Model Notes: What Comes Next?

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Abstract

These notes describes the model underlying the paper "What Comes Next?".

1 The Model

The economy is made up of $F$ industries. I use the letter $j$ to describe individual industries and the letter $i$ to describe individual firms within each industry. With some abuse of notation, I also use $i$ to denote household-level variables.

1.1 Nonlinear model

1.1.1 Households

The economy features two types of households: Ricardian households, who have access to financial markets and non-Ricardian households, who don’t. The share of the two household types is $\omega_r$ and $1 - \omega_r$.

Ricardian Households

There is a continuum of identical households indexed by $i$ (which I suppress when not important). The household’s problem is to choose aggregate and industry-level consumption, investment and capital, household-by-industry level wages and aggregate bond holdings to maximise utility:

$$\sum_{t=0}^{\infty} \beta^t e^{\xi_{c,t}} \log(C^r_t - hC^r_{t-1}) - \frac{A_N}{1+\nu} N^r_t(i)^{1+\nu}$$

subject to the budget constraint:

$$PC_tC^r_t + PI_tI^r_t + B_{t+1}/R_t \leq B_t + \sum_{j=1}^{F} \left( PC_t r^K_{j,t} k_{j,t} u_{j,t} / M + w_{j,t}(i) n^r_{j,t}(t) - a(u_{j,t}) k_{j,t} \right) + T^r_t$$

and capital accumulation constraints for each industry:

$$k_{j,t+1} = (1 - \delta)k_{j,t} + \left( 1 - S \left( \frac{z_{j,t}}{z_{j,t-1}} \right) \right) z_{j,t}$$

where $C_t$ is aggregate consumption, $\xi_{c,t}$ is a consumption preference shifter, $I_t$ is aggregate investment, $k_{j,t}$ is the capital stock of industry $j$, $u_{j,t}$ is the utilisation of capital in industry $j$ and $z_{j,t}$ is gross investment in industry $j$. $w_{j,t}$ is the wage in industry $j$, which is distinct from the wage paid to household $i$ in industry $j$, $w_{j,i}(i)$. Similarly $n_{j,t}(i)$ is hours worked by household $i$ in industry $j$, while $n_{j,t}$ is total hours worked in industry $j$. The household takes industry-level wages and hours worked as given in making its decisions. $T_t$ are lump sum transfers to the government. $\mathcal{M}$ is a wedge between the return on capital paid by firms and the amount received by households. It can

\[\text{1}^1\text{Bank for International Settlements}\]
be viewed as a reduced form for firm defaults or other factors that cause investors to demand a risk premium on lending to corporates.

Aggregate consumption and investment consist of bundles of consumption and investment goods sourced from each industry:

\[ C'_t = \left[ \sum_{j=1}^{F} \omega_{cj} c_{j,t} \right]^{\frac{1}{\eta}} \]  

(4)

\[ I'_t = \left[ \sum_{j=1}^{F} \omega_{ij} i_{j,t} \right]^{\frac{1}{\eta}} \]  

(5)

The price indices accompanying the consumption and investment aggregates are:

\[ P_{C,t} = \left[ \sum_{j=1}^{F} \omega_{c,j} p_{j,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \]  

(6)

\[ P_{I,t} = \left[ \sum_{j=1}^{F} \omega_{i,j} p_{j,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \]  

(7)

where \( p_{j,t} \) is the price of the good produced by industry \( j \).

It follows that the demand functions for the output of individual industries are:

\[ c_{j,t} = \omega_{cj} \left( \frac{p_{j,t}}{P_{C,t}} \right)^{-\eta} C_t \]  

(8)

\[ i_{j,t} = \omega_{ij} \left( \frac{p_{j,t}}{P_{I,t}} \right)^{-\eta} I_t \]  

(9)

Similarly, total labour supply, \( N_t(\iota) \) is a bundle of labour supplied to each sector:

\[ N'_t(\iota) = \left[ \sum_{j=1}^{F} \omega_{nj} n_{j,t}(\iota) \right]^{\frac{1}{\xi}} \]  

(10)

A labour packer aggregates the labour supply of individual households in each industry according to:

\[ n_{j,t} = \left( \int_{0}^{1} n_{j,t}(\iota) \frac{u^{1-\xi}}{\xi} \, du \right)^{\frac{1}{\xi}} \]  

(11)

Consequently, demand for different types of labour is given by:

\[ n_{j,t}(\iota) = \left( \frac{w_{j,t}(\iota)}{w_{j,t}} \right)^{-\xi \omega} n_{j,t} \]  

(12)

where \( w_{j,t} \) is the aggregate wage index in industry \( j \). The household takes this labour demand function into account when making its wage decisions.

Price and wage inflation is given by:

\[ \Pi_{C,t} = \frac{P_{C,t}}{P_{C,t-1}} \]  

(13)

\[ \Pi_{I,t} = \frac{P_{I,t}}{P_{I,t-1}} \]  

(14)
Market clearing for investment goods requires the aggregate volume of investment goods demanded by households to equal the sum of investment in all of the industries, that is:

$$I^r_t = \sum_{j=1}^{F} z_{j,t}$$  \hspace{1cm} (15)$$

Letting the Lagrange multipliers for the constraints on the budget constraint and the capital accumulation condition be \(\Lambda_t/P_{C,t}\) and \(\Lambda_t q_{j,t}\), the first order conditions for the household’s problem are:

$$\frac{e^{x_{C,t}}}{C_{t}^r - hC_{t-1}^r} = \Lambda_t + \beta E_t \left\{ \frac{h e^{x_{C,t+1}}}{C_{t+1}^r - hC_t^r} \right\}$$  \hspace{1cm} (16)$$

$$\Lambda_t = \beta R_t E_t \left\{ \frac{\Lambda_{t+1}^r}{\Pi_{C,t+1}} \right\}$$  \hspace{1cm} (17)$$

$$\Lambda_t q_{j,t} = \beta E_t \left\{ (1 - \delta_j) \Lambda_{t+1}^r q_{j,t+1} + \Lambda_{t+1}^r \frac{r_{j,t+1}^{k} u_{j,t+1}}{M} \right\}$$  \hspace{1cm} (18)$$

$$\Lambda_t = \Lambda_t q_{j,t} \left[ 1 - S \left( \frac{z_{j,t}}{z_{j,t-1}} \right) - S' \left( \frac{z_{j,t}}{z_{j,t-1}} \right) \frac{z_{j,t}}{z_{j,t-1}} \right] + \beta E_t \left\{ \Lambda_{t+1}^r q_{j,t+1} S' \left( \frac{z_{j,t+1}}{z_{j,t}} \right) \frac{z_{j,t}^2}{z_{j,t+1}^2} \right\}$$  \hspace{1cm} (19)$$

$$r_{j,t}^k = a(u_{j,t})$$  \hspace{1cm} (20)$$

**Non-Ricardian households**

Non-Ricardian households maximise the utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[ e^{x_{C,t}} \log(C_{t}^r - hC_{t-1}^r) - \frac{A_N}{1+\nu} N_t^{nr} (i)^{1+\nu} \right]$$  \hspace{1cm} (21)$$

subject to the budget constraint:

$$P_{C,t} C_{t}^{nr} \leq \sum_{j=1}^{F} w_{j,t} n_{j,t}^{nr} + T_t^{nr}$$  \hspace{1cm} (22)$$

The first order conditions for their problem are:

$$\frac{e^{x_{C,t}}}{C_{t}^{nr} - hC_{t}^{nr}} = P_{C,t} \Lambda_{t}^{nr} + \beta E_t \left\{ \frac{e^{x_{C,t+1}} h}{C_{t+1}^{nr} - hC_{t}^{nr}} \right\}$$  \hspace{1cm} (23)$$

which defines the marginal utility of consumption for non-Ricardian households.

### 1.1.2 Aggregate consumption and marginal utility of consumption

The ‘aggregate’ marginal utility of consumption, \(\Lambda_t\) is a weighted average of the marginal utilities of the Ricardian and non-Ricardian households:

$$\Lambda_t = \omega_r \Lambda_t^r + (1 - \omega_r) \Lambda_t^{nr}$$  \hspace{1cm} (24)$$

Similarly, aggregate consumption is a weighted average of Ricardian and non-Ricardian consump-
condition:
\[ C_t = \omega_r C_t^* + (1 - \omega_r) C_t^{nr} \tag{25} \]

1.1.3 Labour market:

In each industry, a continuum of perfectly competitive labour hiring firms combine the specialised labour types according to:

\[ n_{j,t} = \left( \int_0^1 n_{j,t}(s) \frac{w_{j,t}(s)}{w_{j,t}} ds \right) ^{\frac{1}{1-\nu}} \tag{26} \]

The hiring firm’s demand for each labour type \( j \) is given by:

\[ n_{j,t}(s) = \left( \frac{w_{j,t}(s)}{w_{j,t}} \right) ^{-\nu} n_{j,t} \tag{27} \]

where \( w_{j,t} \) is the industry wage index given by:

\[ w_{j,t} = \left( \int_0^1 w_{j,t}(s) ^{1-\nu} ds \right) ^{\frac{1}{1-\nu}} \tag{28} \]

Workers of type \( s \) unionise in order to take advantage of their monopoly power. These unions set nominal wages subject to the labour demand constraint and a Calvo friction that means that a random proportion, \( \theta_{w,j} \) of households cannot re-optimise their wage each period.

Unions that do not re-optimise their wages re-scale them according to the indexation rule that depends on industry-specific lagged wage inflation \( \pi_{w,j}^{w,-1} \):

\[ w_{j,t}(s) = (\pi_{w,j}^{w,-1})^{\chi_{w,j}} w_{j,t-1}(s) \]

Define:

\[ \Omega_{j,t,t+s} = \prod_{m=t}^{t+s-1} (\pi_{j,m}^{w})^{\chi_{w,j}} \]

to be the total indexation in period \( s \) of a union that last updated its wage in period \( t \).

Unions choose \( w_{j,t}(s) \) to maximise:

\[ L = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \theta_{w,j})^s \left[ \Upsilon_j^w \frac{\Lambda_{j,s+1}}{\Pi_{j,t+s}} w_{j,t}(s) \Omega_{j,t,t+s} \left( \frac{w_{j,t}(s) \Omega_{j,t,t+s}}{w_{j,t+s}} \right) ^{-\nu} n_{j,t+s} \right. \]

\[ \left. - \frac{A_N}{1 + \nu} \left[ \sum_{j=1}^{\mathcal{F}} \left( \int_0^1 \left( \frac{w_{j,t}(k) \Omega_{j,t,t+s}}{w_{j,t+s}} \right) ^{-\nu} n_{j,t+s} dk \right) \right] \right] ^{\frac{1}{1-\nu}} \tag{29} \]

where \( \Upsilon_j^w \) is a wage subsidy calibrated to offset the effect of imperfect labour market competition on employment.

The first order condition for this problem is:

\[ 0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \theta_{w,j})^s \left[ - (1 - \nu) \Upsilon_j^w \frac{\Lambda_{j,s+1}}{\Pi_{j,t+s}} \Omega_{j,t,t+s} \left( \frac{w_{j,t}(s) \Omega_{j,t,t+s}}{w_{j,t+s}} \right) ^{-\nu} n_{j,t+s} \right. \]

\[ + \left. \epsilon_w A_N \frac{1}{1-\nu} \frac{1}{n_{j,t+s}} \left( \Omega_{j,t+s} \right) ^{-\epsilon_w} \left( w_{j,t}(k) \right) ^{-\epsilon_w} \right] \tag{30} \]
which we can re-arrange to:
\[
\left[ \frac{w_{j,t}(k)}{w_{j,t}} \right]^{\frac{\xi(x)}{\xi}} = \frac{H_{w1,t}}{H_{w2,t}}
\]  
(31)

where:
\[
H_{w1,t} = \sum_{s=0}^{\infty} (\beta \theta_{w,j})^s A_N \nu^{-\frac{1}{s}} n_{j,t}^{1-s} \left( \Omega_{j,t+s} \frac{\pi_{w,j,t+s}}{\pi w_{j,t+s}} \right)^{-\epsilon w 1-\frac{s}{s}}
\]  
(32)
\[
H_{w2,t} = \sum_{s=0}^{\infty} (\beta \theta_{w,j})^s A_t^{s} \frac{w_{j,t+s}}{P_{C,t+s}} n_{j,t}^{s} \left( \Omega_{j,t+s} \frac{\pi_{w,j,t+s}}{\pi w_{j,t+s}} \right)^{1-\epsilon w}
\]  
(33)

We can re-write \( H_{w1,t} \) and \( H_{w2,t} \) as:
\[
H_{w1,t} = A_N \nu^{-\frac{1}{s}} n_{j,t}^{1-s} + \beta \theta_{w,j} E_t \left\{ \left( \frac{\pi_{w,j,t}}{\pi w_{j,t+1}} \right)^{-\epsilon w 1-\frac{s}{s}} H_{w1,t+1} \right\}
\]  
(34)
\[
H_{w2,t} = A_t^{s} \frac{w_{j,t+s}}{P_{C,t+s}} n_{j,t}^{s} + \beta \theta_{w,j} E_t \left\{ \left( \frac{\pi_{w,j,t}}{\pi w_{j,t+1}} \right)^{1-\epsilon w} H_{w2,t+1} \right\}
\]  
(35)

From the definition of the wage index, we also know that:
\[
1 = (1 - \theta_{w,j}) \left( \frac{w_{j,t}(k)}{w_{j,t}} \right)^{1-\epsilon w} + \theta_{w,j} \left( \frac{\pi_{w,j,t}}{\pi w_{j,t+1}} \right)^{1-\epsilon w}
\]  
(36)

1.1.4 Firms:

Firms in industry \( j \) produce output using capital, labour and intermediate goods according to the multi-layered production function:

\[
y_{j,t}^{va}(i) = \left[ \frac{1}{\omega_{n,j,t}(i)} + (1 - \omega_{n,j,t}) \frac{1}{k_{j,t}(i)} \right]^{\frac{\xi(x)}{\xi}}
\]  
(37)
\[
x_{j,t}(i) = \sum_{k=1}^{x} \omega_{j,k,t} x_{k,j,t}(i) \left[ \frac{\omega_{j,k,t}}{\omega_{j,t}} \right]^{\frac{\xi(x)}{\xi - 1}}
\]  
(38)
\[
y_{j,t}(i) = a_j \left[ \omega_{j,y,t} y_{j,t}(i) \left[ \frac{\omega_{j,y,t}}{\omega_{j,t}} \right]^{\frac{\xi(x)}{\xi - 1}} + (1 - \omega_{y,j}) \frac{1}{\xi(x) - 1} x_{j,t}(i) \right]^{\frac{\xi(x)}{\xi}}
\]  
(39)

where \( y_{j,t}^{va}(i) \) is the value added of firm \( i \) in industry \( j \), \( k_{j,t}(i) \) is the amount of capital hired by the firm, \( x_{j,t}(i) \) is the amount of intermediate goods used by the firm and \( y_{j,t}(i) \) is gross output of the firm. The total capital hired by industry \( j \) and total capital available to be hired is related by:

\[
k_{j,t} = u_{j,t} k_{j,t}
\]  
(40)

Marginal costs (deflated by industry-specific final prices) and the resulting demand functions are:

\[
mc_{j,t}(i) = \frac{1}{a_j} \left[ \omega_{y,j} (y_{j,t}^{va}(i)/p_{j,t})^{1-\phi} + (1 - \omega_{y,j}) (p_{j,t}^{va}(i)/p_{j,t})^{1-\phi} \right]^{\frac{1}{1-\phi}}
\]  
(41)
\[
(p_{j,t}^{va}(i))^{\phi} y_{j,t}^{va}(i) = \frac{1}{1-\omega_{y,j} (p_{j,t}^{va}(i))^{\phi} x_{j,t}(i)}
\]  
(42)
where the price indices for value-added and intermediate goods in each industry are given by:

\[ p_{t}^{\text{vva}} = \left[ \omega_{n}w_{j,t}^{1-\zeta} + (1 - \omega_{n,j})r_{j,t}^{k1-\zeta} \right]^{\frac{1}{1-\zeta}} \]  
(43)

\[ p_{j,t}^{x} = \left[ \sum_{k=1}^{x} \omega_{k,j}p_{k,t}^{1-\psi} \right]^{\frac{1}{1-\psi}} \]  
(44)

These imply the demand functions:

\[ n_{j,t} = \omega_{n,j} \left( \frac{w_{j,t}}{p_{j,t}^{\text{vva}}} \right)^{-\zeta} y_{j,t} \]  
(45)

\[ k_{j,t} = (1 - \omega_{n,j}) \left( \frac{r_{j,t}^{k}}{p_{j,t}^{\text{vva}}} \right)^{-\zeta} y_{j,t} \]  
(46)

\[ x_{k,j,t} = \omega_{k,j} \left( \frac{p_{k,t}}{p_{j,t}^{\text{vva}}} \right)^{-\psi} x_{j,t} \]  
(47)

In each industry, individual firms face price stickiness a la Calvo. Each period a fraction of firms, \( 1 - \theta_{pj} \), are able to change their prices. The remainder follow an indexing rule:

\[ p_{j,t}(\iota) = (\pi_{j,t+1})^{\varphi_{j,p}} p_{j,t-1}(\iota) \]

Define:

\[ \Omega_{j,t,t+s} = \prod_{m=t}^{t+s-1} (\pi_{j,m})^{\varphi_{j,p}} \]

as the cumulative change in prices between \( t \) and \( m \), conditional on not re-optimising.

The problem for a firm that is able to reset its prices at time \( t \) is:

\[
\max_{p_{j,t}(\iota)} \mathbb{E}_{t} \sum_{s=0}^{\infty} (\beta \theta_{pj})^{s} \left\{ \Lambda_{t+s} \left[ \frac{p_{j,t}(\iota)\gamma_{j,t+s}\Omega_{j,t,t+s}}{p_{j,t+s}} y_{j,t+s}(\iota) - \frac{1}{1 + \phi_{pj}} m_{c,j,t+s}\gamma_{j,t+s}y_{j,t+s}(\iota) \right] \right\} 
\]

where \( \gamma_{j,t+s} = \frac{p_{j,t+s}}{p_{C,t+s}} \) is the relative price of goods in industry \( j \), subject to the demand condition given above. The parameter \( \phi_{j} \) is a production subsidy to offset the steady-state distortion from imperfect competition. The first order condition for this problem is:

\[
\mathbb{E}_{t} \sum_{s=0}^{\infty} (\beta \theta_{pj})^{s} \left\{ \Lambda_{t+s} \left[ \frac{1 - \epsilon_{jp}\gamma_{j,t+s}}{p_{j,t}(\iota)} \left( \frac{p_{j,t}(\iota)\Omega_{j,t,t+s}}{p_{j,t+s}} \right)^{1-\epsilon_{jp}} y_{j,t+s} \right] + \frac{\epsilon_{jp}}{1 + \phi_{jp}} m_{c,j,t+s} \left( \frac{p_{j,t}(\iota)\Omega_{j,t,t+s}}{p_{j,t+s}} \right)^{-\epsilon_{jp}} \gamma_{j,t+s}y_{j,t+s} \right\} = 0 \]  
(48)

Re-arranging gives:

\[ \frac{p_{j,t}(\iota)}{p_{j,t}} = \frac{\epsilon_{jp}h_{j,p1,t}}{(1 + \phi_{jp})(\epsilon_{jp} - 1)h_{j,p2,t}} \]
where

\[ h_{j,p1,t} = E_t \sum_{s=0}^{\infty} (\beta \theta_{pj})^s \Lambda_{t+s} \left( \frac{\Omega_{j,t,t+s}}{\pi_{j,t,t+s}} \right)^{-\epsilon_{jp}} mc_{j,t+s} y_{j,t+s} \]  

(49)

\[ h_{j,p2,t} = E_t \sum_{s=0}^{\infty} (\beta \theta_{pj})^s \Lambda_{t+s} \left( \frac{\Omega_{j,t,t+s}}{\pi_{j,t,t+s}} \right)^{1-\epsilon_{jp}} y_{j,t+s} \]  

(50)

Note that:

\[ h_{j,p1,t} = \Lambda_{t} mc_{j,t} y_{j,t} + \beta \theta_{pj} E_t \left( \frac{\Omega_{j,t+1,t}}{\pi_{j,t+1,t}} \right)^{-\epsilon_{jp}} h_{j,p1,t+1} \]  

(51)

\[ h_{j,p2,t} = \Lambda_{t} y_{j,t} + \beta \theta_{pj} \left( \frac{\Omega_{j,t+1,t}}{\pi_{j,t+1,t}} \right)^{1-\epsilon_{jp}} h_{j,p2,t+1} \]  

(52)

Note also that the domestic price index can be expressed as:

\[ 1 = (1 - \theta_{jp}) \left( \frac{p_{j,t}(t)}{p_{j,t}} \right)^{1-\epsilon_{p}} + \theta_{jp} \left( \frac{\pi_{j,t-1}}{\pi_{j,t}} \right)^{1-\epsilon_{jp}} \]  

(53)

1.1.5 Market clearing and aggregate price indices:

Goods market clearing requires that:

\[ y_{j,t} = c_{j,t} + i_{j,t} + \sum_{k=1}^{F} x_{j,k} \]  

(54)

1.1.6 Monetary policy:

The monetary policy authority follows the policy rule:

\[ \frac{R_t}{R} = \left[ \frac{R_{t-1}}{R} \right]^{\rho_R} \left[ \left( \Pi_{Y}^{\epsilon_{R}} \right)^{\phi_{p}} \left( \frac{Y_{t}}{\bar{Y}} \right)^{\phi_{y}} \right]^{1-\rho_{R}} e^{\epsilon_{R},t} \]  

(55)

1.1.7 Fiscal policy:

The government budget constraint is:

\[ \frac{B_{t+1}}{R_t} = B_t + P_{G,t} G_t - T_t \]  

(56)

I assume that in steady state, government bonds are in zero net supply, so that \( B_t = 0 \forall t \).

Fiscal policy purchases goods and services, \( G_t \), according to the aggregate:

\[ G_t = \left[ \sum_{j=1}^{F} \omega_{g,j} \theta_{g,j,t} \right]^{\eta_{g}} \]  

(57)

Implying the price index:

\[ P_{t}^{g} = \left[ \sum_{j=1}^{F} \omega_{g,j}^{1-\eta_{g}} p_{j,t}^{1-\eta_{g}} \right]^{\frac{1}{1-\eta_{g}}} \]  

(58)
and final output demands:

\[ g_{j,t} = \omega_{g,j} \left( \frac{p_{j,t}}{P_{t}} \right)^{-\eta} G_{t} \]  

(59)

Aggregate government spending evolves according to:

\[ \frac{G_{t}}{G} = \left[ \frac{G_{t-1}}{G} \right]^{\rho_{G}} \exp(\xi_{t}) \]  

(60)

Transfers consist of transfers to Ricardian and non-Ricardian households:

\[ T_t = T^r_t + T^{nr}_t \]  

(61)

1.2 Steady state

The steady state of the system is given by:

From the first order condition for bond holdings:

\[ R = 1/\beta \]  

(62)

From the first order condition for investment

\[ q_j = 1 \]  

(63)

From the first order condition for capital:

\[ r^k_j = M \left( \frac{1}{\beta} - 1 + \delta \right) \]  

(64)

From the consumption choice for Ricardian households:

\[ \frac{1 - \beta h}{C^r(1-h)} = \frac{\Lambda^r}{\epsilon^{c_r}} \]  

(65)

From the budget constraint for for non-Ricardian households:

\[ C^{nr}_t = W_t N^{nr}_t + T^{nr}_t \]  

(66)

From the consumption choice of non-Ricardian households

\[ \frac{1 - \beta h}{C^{nr}(1-h)} = \frac{\Lambda^{nr}}{\epsilon^{c_{nr}}} \]  

(67)

I set the level of transfers so that the marginal utility of consumption for Ricardian and non-Ricardian households are equal, i.e. \( \Lambda^r = \Lambda^{nr} = \Lambda \).

From the definition of aggregate marginal utility:

\[ \Lambda = \omega_r \Lambda^r + (1 - \omega_r) \Lambda^{nr} \]  

(68)

From the definition of aggregate consumption:

\[ C = \omega_r C^r + (1 - \omega_r) C^{nr} \]  

(69)
From the wage choice:

\[ \omega_j^\frac{1}{1+\xi} A_n N^{\nu + \frac{1}{1+\xi}} n_j^{-\frac{1}{1+\xi}} = \Lambda w_j \]  

(70)

From the definition of aggregate labour supply:

\[ N = \left[ \sum_{j=1}^F \omega_{nj} n_j^{\frac{1}{1+\xi}} \right]^{\frac{1}{1+\xi}} \]

(71)

From the capital accumulation condition:

\[ z_j = \delta k_j \]  

(72)

From the market clearing condition for investment:

\[ I = \sum_{j=1}^F z_j \]  

(73)

From the demand function for consumption:

\[ c_j = \omega c_j \gamma_j^{-\eta} C \]  

(74)

From the demand function for investment:

\[ i_j = \omega_i j \left( \frac{\gamma_j}{\gamma_I} \right)^{-\eta} I \]  

(75)

From the demand function for government expenditure:

\[ g_j = \omega g_j \left( \frac{\gamma_j}{\gamma_G} \right)^{-\eta} G \]  

(76)

From the price index for consumption:

\[ 1 = \left[ \sum_{j=1}^N \omega_n j \gamma_j^{1-\eta} \right] \]

(77)

From the price index for investment:

\[ \gamma_I = \left[ \sum_{j=1}^N \omega_i j \gamma_j^{1-\eta} \right] \]  

(78)

From the price index for government expenditure:

\[ \gamma_G = \left[ \sum_{j=1}^F \omega g_j \gamma_j^{1-\eta} \right] \]  

(79)

From the definition of aggregate wages:

\[ W = \left[ \sum_{j=1}^N \omega l_j w_j^{1+\xi} \right]^{\frac{1}{1+\xi}} \]  

(80)
From the production function:

\[ y_j = a_j \left[ \omega_{y,j} y_j^\alpha + (1 - \omega_{y,j}) y_j^{\frac{\alpha - 1}{\alpha}} \right]^{\frac{\alpha}{\alpha - 1}} \]  

(81)

From the demand functions for intermediate goods:

\[ x_{kj} = \omega_{kj} \left( \frac{\gamma_k}{\gamma_j} \right)^{-\psi} x_j \]  

(82)

From the demand function for capital:

\[ k_j = (1 - \omega_n) \left( \frac{x_j^{\kappa_j}}{y_j^{\psi_j}} \right)^{-\zeta} y_j^{\psi_j} \]  

(83)

From the demand function for labour:

\[ n_j = \omega_{nj} \left( \frac{w_j}{y_j^{\psi_j}} \right)^{-\zeta} y_j^{\psi_j} \]  

(84)

From the definition of the price index of intermediate goods:

\[ \gamma^x_j = \left[ \sum_{j=1}^{J} \omega_{kj} \gamma_k^{1-\psi} \right]^{\frac{1}{1-\psi}} \]  

(85)

From the definition of the price of value added:

\[ \gamma_j^{yva} = \left[ \omega_{nj} w_j^{1-\zeta} + (1 - \omega_{nj}) x_j^{k_1-\zeta} \right]^{\frac{1}{1-\zeta}} \]  

(86)

From the relative demand for inputs:

\[ (\gamma_j^{yva})^{\phi_j^{yva}} = \frac{\omega_{y,j} (\gamma_j^x)^{\phi_j^x}}{1 - \omega_{y,j}} \]  

(87)

From goods market clearing:

\[ y_j = c_j + i_j + g_j + \sum_{k=1}^{K} x_{j,k} \]  

(88)

From the definition of \( h_{j,p2} \):

\[ h_{j,p2} = \frac{\Lambda \gamma_j y_j}{1 - \beta \theta_{pj}} \]  

(89)

From the price index for good \( j \):

\[ h_{j,p2} = h_{j,p1} \]  

(90)

From the definition of \( h_{j,p1} \):

\[ mc_j = 1 \]  

(91)

From the definition of marginal costs:

\[ a_j = \left[ \omega_{y,j} (\gamma_j^{yva})^{1-\phi_j^{yva}} + (1 - \omega_{y,j}) (\gamma_j^x)^{1-\phi_j^x} \right]^{\frac{1}{1-\phi_j^{yva}}} \]  

(92)
1.3 Linearised equations

Capital accumulation:
\[ \hat{k}_{j,t+1} - \delta \hat{z}_{j,t} = (1 - \delta) \hat{k}_{j,t} \] (93)

Consumption price index:
\[ 0 = \sum_{j=1}^{F} \omega_{c,j} \gamma_j^{1-\eta} \hat{c}_{j,t} \] (94)

Investment price index:
\[ \gamma_1^{1-\eta} - \sum_{j=1}^{F} \omega_{i,j} \gamma_j^{1-\eta} \hat{i}_{j,t} = 0 \] (95)

Government price index:
\[ \gamma_G^{1-\eta} - \sum_{j=1}^{F} \omega_{g,j} \gamma_j^{1-\eta} \hat{g}_{j,t} = 0 \] (96)

Consumption variety choice:
\[ \hat{c}_{j,t} = \hat{C}_t - \eta \hat{c}_{j,t} \] (97)

Investment variety choice:
\[ \hat{i}_{j,t} = \hat{I}_t - \eta (\hat{c}_{j,t} - \hat{c}_{I,t}) \] (98)

Government expenditure variety choice:
\[ \hat{g}_{j,t} = \hat{G}_t - \eta (\hat{c}_{j,t} - \hat{c}_{G,t}) \] (99)

Aggregate labour supply:
\[ N^{\xi+1} \hat{n}_t - \sum_{j=1}^{F} \omega_{n,j} \gamma_j \hat{n}_{j,t} = 0 \] (100)

Investment price inflation:
\[ \hat{\pi}_{I,t} - \hat{\pi}_t - \gamma_{I,t} = -\gamma_{I,t-1} \] (101)

Wage inflation:
\[ \hat{\pi}_{W,t} - \hat{\pi}_t - \hat{w}_t = -\hat{w}_{t-1} \] (102)

Aggregate wage index:
\[ W^{1+\xi} \hat{w}_t = \sum_{j=1}^{F} \omega_{n,j} W_j^{1+\xi} \hat{w}_{j,t} \] (103)

Investment market clearing:
\[ I_\hat{t} - \sum_{j=1}^{F} z_j \hat{z}_{j,t} = 0 \] (104)

Consumption choice for Ricardian consumers:
\[ h \hat{c}_{t-1} + \beta h E_t \{ \hat{c}_{t+1} \} = (1 + \beta h^2) \hat{c}_t + (1 - h)(1 - \beta h) \hat{\lambda}_t - (1 - h)(\hat{\xi}_c,t - \beta \hat{\xi}_{c,t+1}) \] (105)

Euler equation for Ricardian consumers:
\[ \hat{\lambda}_t = \hat{r}_t + E_t \{ \hat{\lambda}_{t+1} \} - E_t \{ \hat{\pi}_{t+1} \} \] (106)
Capital stock choice for Ricardian consumers:

\[ \dot{\lambda}_t + \dot{q}_{j,t} = E_t\{\dot{\lambda}_{t+1}\} + \beta(1 - \delta)E_t\{\dot{q}_{j,t+1}\} + \frac{\beta r^{K}}{\mathcal{M}} E_t\{\dot{r}^k_{j,t+1}\} \]  

(107)

Relationship between capital supplied to firms and total capital stock:

\[ \dot{k}_{j,t} = u_{j,t} + \dot{k}_{j,t} \]  

(108)

Capital utilisation in industry \( j \):

\[ A^j \dot{r}^k_{j,t} = \dot{u}_{j,t} \]  

(109)

where \( A \) controls the degree of capital utilisation costs.

Investment choice:

\[ (1 + \beta)\dot{z}_{j,t} = \frac{\dot{q}_{j,t}}{S^{nr}} + \beta E_t\{\dot{z}_{j,t+1}\} + \dot{z}_{j,t-1} \]  

(110)

Consumption choice for non-Ricardian consumers

\[ C^{nr}\dot{c}^{nr}_t - WN(\dot{w}_t + \dot{n}_t) - TRANS\dot{r}^{nr}_t = 0 \]  

(111)

Marginal utility of consumption for non-Ricardian consumers

\[ h\dot{c}^{nr}_t - 1 + \beta hE_t\{\dot{c}^{nr}_{t+1}\} = (1 + \beta h^2)\dot{c}^{nr}_t + (1 - h)(1 - \beta h)\dot{\lambda}_t^{nr} - (1 - h)(\dot{\xi}_{c,t} - \beta \dot{\xi}_{c,t+1}) \]  

(112)

Aggregate consumption:

\[ \dot{c}_t - \omega_r C^{r} \dot{c}^{r}_t - (1 - \omega_r)\frac{C^{nr}}{C} \dot{c}^{nr}_t = 0 \]  

(113)

Aggregate marginal utility:

\[ \dot{\lambda}_t - \omega_r \frac{\lambda^r}{\Lambda} \dot{\Lambda}^{nr}_t - (1 - \omega_r)\frac{\Lambda^{nr}}{\Lambda} \dot{\lambda}^{nr}_t = 0 \]  

(114)

Wages choice:

\[ \dot{w}^{nr}_{j,t} - \frac{\beta}{1 + \beta X_w} E_t\{\dot{w}^{nr}_{j,t+1}\} - \frac{\kappa_{w,j}}{(1 + \beta X_w)} \left[-\dot{\lambda}_t + (\nu - 1)\dot{\lambda}_t + \frac{1}{\xi} \dot{\lambda}_{j,t} - \dot{w}_{j,t} \right] = \frac{\chi_w}{1 + \beta X_w} \dot{w}^{nr}_{j,t-1} \]  

(115)

where \( \kappa_{w,j} = \frac{\xi}{1 + \beta X_w}(1 - \beta \theta_{w,j})(1 - \theta_{w,j})/\theta_{w,j} \)

Gross output in sector \( j \):

\[ \frac{\gamma^{VA}_{j}}{y^{\gamma}_{j,t}} \dot{y}^{\gamma}_{j,t} - a^{\gamma}_{j,t} - \omega_{y_{j}}(y^{\gamma}_{j,t})^{\gamma^{\gamma}_{j,t}} \dot{y}^{\gamma}_{j,t} - (1 - \omega_{y_{j}}) \frac{1}{\gamma^{\gamma}_{j,t}} x^{\gamma}_{j,t} \dot{x}^{\gamma}_{j,t} = 0 \]  

(116)

Marginal costs in sector \( j \):

\[ \dot{m}c_{j,t} + \dot{a}_{j,t} - \omega_{y_{j}}(\gamma^{VA}_{j}/a_{j})^{1-\gamma^{VA}_{j,t}} - (1 - \omega_{y_{j}})(\gamma^{\gamma}_{j,t}/a_{j})^{1-\gamma^{\gamma}_{j,t}} + \dot{\gamma}^{\gamma}_{j,t} = 0 \]  

(117)

Factor demand in sector \( j \):

\[ \varphi^{VA}_{j,t} + y^{\gamma}_{j,t} - \varphi^{\gamma}_{j,t} - \dot{\gamma}_{j,t} = 0 \]  

(118)
Value added price index in sector j:

\[ (\gamma_{va}^j)^{1-\zeta} \hat{\gamma}_{va,j,t} - \omega_{n_j} w_{j,t}^{1-\zeta} \hat{w}_{j,t} - (1 - \omega_{n_j}) (r_k^j)^{1-\zeta} \hat{r}_{j,t} = 0 \] (119)

Intermediate good price index in sector j:

\[ (\gamma_x^j)^{1-\psi} \hat{\gamma}_{x,j,t} - \sum_{k=1}^{F} \omega_{k,j} (\gamma_k^j)^{1-\psi} \hat{\gamma}_{k,t} = 0 \] (120)

Labour demand in sector j:

\[ \hat{y}_{va,j,t} - \zeta \hat{w}_{j,t} + \zeta \hat{\gamma}_{va,j,t} - \hat{n}_{j,t} = 0 \] (121)

Capital demand in sector j:

\[ \hat{y}_{va,j,t} - \zeta \hat{r}_{k,j,t} + \zeta \hat{\gamma}_{va,j,t} = \hat{k}_{j,t} \] (122)

Intermediate good k demand in sector j:

\[ \hat{x}_{j,t} - \hat{x}_{k,j,t} - \psi \hat{\gamma}_{k,t} + \psi \hat{\gamma}_{x,j,t} = 0 \] (123)

Definition of relative price in sector j:

\[ \hat{\gamma}_{j,t} - \hat{\pi}_{j,t} + \hat{\pi}_t = \hat{\gamma}_{j,t-1} \] (124)

Phillips curve in sector j:

\[ \hat{\pi}_{j,t} - \frac{\beta}{1 + \beta \chi_p} E_t \{ \hat{\pi}_{j,t+1} \} - \frac{(1 - \theta)(1 - \beta \theta)}{\theta (1 + \beta \chi_p)} \hat{m}_{c,j,t} = \frac{\chi_p}{1 + \beta \chi_p} \hat{\pi}_{j,t-1} \] (125)

Link between wage inflation and real wages in sector j:

\[ \pi_{j,t}^w - w_{j,t}^\pi - \pi_t = -w_{j,t-1} \] (126)

Market clearing in sector j:

\[ y_j \hat{y}_{j,t} - c_j \hat{c}_{j,t} - i_j - \hat{\gamma}_{j,t} - g_j \hat{g}_{j,t} - \sum_{k=1}^{F} x_{j,k} \hat{x}_{j,k,t} = 0 \] (127)

Taylor rule:

\[ \hat{r}_t - (1 - \rho_r) (\phi_{\pi} \hat{\pi}_{t}^{ye} + \phi_y \hat{y}_{t}^{va}) + \varepsilon_{r,t} = \rho_r \hat{r}_{t-1} \] (128)

Additional aggregate variables:

Year-ended inflation:

\[ \hat{\pi}_{t}^{ye} = \hat{\pi}_t + \hat{\pi}_{t-1} + \hat{\pi}_{t-2} + \hat{\pi}_{t-3} \] (129)

Aggregate value added:

\[ y_{t}^{va} - \sum_{j=1}^{F} n_{va,j} \hat{y}_{va,j,t} = 0 \] (130)

where \( n_{va,j} \) is the steady-state share of sector j in nominal GDP.

Shock processes:

Productivity in sector j:

\[ \hat{a}_{j,t} = \rho_{a_j} \hat{a}_{j,t-1} + \varepsilon_{a_{j,t}} \] (131)
Aggregate government expenditure:

\[ \hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_{g_t} \]  

(132)

Transfers:

\[ \text{trans}_t = \rho_{\text{trans}} \text{trans}_{t-1} + \varepsilon_{\text{trans},t} \]  

(133)