4$, and discount factor $\beta = 0.98$; the perturbation solution is computed to fourth order and evaluated at the ergodic moments implied by the discretization of the DGP for home TFP for 477 "countries". Figure 5 shows

[15]The global solution is typically more accurate than "asymptotic" (also known as local) approximations. Global methods entail some approximations (e.g., in discretizing the state space). Yet, the error can be more easily quantified and improved using global methods (e.g., by refining the grid) than perturbation methods. See e.g., Judd (1998) for a discussion and comparison of these methods.

Figure 5: Comparison of solution methods for κ



the value of $\kappa_{A,B}$ (κ for short) obtained using perturbation method (x-axis) against the value obtained using global methods (y-axis). Regressing the global solution on the perturbation solution (both in logs) yields a slope of 1.01 with a correlation of 0.97: the two methods agree one-for-one across economies, with discrepancies concentrated among the economies with the most extreme higher-order moments—precisely where global methods are needed.

J Unpacking the RoW into smaller units

To gauge whether subdividing the large regions into smaller units will affect our results, note that by the equilibrium expression for $\mu_{A,0}$

$$\mu_{A,0} = \left[n_A + (1 - n_A) \sum_{i=1}^{N} N^{-1} Q_{A,i,0}^{\frac{\rho-1}{\rho}} \kappa_{A,i} \right]^{-1} \tag{J.1}$$

a country's consumption share of total output will not vary as we increase N only if $Q_{A,i,0}^{\frac{\rho-1}{\rho}} \kappa_{A,i}$ remains constant. Consumption smoothing should not be affected by this sub-partition of RoW, as the SDF will still be determined by global income. But the regression results in the main text suggest that both κ and Q (via ToT)[16] vary with size, albeit considerably less than one-to-one. We can thus conclude that breaking up RoW into sub-units does have material implications for country A's consumption share and welfare. Quantifying this effect would require solving our model for a sufficiently large number of countries, at a very large computational cost.[17]

K Proof of Propositions 4.5 and 4.6

Some algebra manipulation of the solution of the model yields

$$\frac{\partial\left(\mathcal{W}_{A,0}^{cm}-\mathcal{W}_{A,0}^{au}\right)}{\partial n_A}=\frac{\beta(\theta-1)^2\rho}{(\beta-1)\theta^2}\cdot\Gamma-\frac{\beta(\theta-1)^2(\rho-1)\rho}{2(\beta-1)\theta^2}\cdot\Phi+\frac{\beta(\theta-1)^2\rho\left(-5+\rho^2+\theta(6+3\rho-\rho^2)+\theta^2(2-9\rho+3\rho^2)\right)}{24(\beta-1)\theta^4}\cdot H$$

Rearranging terms yields the results in Proposition 4.5.

By setting $n_A=\frac{1}{2}$, $\nu_A=1-n_B\lambda$ and $\nu_B=1-n_A\lambda$ we have also that

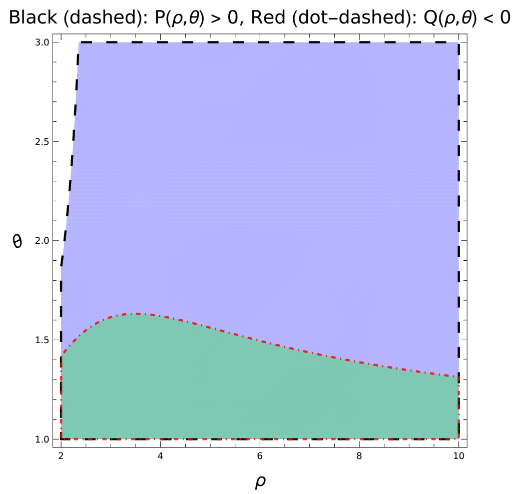
$$\left.\frac{\partial\left(\mathcal{W}_{A,0}^{cm}-\mathcal{W}_{A,0}^{au}\right)}{\partial\lambda}\right|_{\lambda=1}=\frac{\beta\Gamma(\theta-1)^2}{(1-\beta)\theta^3}-\frac{\beta(\theta-1)^2(\rho-1)}{2(1-\beta)\theta^3}\Phi+-\frac{\beta(\theta-1)^2\left[\theta^2(\rho^3-3\rho^2+8\rho-2)-\theta(\rho^3+2\rho+5)+4\right]}{24(1-\beta)\theta^5}H$$

Provided $\Phi<0$, Figure 6 shows the range of values for ρ and θ for which the conditions in Proposition 4.5 and 4.6 are always satisfied. The effect of size under PPP is robust, provided the degree of risk aversion is sufficiently large. The sign of $Q(\rho,\theta)$ changes for sufficiently large trade elasticity. In that case kurtosis mitigates the effect of openness on

[16]Note that $ToT_{au,0}=1$.

[17]Solving our three country model starting from a reasonable guess of the state-space (e.g., solving the equal-size case starting from the heterogeneous-size solution) takes approximately 19 hours on a Laptop with 4-core Intel(R) Core(TM) i7-4810MQ CPU @ 2.80GHz (3.80 max).

Figure 6: Regions of $P(\rho, \theta) > 0$ and $Q(\rho, \theta) < 0$



the welfare gains (around PPP), and in extreme cases it can also overturn it.

L Smoothing and Level Effects in a SOE

We have argued that smoothing and level effects can have the same sign in a SOE, i.e., when $n_A \to 0$, under the assumption that Country B output is deterministic and constant: $\gamma_B = \phi_B = \eta_B = 0$. Under this assumption, perfect risk sharing implies that the SDF of country A is determined solely by foreign factors, and constant. Now, writing the country A's expected income stream ($E\left(p_{A,t}Y_{A,t}\right)$) as a function of cumulants and taking its derivative with respect to γ_A, ϕ_A and η_A yields:

$$\lim_{n_A \to 0} \frac{\partial E\left(p_{A,t}Y_{A,t}\right)}{\partial \gamma_A} = \frac{1}{2}\left(\frac{\theta - 1}{\theta}\right)^2 + \mathcal{O}\left(\omega^5\right), \tag{L.1a}$$

$$\lim_{n_A \to 0} \frac{\partial E\left(p_{A,t}Y_{A,t}\right)}{\partial \phi_A} = \frac{1}{6}\left(\frac{\theta - 1}{\theta}\right)^3 + \mathcal{O}\left(\omega^5\right), \tag{L.1b}$$

$$\lim_{n_A \to 0} \frac{\partial E\left(p_{A,t}Y_{A,t}\right)}{\partial \eta_A} = \frac{1}{24}\left(\frac{\theta - 1}{\theta}\right)^4 + \mathcal{O}\left(\omega^5\right). \tag{L.1c}$$

In expectations, country A's income is increasing in the variance and kurtosis of the country A's TFP, as well as in its skewness provided $\theta > 1$. As expected income changes with cumulants, so do expected consumption and κ. Specifically:

$$\lim_{n_A \to 0} \frac{\partial \kappa_{A,B}}{\partial \gamma_A} = -\frac{\delta}{4} \frac{(\theta-1)^2}{\theta^4} \left(2\theta^2 - \delta\gamma_A(\theta-1)^2\right) + \mathcal{O}\left(\omega^5\right), \tag{L.2a}$$

$$\lim_{n_A \to 0} \frac{\partial \kappa_{A,B}}{\partial \phi_A} = -\frac{\delta}{6} \left(\frac{\theta-1}{\theta}\right)^3 + \mathcal{O}\left(\omega^5\right), \tag{L.2b}$$

$$\lim_{n_A \to 0} \frac{\partial \kappa_{A,B}}{\partial \eta_A} = -\frac{\delta}{24} \left(\frac{\theta-1}{\theta}\right)^4 + \mathcal{O}\left(\omega^5\right). \tag{L.2c}$$

Given the convexity of the underlying functions, the results above reflect the distribution of the log of TFP we have assumed in our analysis.[18] Observe that, in standard calibrations with $\gamma_A << 1$ and $\theta >> 0$ (including our baseline parameterization in the main text), an increase in country A's variance reduces $\kappa_{A,B}$. When n_A is very small, $\kappa_{A,B}$ becomes predominantly driven by the curvature of the functions that map output into domestic income.

M Country clustering with GMM

Model–based clustering. To synthesize the country characteristics that drive the cross–country welfare ranking, we group countries by *model–based clustering* via finite Gaussian mixture models, estimated by EM as implemented in the `mclust` package (Scrucca et al., 2023). The feature vector contains five elements per country: population size, openness (the consumption import share), and the standard deviation, skewness and kurtosis of the ergodic distribution of the discretized TFP process—i.e., exactly the size, openness and moment inputs of the quantitative model. Throughout we use the EEI covariance specification (diag-

[18] Setting $\tau = \dfrac{\theta-1}{\theta}$ in the main text, clarifies that all the three moments contribute to an increase in the average TFP (even in the case of a one-good economy), with $\tau = 1$ ($\theta \to \infty$).

onal covariance with equal volume and shape across components); with 39 observations and five features, richer covariance parameterizations are weakly identified.

Two–stage procedure. The grouping proceeds in two stages.

Stage 1 (global). All features are standardized to mean zero and unit variance across the full sample, and a mixture with $G = 4$ components is fitted. The BIC profile over G has a pronounced local optimum at $G = 4$; higher BIC values exist at larger G but are attained by proliferating thinly populated components (at the BIC maximum, $G = 7$, four of the seven components contain three or fewer countries), which we avoid. The $G = 4$ solution isolates the extremes of the sample: two clusters of large, relatively closed economies (CHN–IND and BRA–IDN–USA) and one cluster of four economies with markedly fat-tailed TFP (AUT, DEU, NLD, TWN), leaving a fourth cluster with the remaining 30 countries. Assignments are sharp: the maximum posterior classification uncertainty is 0.11.

Stage 2 (within the large cluster). The global standardization is dominated by the extreme observations—the fat-tailed cluster has mean kurtosis near 18, against about 5.5 for the remaining countries—so the 30 countries of the fourth cluster are compressed near the origin of the global z-scores, which masks their internal heterogeneity. We therefore re-standardize the features within this subsample and fit a second EEI mixture. The BIC selects $G = 3$, splitting the subsample into a core of 21 countries, a group of five small open economies (BEL, GRC, MYS, PHL, SGP), and a group of four economies with moderately fat tails (FRA, ITA, JPN, ZAF). Assignments are again sharp, except for Malaysia (posterior uncertainty 0.41, between the small-open and core groups).

In the figure, the two stage-1 large-economy clusters are merged into a single "Large" group, as they share low openness and low welfare gains; this yields the five color-coded groups. Table 4 reports the average standardized features of each group. The welfare gains are not an input of the clustering at either stage, so the alignment between groups and the gains ranking in the main text is an out-of-sample property of the grouping.

Stability. We assess the stability of the grouping with the cluster-wise bootstrap of Hennig (2007): the full two-stage procedure is refitted on 500 bootstrap resamples and each original group is compared, via the Jaccard coefficient, with the most similar cluster in each resample. The extreme groups are the most stable (mean Jaccard 0.87 for CHN–IND and 0.73 for the fat-tailed group); the remaining groups lie between 0.55 and 0.66, consistent with a continuum of intermediate countries together with one sharply distinct fat-tailed extreme. The grouping is accordingly descriptive: it organizes the cross-country evidence, and no inference is attached to group boundaries.

Table 4: Standardised group means for population size, openness, standard deviation, skewness and kurtosis. "Large" merges the two stage-1 large-economy components.

Group	**Population size**	**Openness**	**Standard deviation**	**Skewness**	**Kurtosis**
■ Large	1.93	−1.36	−0.38	0.39	−0.54
■ Core	−0.30	−0.05	−0.52	0.36	−0.46
■ Small open	−0.30	1.41	1.32	0.57	−0.27
■ Moderate tails	−0.16	−0.59	−0.10	−0.91	0.93
■ Fat-tailed	−0.31	0.79	1.64	−2.16	2.52

N Distribution of gains and moments under CRRA preferences

For brevity, the main text reports the two exercises of Section 5.2 under GHH preferences. Tables 5 and 6 reproduce them under separable CRRA preferences, for the same 39 countries and the same two calibrations (mixed-Gaussian baseline versus PPP and equal size). The message is unchanged: cross-country asymmetries in size and home bias are the dominant driver, and removing them collapses both the gains—the median gain in permanent-consumption units falls from about 0.15% to 0.04%—and the complete-markets κ's. Gains are uniformly smaller than under GHH (median 0.15% against 0.34%), consistent with the labor wealth effect that partly self-insures households under CRRA. That same wealth effect

Table 5: Percentiles of PCU and κ_s — CRRA preferences

	mixed-Gaussian baseline			**mixed-Gaussian PPP and equal size**		
Variable[†]	min	median	max	min	median	max
PCU	0.05	0.15	1.13	0.02	0.04	0.30
$\kappa^{cm}_{j,AE}$	-8.38	0.17	12.16	-0.01	0.02	0.23
$\kappa^{cm}_{j,EME}$	-29.94	-23.34	-6.62	-0.01	0.03	0.23

[†] PCU $= (\xi^{CM} - \xi^{AU}) \times 100$, percent of permanent consumption; κ's in percent, in deviation from unity, i.e., $(x-1)\times 100$. Separable CRRA felicity. Each column reports the minimum, median and maximum across the same 39 countries.

keeps the labor moments responsive even in the PPP equal-size calibration, whereas under GHH they are identically zero.

O Openness and trade elasticity

In the main text we argue that openness and trade elasticity affect the gains from risk sharing—and their composition into LE and SE—by varying the exposure of countries to foreign shocks. To visualize this result, Figure 7 compares welfare gains and κ_s under different relative values of the home-bias parameters $\nu_A - \nu_B$ (along the horizontal axis) and for different values of the trade elasticity θ. We use the generated sample of 477 countries discussed in the main text and in this Appendix. This number results from selecting simulations yielding accurate solutions and displaying strongly independent moments. Since the baseline country in the sample has a particular degree of risk, we also show the case in which country B's production is not stochastic. Recall also that country B has $\nu_B = 1 - n_A\lambda$, with $\lambda = 0.3$.

Figures 7 and 8, for CRRA and GHH preferences respectively, show that the more closed country A is, the smaller are its gains from risk sharing. The gains are larger for larger

Table 6: Percentiles of (log) consumption and labor moments — CRRA preferences

	mixed-Gaussian baseline			**mixed-Gaussian PPP and equal size**		
Moment†	min	median	max	min	median	max
			Consumption			
$Stdev_C^{cm-au\ddagger}$	−4.23	−1.35	−0.37	−2.34	−0.68	−0.04
$Skew_C^{cm-au}$	−0.54	0.17	1.34	−0.60	0.16	0.95
$Kurt_C^{cm-au}$	−9.37	−1.14	0.09	−6.52	−0.78	0.53
			Labor			
$Stdev_L^{cm-au\ddagger}$	−6.04	−1.63	−0.36	−3.58	−0.89	0.08
$Skew_L^{cm-au}$	−2.38	−0.15	1.16	−1.98	−0.15	0.93
$Kurt_L^{cm-au}$	−16.15	−0.72	0.71	−13.17	−0.69	2.50

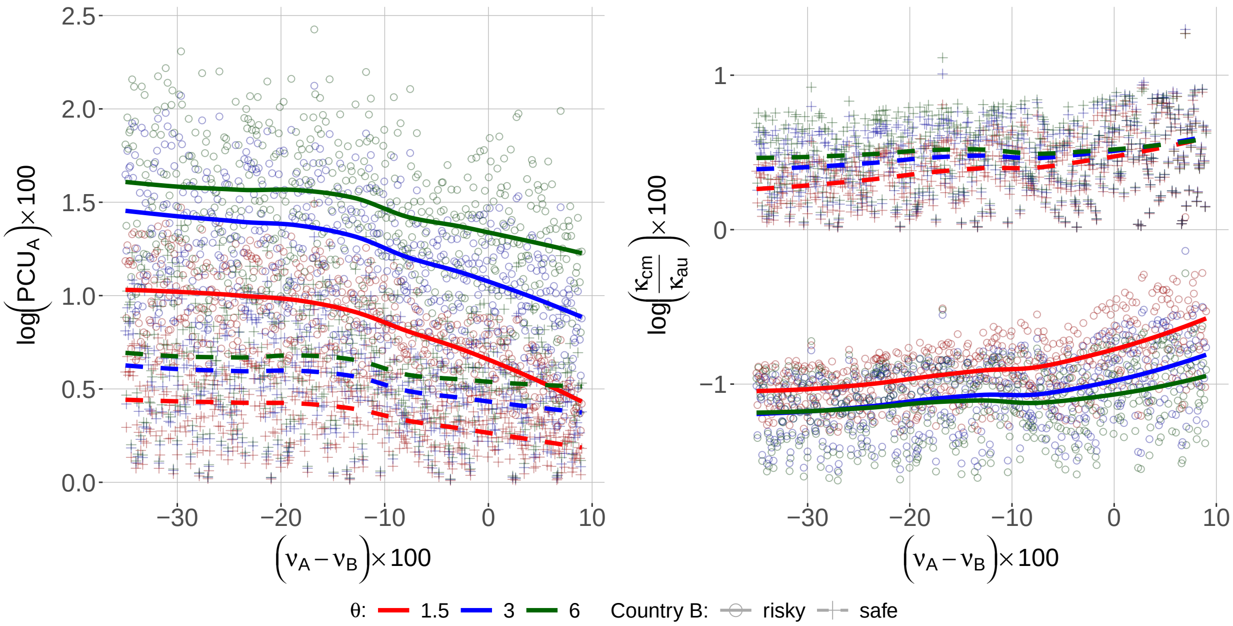
‡ Percentage points. † $moment_X^{cm-au}$, with $X = \{C, L\}$, is the complete-markets minus autarky difference of the moment of (log) X under the Markov-process ergodic distribution, separable CRRA preferences. Skewness and kurtosis are standardized. Each column reports the minimum, median and maximum across the same 39 countries. Unlike GHH, the labor moments respond to risk sharing even under PPP and equal size, reflecting the CRRA wealth effect on labor.

values of the trade elasticity. Along the openness dimension, gains and κ move in opposite directions: as country A becomes more open its gains rise while κ falls, consistent with the transfer mechanism discussed in the main text. These results are qualitatively independent of whether country B's production is risky. That said, the implicit transfer captured by κ is in absolute value larger, the larger is the trade elasticity: a country on the receiving end of the transfer receives more, the higher is θ.

P Consumption import shares

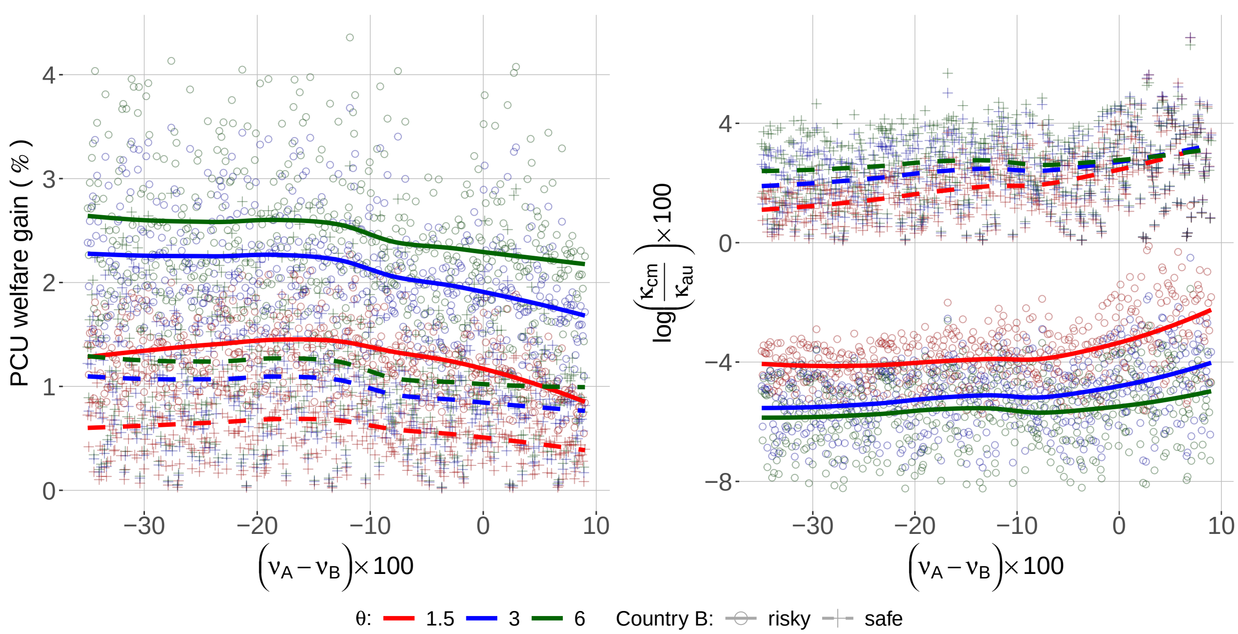
For the evidence-based exercise of Section 5.2 we calibrate each country's consumption home bias $1-\nu_j$ from its consumption import share. We measure consumption-import penetration as the foreign value-added content of final domestic demand, computed from the OECD

Figure 7: Openness, welfare gains and LE: the role of trade elasticity — CRRA preferences



Note: circles (solid fitting line) denote welfare gains for different countries "A" (left panel) and $\kappa_{A,B}$ relative to autarky (right panel) when country B's production is risky. Crosses (dashed fitting line) denote the case of safe country B's production.

Figure 8: Openness, welfare gains and LE: the role of trade elasticity — GHH preferences



Note: as the figure above, under GHH preferences. Left panel: welfare gain in permanent-consumption units (PCU); right panel: $\log(\kappa_{cm}/\kappa_{au}) \times 100$. Circles (solid line): country B risky; crosses (dashed line): country B safe.

Inter-Country Input–Output (ICIO) tables and averaged over 2016–2022: $cshare_j = 1 - \sum_s d_{j,s} \left[LF_j\right]_{j,s} / F_j^{\text{tot}}$, where $d_{j,s}$ is the domestic value added per unit of gross output of industry s, $L = (I - A)^{-1}$ is the global Leontief inverse, F_j is country j's final-demand vector, and F_j^{tot} its total final use. Relative to gross import shares, the value-added measure nets out re-exported content, which would otherwise overstate consumption openness for entrepôt economies. For the few sample economies not covered by the ICIO tables we use the direct measure $M/(C + I + G)$ from the World Bank's World Development Indicators, averaged over the same period.

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