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**A NOTE ON THE GORDON GROWTH MODEL  
WITH NONSTATIONARY DIVIDEND GROWTH**

by

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**Abstract**

Researchers have sometimes argued that the recent ascent in stock prices could be explained in some measure by changes in expectations about long-run future dividend growth. For example, Barsky and De Long (1993) argue that a small random walk component in the growth rate of dividends, when extrapolated into the future, is capable of reproducing the large swings in US stock prices over the period 1880–1990. I show that the hypothesis of a nonstationary permanent growth rate of dividends is inconsistent with the Gordon growth model.

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## 1. Introduction

To explain long-run swings in the US stock market, Barsky and De Long (1993) – hereafter BD – computed “warranted” stock prices on the assumption that investors extrapolate a long distributed lag of realised dividend growth rates into the future. Their approach combines the standard Gordon model of stock price determination with a model of dividend changes characterised by a time-varying, albeit small, persistent component. The thrust of their argument is that small but permanent shocks to dividend growth rates may have a dramatic impact on stock prices. This goes some way towards explaining the documented “overreaction” of asset prices to long swings in dividends, thus restoring the power of the conventional present-value model.

In contrast to Mankiw, Romer and Shapiro (1985, 1991) and Kleidon (1986), who emphasise that the log dividend process is close to a random walk, BD postulate an environment in which it is the growth rate of dividends that contains a random walk. The two approaches are distinct, since in the former case an innovation in the growth rate of dividends is assumed to be temporary, whereas in the latter it is expected to persist indefinitely. Despite this difference, BD predicate their analysis on the Gordon valuation formula, a variant of the present-value model which obtains when the rate of dividend growth is constant.

BD’s methodology has been followed by other researchers. Carlson and Sargent (1998), for instance, consider a similar setup where the growth of earnings replaces that of dividends. A reason for this may be that it has an important bearing on policy. Indeed, if investors expect part of the recent rise in dividends to be permanently incorporated in future dividend growth, the recent ascent in stock prices might be deemed appropriate. By contrast, stationary models of dividend growth rates would generally fail to conclude that stocks are rationally priced, given current trends of dividend payments.

Applying a formula which does not hold when the growth rate of dividends is nonstationary is logically inconsistent. Indeed, BD recognise (p. 299) that the formula

*“does not hold exactly in a stochastic model for the dividend process given by (3): [it] does not allow prices today to be influenced by investors’ knowledge that they will be revising their estimate of  $g$  in the future.”*

They continue:

*“But for simplicity we ignore these higher-order corrections here and use [it] as our formula for the log of warranted stock index price.”*

However, the situation is worse than BD acknowledge. In their model, outlined below, the sum of expected discounted dividends fails to converge, and Gordon’s valuation formula is not applicable.

## 2. Model

With a constant required rate of return, the standard version of the present-value model is  $P_t = E_t \left[ \sum_{j=1}^{\infty} (1+r)^{-j} D_{t+j} \right]$ , where  $P_t$  is the real stock price,  $D_{t+j}$  the real dividend paid at time  $t+j$  and  $r$  the real rate of return (or discount rate). It is useful to rewrite the formula in terms of future dividend growth rates as:

$$(1) \quad P_t = D_t E_t \left[ \sum_{j=1}^{\infty} \exp \left\{ -j(\rho - g_t^j) \right\} \right]$$

where  $g_t^j = (d_{t+j} - d_t) / j$  denotes the average dividend growth rate between  $t$  and  $t+j$ ,  $d_t$  is the log of dividends, and the discount rate has been renormalised as  $\rho = \ln(1+r)$ . In the special case of a constant growth rate with  $\exp g_t^j = 1+g$ , formula (1) simplifies to:

$$(2) \quad P_t = D_t \frac{1+g}{r-g}$$

which is the well-known Gordon growth model (1962).

If the dividend growth rate varies over time, (2) is not directly applicable. However, one may wish to consider (2) as an approximation if dividends are expected to grow at a constant rate in the future, i.e. if  $E g_t^j = g_t$  for all  $j$ . As an extreme case, an investor certain that the mean dividend growth rate is a fixed but unknown parameter of the economy could extrapolate it from period to period and substitute the resulting estimate  $g_t$  for  $g$  in the Gordon formula. However, Jensen's inequality shows that substituting *expected* for *realised* growth in (1) induces a downward bias. The empirical relevance of that bias depends in turn on the specification of the underlying process governing dividends.

It is important at this point to understand intuitively the complications that arise when the dividend process is stochastic. This assumption implies that the average growth of dividends  $g_t^j$  from  $t$  to  $t+j$  may be greater than  $\rho$  with positive probability, rendering the exponential in (1) greater than unity. If this occurs only finitely many times in the future, there is no convergence problem and  $P_t$  in (1) is well-defined. If not, the probability that this happens must decrease sufficiently fast as  $j$  increases. Failing this,  $P_t$  would not be well-defined, as more and more terms would contribute in a non-negligible way to the infinite sum.<sup>1</sup>

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<sup>1</sup> Note that  $E_t \left[ \exp \left\{ -j(\rho - g_t^j) \right\} \right] \geq E_t \left[ \exp \left\{ -j(\rho - g_t^j) \right\}; \{g_t^j \geq \rho\} \right] \geq E_t \left[ 1; \{g_t^j \geq \rho\} \right] = \text{Prob} \left( g_t^j \geq \rho \right)$ . A necessary condition for convergence is thus that  $\text{Prob} \left( g_t^j \geq \rho \right) \rightarrow 0$  as  $j \rightarrow \infty$ .

Assume, for example, that  $\Delta d_t$  contains a unit root. As the length of the horizon  $j$  increases, the range of values taken by future growth rates will increase and so will the variance of their arithmetic mean  $g_t^j$ . This holds even though, as of time  $t$ ,  $g_t^j$  itself has a constant expectation equal to  $g_t$ . With the tails of the distribution growing fatter around its fixed mean, the probability that  $g_t^j$  overshoots the constant discount rate  $\rho$  keeps increasing and drives the stock price to infinity.

Returning to BD's model, one notes that the specification of their underlying dividend process relies on two assumptions. First, dividend growth is modelled as the sum of a white-noise innovation,  $\varepsilon_t$ , and a permanent growth component  $g_{t-1}$ :

$$(3) \quad \Delta d_t = g_{t-1} + \varepsilon_t$$

Second, the permanent growth component  $g_t$  is revised each period by a small fraction of current dividend changes:

$$(4) \quad g_t = \theta g_{t-1} + (1 - \theta) \Delta d_t$$

where  $\theta$  is a parameter close to one. In other words, the permanent growth component is a geometric average of past dividend changes, with rate of decay  $1 - \theta$ . Equations (3–4) are identical to (3–4) in BD.<sup>2</sup> The two assumptions taken together imply that  $g_t = g_{t-1} + (1 - \theta) \varepsilon_t$  is itself a random walk. By (3), this implies that  $\Delta d_t$  contains a unit root. As argued above, it follows that the stock price is infinite. A formal proof is given in the Appendix.

It is the fast rate at which the volatility of future average growth increases that prevents the infinite sum defining present value in (1) from converging. The following back-of-the-envelope calculation casts light on this issue. We need the following property of random variables: if  $\ln x$  is normal with mean  $\mu$  and variance  $\sigma^2$ , then  $\ln E(x) = \mu + \sigma^2/2$ . Here,  $g_t^j$  tends to a conditionally normal distribution, with mean  $g_t$  and variance  $j\sigma^2$ , where  $\sigma^2$  is an empirically “small” number depending on the value of  $\theta$ . Accordingly, the  $j$ th term in (1) is approximately log-normal for large  $j$ , with mean  $\exp\{-j(\rho - g_t) + j^3 \sigma^2/2\}$ . We see that, when  $j \rightarrow \infty$ , the variance term in  $j^3$  causes the expectation to explode. (The argument is only heuristic, since  $g_t^j$  does not have a well-defined limit distribution.)

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<sup>2</sup> In a footnote BD point out that, instead of treating (2) as an approximation to the pricing rule, one could treat (3–4) as an approximation to the dividend process if permanent growth is governed by the following stochastic differential equation:

$$dg_t = -\frac{(1-\theta)^2 \sigma_\varepsilon^2}{r - g_t} dt + (1-\theta) \sigma_\varepsilon dW_t$$

However, Ito's formula shows that  $y_t = (r - g_t) / \sigma_\varepsilon (1 - \theta)$  satisfies:

$$dy_t = \frac{dt}{y_t} - dW_t$$

and is thus a Bessel process of dimension 3. It follows that  $y_t \rightarrow \infty$  as  $t \rightarrow \infty$ . The fact that growth rates become arbitrarily large and negative raises doubts about the relevance of this formulation.

### 3. Conclusion

This note has investigated whether the assumption of a “permanent” growth rate of dividends is consistent with a constant discount rate. It was found that the infinite sum defining present-value diverges when the dividend growth rate  $g_t$  has a random walk component. Of course, it may be possible to restore convergence in the present-value model by assuming that the forcing process for dividends contains a very persistent but stationary autoregressive root instead of an exact unit root. The inferences above, however, imply that as this root tends to one, the associated stock price tends to infinity and cannot be approximated by the Gordon valuation formula. On the other hand, a lower autoregressive root would mean that  $g_t$  is less correlated with past dividend growth rates. This runs up against the very intuition of the model, which views past dividend growth rates as the main driving force behind revisions in expected future dividend growth. Whether the model retains considerable power under the present context remains uncertain, since one cannot embed the assumption of nonstationarity in dividend growth in the present-value model without considering its long-run theoretical consequences.

This leaves us with the issue of which assumption of the Gordon model should be relaxed for the present-value model with nonstationary dividend growth to remain meaningful. One way to restore convergence might be to drop the assumption of a constant discount rate and derive instead the required rate of return endogenously within an equilibrium asset pricing model. I have not been able to find a simple model, with standard specification of preferences, that would be consistent with nonstationarity in dividends. How to value equity in the present context seems to remain an open question.

## Annex

### Proposition 1

When  $j \rightarrow \infty$ , the variable  $j^{-1/2} g_t^j$  converges in distribution to a normal variable with mean 0 and variance  $(1-\theta)^2 \sigma_\varepsilon^2 / 3$ .

*Proof.* Let  $X_n^u = \sqrt{u} \Delta d_{t+n}$  for some small  $u > 0$ . From (3–4), one gets the scalar difference equation

$$X_n^u = X_{n-1}^u + \sqrt{u} (\varepsilon_{t+n} - \theta \varepsilon_{t+n-1})$$

It is readily verified that the spectrum at 0 of the innovation  $\varepsilon_{t+n} - \theta \varepsilon_{t+n-1}$ , defined as the infinite sum of its autocovariances at all leads and lags, is  $(1-\theta)^2 \sigma_\varepsilon^2$ . It follows from weak convergence arguments that the process  $X_n^u$  converges weakly to the process  $x = (1-\theta) \sigma_\varepsilon w$ , where  $w$  is a standard Brownian motion, as  $u \rightarrow 0$ ; e.g. Kushner (1984), pp. 91–3. In particular, choosing  $uj = 1$ , one sees that:

$$\begin{aligned} j^{-1/2} g_t^j &= j^{-3/2} (d_{t+j} - d_t) \\ &= u \sum_{n=1}^j X_n^\varepsilon \\ &\Rightarrow \int_0^1 x_s ds = (1-\theta) \sigma_\varepsilon \int_0^1 w_s ds \end{aligned}$$

The variable  $\int_0^1 w_s ds$  has mean zero and variance

$$\begin{aligned} E \int_0^1 \int_0^1 w_s w_t ds dt &= \int_0^1 \int_0^1 s \wedge t ds dt \\ &= \int_0^1 \left[ \int_0^t s ds + \int_t^1 t ds \right] dt \\ &= \int_0^1 \left[ \frac{t^2}{2} + t(1-t) \right] dt = 1/3 \end{aligned}$$

hence the result.

To see that the standard present-value model breaks down, consider the  $j$ th element in the infinite sum of expected future dividends. If  $U$  is a random variable with the limit distribution of proposition 1, and  $\varphi$  some strictly positive, bounded continuous function<sup>3</sup> such that  $\varphi(x) \leq \exp \{-j\rho + j^{3/2} x\}$  for all  $j$ , one has:

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<sup>3</sup> For example, the envelope  $\varphi(x) = 1_{\{x > 0\}} \exp \left\{ -\frac{4\rho^3}{27x^2} \right\}$  of all functions  $\varphi_\alpha(x) = \exp \{-\alpha\rho + \alpha^{3/2} x\}$  for  $\alpha > 0$ .

$$\begin{aligned}
E\left[(1+r)^{-j} \frac{D_{t+j}}{D_t}\right] &= E\left[\exp\{-j(\rho - g_t^j)\}\right] \\
&= E\left[\varphi(j^{-1/2} g_t^j)\right] \\
&= E[\varphi(U)]
\end{aligned}$$

a strictly positive number. Since discounted expected future dividends are bounded away from 0, their infinite sum defining present value diverges.

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