Online Appendix

A. Appendix: Omitted Derivations and Proofs

This appendix presents the derivations and proofs omitted from the main text.

A.1. Omitted derivations in Section 2

Most of the analysis is provided in the main text. Here, we formally state the investor’s problem and derive the optimality conditions. Recall that the market portfolio is the claim to all output at date 1. Let \( r^k(z_1) = \log \left( \frac{z_1}{Q} \right) \) denote the log return on this portfolio if the productivity is realized to be \( z_1 \). Since the payoff distribution is log normal, the return distribution is also log normal,

\[
    r^k (z_1) \sim N \left( g - \log Q - \frac{\sigma^2}{2}, \sigma^2 \right) .
\]  

(A.1)

Recall that there are two types of investors, \( i = o \) and \( i = p \), that differ only in their beliefs about the expected growth rate, denoted by \( g^i \), and their wealth shares, denoted by \( \alpha^i \). Type \( i \) investors solve the following problem,

\[
    \max_{c_0, a_0, \omega^k} \log c_0 + e^{-\rho} \log U_1 \quad \text{(A.2)}
\]

where

\[
    U_1 = \left( E^i \left[ c_1 (z_1)^{1-\gamma} \right] \right)^{1/(1-\gamma)}
\]

s.t. \( c_0 + a_0 = \alpha^i (y_0 + Q) \)

and \( c_1 (z_1) = a_0 (\omega^k \exp (r^k (z_1)) + (1 - \omega^k) \exp (r^f)) \).

Here, \( c_1(z_1) \) denotes total financial wealth, which equals consumption (since the economy ends at date 1). Note that investors have Epstein-Zin preferences with EIS coefficient equal to one and the RRA coefficient equal to \( \gamma > 0 \). The case with \( \gamma = 1 \) is equivalent to log utility as in the dynamic model.

In view of the Epstein-Zin functional form, investors’ problem naturally splits into two steps. Conditional on savings, \( a_0 \), she solves a portfolio optimization problem, that is, \( U_1 = R^{CE,i} a_0 \), where

\[
    R^{CE,i} = \max_{\omega^k} \left( E^i \left[ \left( R^p(z_1) \right)^{1-\gamma} \right] \right)^{1/(1-\gamma)}
\]

and \( R^p(z_1) = (\omega^k \exp (r^k(z_1))) + (1 - \omega^k) \exp (r^f) \).

(A.3)

Here, we used the observation that the portfolio problem is linearly homogeneous. The variable, \( R^p(z_1) \), denotes the realized portfolio return per dollar, and \( R^{CE,i} \) denotes the optimal certainty-equivalent portfolio return perceived by type \( i \) investors. In turn, these investors choose asset holdings, \( a_0 \), that solve the intertemporal problem,

\[
    \max_{a_0} \log (\alpha^i (y_0 + Q) - a_0) + e^{-\rho} \log \left( R^{CE,i} a_0 \right) .
\]

The first order condition for this problem implies Eq. (4) in the main text. That is, regardless of her certainty-equivalent portfolio return, the investor consumes and saves a constant fraction of her lifetime wealth.
It remains to characterize the optimal portfolio weight, $\omega^k$, as well as the certainty-equivalent return, $R^{CE,i}$. Even though the return on the market portfolio is log-normally distributed (see Eq. (4.1)), the portfolio return, $R^p(z_1)$, is in general not log-normally distributed (since it is the sum of a log-normal variable and a constant). Following Campbell and Viceira (2002), we assume investors solve an approximate version of the portfolio problem (A.3) in which the log portfolio return is also normally distributed. Moreover, the mean and the variance of this distribution are such that the following identities hold,

$$\pi^{p,i} = \omega^k \pi^{k,i} \text{ and } \sigma^{p,i} = \omega^k \sigma^k,$$

where $\pi^{p,i} = \log E^i [R^p] - r^f$ and $(\sigma^{p,i})^2 = \text{var} (\log R^p),$ and

$$\pi^{k,i} = \log (E^i [\exp (r^k)]) - r^f = E^i [r^k] - r^f + \frac{\sigma^2}{2}.$$

Here, the first line says that the risk premium on the investor’s portfolio (measured in log difference of expected gross returns) depends linearly on the investor’s portfolio weight and the risk premium on the market portfolio. The third line says that the standard deviation of the (log) portfolio return depends linearly on the investor’s portfolio weight and the risk premium on the market portfolio. These identities hold exactly in continuous time. In the two period model, they hold approximately when the period time-length is small.

Taking the log of the objective function in problem (A.3), and using the log-normality assumption, the problem can be equivalently rewritten as,

$$\log R^{CE,i} - r^f = \max_{\omega^k} \pi^{p,i} - \frac{1}{2} \gamma (\sigma^{p,i})^2,$$

where $\pi^{p,i}$ and $\sigma^{p,i}$ are defined in Eq. (A.4). It follows that, up to an approximation (that becomes exact in equilibrium), the investor’s problem turns into standard mean-variance optimization. Taking the first order condition, we obtain Eq. (7) in the main text. Substituting $E^i [r^k] = g^i - \log Q - \frac{\sigma^2}{2}$ [cf. Eq. (4.1)] into this expression, the optimality condition for type $i$ investors can also be written as,

$$\omega^{k,i} \sigma = \frac{1}{\gamma} \left( g^i - \log Q - r^f \right).$$

Combining this with the asset market clearing condition (6), we further obtain Eq. (8) in the main text.

### A.2. Omitted derivations in Section 3

#### A.2.1. Portfolio problem and its recursive formulation

The investor’s portfolio problem (at some time $t$ and state $s$) can be written as,

$$V_{t,s}^i(a_{t,s}^i) = \max_{[\tilde{c}_{t,s}, \tilde{\omega}_{t,s}^k, \tilde{\omega}_{t,s}^\nu]} E_{t,s}^i \left[ \int_t^\infty e^{-\rho^t \log \tilde{c}_{t,s}^i} dt \right]$$

s.t. \[
\begin{cases}
\frac{d a_{t,s}^i}{dt} = \left( a_{t,s}^i \left( r_{t,s}^i + \tilde{\omega}_{t,s}^k \left( r_{t,s}^k - r_{t,s}^p \right) - \tilde{c}_{t,s}^i \right) - \tilde{\omega}_{t,s}^\nu \right) + \tilde{\omega}_{t,s}^k \sigma_s \sigma_{s} dZ_t - \tilde{\omega}_{t,s}^\nu \frac{Q_s - Q_{s'}}{Q_{t,s}} \\
\frac{d \tilde{\omega}_{t,s}^k}{dt} = \tilde{\omega}_{t,s}^k \left( 1 + \frac{\tilde{\omega}_{t,s}^k}{\tilde{\omega}_{t,s}^\nu} \right) + \tilde{\omega}_{t,s}^\nu \frac{1}{p_{t,s}} \\
\text{absent transition,}\\
\frac{d \tilde{\omega}_{t,s}^\nu}{dt} = \tilde{\omega}_{t,s}^\nu \left( \frac{Q_s - Q_{s'}}{Q_{t,s}} \right) + \omega_{t,s}^\nu \frac{1}{p_{t,s}}, \\
\text{if there is a transition to state } s' \neq s.
\end{cases}
\]
Here, \( E^i_{t,s} [ \cdot ] \) denotes the expectations operator that corresponds to the investor \( i \)'s beliefs for state transition probabilities. The HJB equation corresponding to this problem is given by,

\[
\rho V^i_{t,s} (a^i_{t,s}) = \max_{\omega^k, \tilde{\omega}'^k} \log \tilde{\omega} + \frac{\partial V^i_{t,s}}{\partial a} (a^i_{t,s} (r^f_{t,s} + \omega^k (r^k_{t,s} - r^f_{t,s}) - \tilde{\omega}^s) - \tilde{\omega}) + \frac{1}{2} \frac{\partial^2 V^i_{t,s}}{\partial a^2} (\omega^k a^i_{t,s} \sigma^2) + \frac{\partial V^i_{t,s} (a^i_{t,s})}{\partial t} + \lambda^i_s \left( V^i_{t,s} \left( a^i_{t,s} (1 + \tilde{\omega}^k Q_{t,s} - \tilde{Q}_{t,s} + \tilde{\omega}^s p^s_{t,s}) + V^i_{t,s} (a^i_{t,s}) \right) - V^i_{t,s} (a^i_{t,s}) \right).
\]

(A.6)

In view of the log utility, the solution has the functional form in [42], which we reproduce here,

\[
V^i_{t,s} (a^i_{t,s}) = \frac{\log (a^i_{t,s} / Q_{t,s})}{\rho} + v^i_{t,s}.
\]

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investors consumption proportionally at all times and states. The second term, \( v^i_{t,s} \), is the normalized value function when the investor holds one unit of the capital stock (or wealth, \( a^i_{t,s} = Q_{t,s} \)). This functional form also implies,

\[
\frac{\partial V^i_{t,s}}{\partial a} = \frac{1}{\rho a_{t,s}^2} \quad \text{and} \quad \frac{\partial^2 V^i_{t,s}}{\partial a^2} = -\frac{1}{\rho (a_{t,s})^2}.
\]

The first order condition for \( \tilde{\omega} \) then implies Eq. (17) in the main text. The first order condition for \( \omega^k \) implies,

\[
\frac{\partial V^i_{t,s}}{\partial a} a^i_{t,s} (r^k_{t,s} - r^f_{t,s}) + \lambda^i_s \frac{\partial V^i_{t,s}}{\partial a} (a^i_{t,s} Q_{t,s} - \tilde{Q}_{t,s}) = -\frac{\partial^2 V^i_{t,s}}{\partial a^2} \omega^k_{t,s} (a^i_{t,s} \sigma^2) + \lambda^i_s Q_{t,s} (a^i_{t,s} \sigma^2).
\]

After substituting for \( \frac{\partial V^i_{t,s}}{\partial a} \), \( \frac{\partial V^i_{t,s}}{\partial a^2} \) and rearranging terms, this also implies Eq. (18) in the main text. Finally, the first order condition for \( \tilde{\omega}' \) implies,

\[
\frac{p^i_{t,s}}{\lambda^i_s} = \frac{\partial V^i_{t,s}}{\partial a} (a^i_{t,s}) = \frac{1}{a^i_{t,s}}.
\]

which is Eq. (19) in the main text. This completes the characterization of the optimality conditions.

**A.2.2. Description of the New Keynesian production firms**

The supply side of our model features nominal rigidities similar to the standard New Keynesian setting. There is a continuum of measure one of production firms denoted by \( \nu \). These firms rent capital from the investment firms, \( k_{t,s} (\nu) \), and produce differentiated goods, \( y_{t,s} (\nu) \), subject to the technology,

\[
y_{t,s} (\nu) = A \eta_{t,s} (\nu) k_{t,s} (\nu). \quad \text{(A.7)}
\]

Here, \( \eta_{t,s} (\nu) \in [0, 1] \) denotes the firm’s choice of capital utilization. We assume utilization is free up to \( \eta_{t,s} (\nu) = 1 \) and infinitely costly afterwards (see our extended working paper version, in which we relax this assumption and allow for excess utilization at the cost of excess depreciation). The production firms sell their output to a competitive sector that produces the final output according to the CES technology,
\[ y_{t,s} = \left( \int_0^1 y_{t,s}(\nu)^{\frac{1}{\varepsilon}} \, d\nu \right)^{\varepsilon/(\varepsilon-1)}, \text{ for some } \varepsilon > 1. \] Thus, the demand for the firms’ goods is given by,

\[ y_{t,s}(\nu) = p_{t,s}(\nu)^{-\varepsilon} y_{t,s}, \text{ where } p_{t,s}(\nu) = P_{t,s}(\nu)/P. \] (A.8)

Here, \( p_{t,s}(\nu) \) denotes the firm’s relative price, which depends on its nominal price, \( P_{t,s}(\nu) \), as well as the ideal nominal price index, \( P_{t,s} = \left( \int P_{t,s}(\nu)^{1-\varepsilon} \, d\nu \right)^{1/(1-\varepsilon)}. \)

We also assume there are subsidies designed to correct the inefficiencies that stem from the firm’s monopoly power and markups. In particular, the government taxes the firm’s profits lump sum, and redistributes these profits to the firms in the form of a linear subsidy to capital. Formally, we let \( \Pi_{t,s}(\nu) \) denote the equilibrium pre-tax profits of firm \( \nu \) (that will be characterized below). We assume each firm is subject to the lump-sum tax determined by the average profit of all firms,

\[ T_{t,s} = \int \Pi_{t,s}(\nu) \, d\nu. \] (A.9)

We also let \( R_{t,s} - \tau_{t,s} \) denote the after-subsidy cost of renting capital, where \( R_{t,s} \) denotes the equilibrium rental rate paid to investment firms, and \( \tau_{t,s} \) denotes a linear subsidy paid by the government. We assume the magnitude of the subsidy is determined by the government’s break-even condition,

\[ \tau_{t,s} \int k_{t,s}(\nu) \, d\nu = T_{t,s}. \] (A.10)

Without price rigidities, the firm chooses \( p_{t,s}(\nu), k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu) \), to maximize its (pre-tax) profits,

\[ \Pi_{t,s}(\nu) \equiv p_{t,s}(\nu) y_{t,s}(\nu) - (R_{t,s} - \tau_{t,s}) k_{t,s}(\nu), \] (A.11)

subject to the supply constraint in (A.7) and the demand constraint in (A.8). The optimality conditions imply,

\[ p_{t,s}(\nu) = \frac{\varepsilon}{\varepsilon - 1} \frac{R_{t,s} - \tau_{t,s}}{A} \text{ and } \eta_{t,s}(\nu) = 1. \]

That is, the firm charges a markup over its marginal costs, and utilizes its capital at full capacity. In a symmetric-price equilibrium, we further have, \( p_{t,s}(\nu) = 1 \). Using Eqs. (A.7–A.10), this further implies,

\[ y_{t,s}(\nu) = y_{t,s} = Ak_{t,s} \text{ and } R_{t,s} = \frac{\varepsilon - 1}{\varepsilon} A + \tau_{t,s} = A. \] (A.12)

That is, output is equal to potential output, and capital earns its marginal contribution to potential output (in view of the linear subsidies).

We focus on the alternative setting in which the firms have a preset nominal price that is equal to one another, \( P_{t,s}(\nu) = P \). In particular, the relative price of a firm is fixed and equal to one, \( p_{t,s}(\nu) = 1 \). The firm chooses the remaining variables, \( k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu) \), to maximize its (pre-tax) profits, \( \Pi_{t,s}(\nu) \). We conjecture a symmetric equilibrium in which all firms choose the same allocation, \( k_{t,s}, \eta_{t,s}, y_{t,s} \), output is determined by aggregate demand,

\[ y_{t,s} = \eta_{t,s}Ak_{t,s} = \int c_{t,s}(i) \, d\nu + k_{t,s}t_{t,s}, \text{ for } \eta_{t,s} \in [0,1], \] (A.13)

and the rental rate of capital is given by,

\[ R_{t,s} = A\eta_{t,s}. \] (A.14)
To verify that the conjectured allocation is an equilibrium, first consider the case in which aggregate demand is below potential output, so that \( y_{t,s} < A k_{t,s} \) and \( \eta_{t,s} < 1 \). In this case, firms can reduce their capital input, \( k_{t,s} (\nu) \), and increase their factor utilization, \( \eta_{t,s} (\nu) \), to obtain the same level of production. Since factor utilization is free (up to \( \eta_{t,s} (\nu) = 1 \)), after tax cost of capital must be zero, \( R_{t,s} - \tau_{t,s} = 0 \). Since its marginal cost is zero, and its relative price is one, it is optimal for each firm to produce according to the aggregate demand, which verifies Eq. (A.13). Using Eqs. (A.9) and (A.10), we further obtain, \( \tau_{t,s} = A \eta_{t,s} \). Combining this with the requirement that \( R_{t,s} - \tau_{t,s} = 0 \) verifies Eq. (A.14).

Next consider the case in which aggregate demand is equal to potential output, so that \( y_{t,s} = A k_{t,s} \) and \( \eta_{t,s} = 1 \). In this case, a similar analysis implies there is a range of equilibria with \( \frac{R_{t,s} - \tau_{t,s}}{A} \leq 1 \) and \( R_{t,s} = A \). Here, the first equation ensures it is optimal for the firm to meet the aggregate demand. The second equation follows from the subsidy and the tax scheme. In particular, the frictionless benchmark allocation (A.12), that features \( \frac{R_{t,s} - \tau_{t,s}}{A} = \frac{\varepsilon - 1}{\varepsilon} \) and \( R_{t,s} = A \), is also an equilibrium with nominal rigidities as long as the aggregate demand is equal to potential output.

### A.3. Omitted derivations and proofs in Section 4

**Proof of Proposition 1**. Most of the proof is provided in the main text. It remains to show that Assumptions 1-3 ensure there exist a unique solution, \( q_2 < q^* \) and \( r_1^f \geq 0 \), to Eqs. (30) and (31). To this end, we define the function,

\[
 f (q_2, \lambda_2) = \rho + \psi q_2 - \delta + \lambda_2 \left( 1 - \frac{\exp (q_2)}{Q^*} \right) - \sigma_2^2. 
\]

The equilibrium price is the solution to, \( f (q_2, \lambda_2) = 0 \) (given \( \lambda_2 \)). Note that \( f (q_2, \lambda_2) \) is a concave function of \( q_2 \) with \( \lim_{q_2 \to -\infty} f (q_2, \lambda_2) = \lim_{q_2 \to -\infty} f (q_2, \lambda_2) = -\infty \). Its derivative is,

\[
 \frac{\partial f (q_2, \lambda_2)}{\partial q_2} = \psi - \lambda_2 \exp (q_2 - q^*). 
\]

Thus, for fixed \( \lambda_2 \), it is maximized at,

\[
 q_2^{\max} (\lambda_2) = q^* + \log (\psi / \lambda_2). 
\]

Moreover, the maximum value is given by

\[
 f (q_2^{\max} (\lambda_2), \lambda_2) = \rho - \delta + \psi (q^* + \log (\psi / \lambda_2)) + \lambda_2 (1 - \exp (\log (\psi / \lambda_2))) - \sigma_2^2 \\
 = \rho - \delta + \psi q^* + \psi \log (\psi / \lambda_2) + \lambda_2 - \psi - \sigma_2^2. 
\]

Next note that, by Assumption 1, the maximum value is strictly negative when \( \lambda_2 = \psi \), that is, \( f (q_2^{\max} (\psi), \psi) < 0 \). Note also that \( \frac{\partial f (q_2^{\max} (\lambda_2), \lambda_2)}{\partial \lambda_2} = 1 - \frac{\psi}{\lambda_2} \), which implies that the maximum value is strictly increasing in the range \( \lambda_2 \geq \psi \). Since \( \lim_{\lambda_2 \to -\infty} f (q_2^{\max} (\lambda_2), \lambda_2) = \infty \), there exists \( \lambda_2^{\min} > \psi \) that ensures \( f (q_2^{\max} (\lambda_2^{\min}), \lambda_2^{\min}) = 0 \). By Assumption 1, the actual level of optimism satisfies \( \lambda_2 \geq \lambda_2^{\min} \), which implies that \( f (q_2^{\max} (\lambda_2), \lambda_2) \geq 0 \). By Assumption 1, we also have that \( f (q^*, \lambda_2) < 0 \).

It follows that, for any \( \lambda_2 \geq \lambda_2^{\min} \), there exists a unique price level, \( q_2 \in [q_2^{\max}, q^*] \), that solves the equation, \( f (q_2, \lambda_2) = 0 \). Our analysis also implies that the equilibrium price satisfies, \( \frac{\partial f (q_2, \lambda_2)}{\partial q_2} = \psi - \lambda_2 \exp (q_2 / Q^*) \leq 0 \), with equality only if \( \lambda_2 = \lambda_2^{\min} \). This facilitates the comparative statics results in Section 4.
Next consider Eq. (31), which can be rewritten as

\[ r_1'^\ast = \rho + \psi q^\ast - \delta + \lambda_1 \left( 1 - \frac{Q^\ast}{Q_2} \right) - \sigma_1^2. \]

Since \( q_2 < q^\ast \), this expression is decreasing in \( \lambda_1 \). When \( \lambda_1 = 0 \), it is strictly positive by Assumption 1. As \( \lambda_1 \to \infty \), it approaches \(-\infty \). Thus, for any \( q_2 < q^\ast \), there exists \( \lambda_1^{\text{max}} (q_2) \) such that \( r_1'^\ast \geq 0 \) if and only if \( \lambda_1 \in [0, \lambda_1^{\text{max}} (q_2)] \). Note also that for any fixed \( \lambda_1 > 0 \), \( r_1'^\ast \) is increasing in \( q_2 \). This implies that the upper bound for the transition probability, \( \lambda_1^{\text{max}} (q_2) \), is increasing in \( q_2 \), completing the proof. \( \square \)

**Proof of Corollary** Fix some \( \Delta t > 0 \) and let \( s^{\Delta t} \) denote the random variable that is equal to \( s \) if there is no state transition over \([0, \Delta t]\), and \( s' \) if there is at least one state transition. The law of total variance implies,

\[ \text{Var}_{t,s} \left( \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} / \Delta t \right) = E^{s^{\Delta t}} \left[ \text{Var}_{t,s} \left( \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} | s^{\Delta t} \right) \right] + \text{Var}^{s^{\Delta t}} \left( E_{t,s} \left[ \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} | s^{\Delta t} \right] \right). \]

(A.15)

Here, \( E^{s^{\Delta t}} \) and \( \text{Var}^{s^{\Delta t}} \) denote, respectively, the expectations and the variance operator over the random variable, \( s^{\Delta t} \). We next calculate each component of variance.

For the first component, we have,

\[ E^{s^{\Delta t}} \left[ \text{Var}_{t,s} \left( \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} | s^{\Delta t} \right) \right] = e^{-\lambda s \Delta t} \sigma_s^2 \Delta t + (1 - e^{-\lambda s \Delta t}) O (\Delta t). \]

Here, the first term captures the variance conditional on there being no transition, \( s^{\Delta t} = s \). The variance in this case comes from the Brownian motion for \( k_{t,s} \). The second term captures the average variance conditional on there being a transition, \( s^{\Delta t} = s' \). Here, the last term satisfies, \( \lim_{\Delta t \to 0} O (\Delta t) = 0 \). Dividing by \( \Delta t \) and evaluating the limit, we obtain,

\[ \lim_{\Delta t \to 0} E^{s^{\Delta t}} \left[ \text{Var}_{t,s} \left( \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} | s^{\Delta t} \right) \right] = \sigma_s^2. \]

(A.16)

For the second component, we have,

\[ \text{Var}^{s^{\Delta t}} \left( E_{t,s} \left[ \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} | s^{\Delta t} \right] \right) = \text{Var}^{s^{\Delta t}} \left( E_{t,s} \left[ \frac{\Delta Q_{t,s}}{Q_{t,s}} | s^{\Delta t} \right] \right) + O \left( (\Delta t)^2 \right), \]

\[ = (1 - \lambda s \Delta t) \left( \frac{Q_s - \overline{Q}}{Q_s} \right)^2 + \lambda s \Delta t \left( \frac{Q_{s'} - \overline{Q}}{Q_s} \right)^2 + O \left( (\Delta t)^2 \right), \]

where \( \overline{Q} = (1 - \lambda_s \Delta t) Q_s + \lambda_s \Delta t Q_{s'}. \)

Here, \( O \left( (\Delta t)^2 \right) \) denotes terms that satisfy, \( \lim_{\Delta t \to 0} \frac{O (\Delta t)^2}{\Delta t} = 0 \). The first line uses the observation that for small \( \Delta t \) the state transitions change the return only through their impact on the price level. The second line calculates the variance of price changes up to terms that are first order in \( \Delta t \). Dividing the last line by \( \Delta t \) and evaluating the limit, we obtain,

\[ \lim_{\Delta t \to 0} \text{Var}^{s^{\Delta t}} \left( E_{t,s} \left[ \frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}} | s^{\Delta t} \right] \right) = \lambda s \left( \frac{Q_{s'} - Q_s}{Q_s} \right)^2. \]

(A.17)

Combining Eqs. (A.15), (A.16) and (A.17), the unconditional variance is given by, \( \sigma_s^2 + \lambda s \left( \frac{Q_{s'} - Q_s}{Q_s} \right)^2, \)
completing the proof.

A.4. Omitted derivations and proofs in Section 5

We derive the equilibrium conditions that we state and use in Section 5. First note that, using Eq. (19), the optimality condition (18) can be written as,

\[
\omega_{t,s}^{k,i} \sigma_s = \frac{1}{\sigma_s} \left( r_{t,s} - r_{t,s}^t + p_{t,s} Q_{t,s} - Q_{t,s} \right).
\]  

(A.18)

Combining this with the market clearing condition (20), we obtain,

\[
\omega_{t,s}^{k,o} = \omega_{t,s}^{k,p} = 1.
\]  

(A.19)

Next note that by definition, we have

\[
a_{t,s}^o = \alpha_{t,s} Q_{t,s} k_{t,s} \text{ and } a_{t,s}^p = (1 - \alpha_{t,s}) Q_{t,s} k_{t,s} \text{ for each } s \in \{1, 2\}.
\]

After plugging these into Eq. (19), using \( k_{t,s} = k_{t,s'} \) (since capital does not jump), and aggregating over optimists and pessimists, we obtain,

\[
p_{t,s}' = \lambda_{t,s} \frac{Q_{t,s}}{Q_{t,s'}}.
\]  

(A.20)

where \( \lambda_{t,s} \) denotes the wealth-weighted average belief defined in (33). Combining Eqs. (A.18), (A.19), and (A.20), we obtain the risk balance condition (34) in the main text.

We next characterize investors’ equilibrium positions. Combining Eq. (A.5) with Eqs. (A.19) and (A.20), investors’ wealth after transition satisfies,

\[
\frac{a_{t,s}^i}{a_{t,s}^o} = \frac{Q_{t,s}}{Q_{t,s}} \left( 1 + \frac{\omega_{t,s}^{i,s}}{\lambda_{t,s}} \right).
\]  

(A.21)

From Eq. (19), we have \( p_{t,s}' / \lambda_{t,s}' = \frac{1}{a_{t,s}^o} \). Substituting this into the previous expression and using Eq. (A.20) once more, we obtain,

\[
\omega_{t,s}^{i,s} = \lambda_{t,s}' - \lambda_{t,s} \text{ for each } i \in \{o, p\}.
\]  

(A.22)

Combining this with Eq. (35), we obtain Eq. (33) in the main text.

Finally, we characterize the evolution of optimists’ wealth share. After substituting \( a_{t,s}^o = \alpha_{t,s} Q_{t,s} k_{t,s} \) and using Eq. (A.22) (as well as \( k_{t,s} = k_{t,s'} \)), Eq. (A.21) implies

\[
\frac{\alpha_{t,s}'}{\alpha_{t,s}} = \frac{\lambda_{t,s}'}{\lambda_{t,s}}.
\]  

(A.23)

Thus, it remains to characterize the evolution of wealth conditional on no transition. To this end, we combine Eq. (A.5) with Eqs. (A.19), (24), (17) to obtain,

\[
\frac{d a_{t,s}^o}{a_{t,s}^o} = \left( g_{t,s} + \mu_{t,s} Q_{t,s} - \omega_{t,s}^{i,s} \right) dt + \sigma_s dZ_t.
\]

After substituting \( a_{t,s}^o = \alpha_{t,s} Q_{t,s} k_{t,s} \) and using the observation that \( \frac{d \alpha_{t,s}'}{\alpha_{t,s}} = \mu_{t,s} dt \) and \( \frac{d \alpha_{t,s}}{k_{t,s}} = g_{t,s} dt + \sigma_s dZ_t \),
we further obtain,
\[ \frac{d\alpha_{t,s}}{\alpha_{t,s}} = -\omega_{t,s}^{\alpha} dt = -\left(\lambda_{t,s}^\alpha - \lambda_{t,s}\right) dt. \] (A.24)

Combining Eqs. (A.23) and (A.24) implies Eq. (36) in the main text.

**Proof of Proposition 2.** We analyze the solution to the system in (38) using the phase diagram over the range \( \alpha \in [0,1] \) and \( q_2 \in [q_2^p, q_2^o] \). First note that the system has two steady states given by, \( (\alpha_{t,2} = 0, q_{t,2} = q_2^p) \), and \( (\alpha_{t,2} = 1, q_{t,2} = q_2^o) \). Next note that the system satisfies the Lipschitz condition over the relevant range. Thus, the vector flows that describe the law of motion do not cross. Next consider the locus, \( \dot{q}_{t,2} = 0 \). By comparing Eqs. (37) and (34), this locus is exactly the same as the price that would obtain if investors shared the same wealth-weighted average belief, denoted by \( q_2 = q_2^h(\alpha) \). Using our analysis in Section 4, we also find that \( q_2^h(\alpha) \) is strictly increasing in \( \alpha \). Moreover, \( q_2 < q_2^h(\alpha) \) implies \( \dot{q}_{t,2} < 0 \) whereas \( q_2 > q_2^h(\alpha) \) implies \( \dot{q}_{t,2} > 0 \). Finally, note that \( \dot{q}_{t,2} < 0 \) for each \( \alpha \in (0,1) \).

Combining these observations, the phase diagram has the shape in Figure 12. This in turn implies that the system is saddle path stable. Given any \( \alpha_{t,2} \in [0,1] \), there exists a unique solution, \( q_{t,2} \), which ensures that \( \lim_{t \to \infty} q_{t,2} = q_2^p \). We define the price function (the saddle path) as \( q_2(\alpha) \). Note that the price function satisfies \( q_2(\alpha) < q_2^h(\alpha) \) for each \( \alpha \in (0,1) \), since the saddle path cannot cross the locus, \( \dot{q}_{t,2} = 0 \). Note also that \( q_2(1) = q_2^o \), since the saddle path crosses the other steady-state, \( (\alpha_{t,2} = 1, q_{t,2} = q_2^o) \). Finally, recall that \( q_2 < q_2^h(\alpha) \) implies \( \dot{q}_{t,2} < 0 \). Combining this with \( \dot{q}_{t,2} < 0 \), we further obtain \( \frac{dq_2(\alpha)}{d\alpha} > 0 \) for each \( \alpha \in (0,1) \).

Next note that, after substituting \( \dot{q}_{t,2} = q_2^h(\alpha) \dot{\alpha}_{t,2} \), Eq. (38) implies the differential equation (39) in \( \alpha \)-domain. Thus, the above analysis shows there exists a solution to the differential equation with \( q_2(0) = q_2^p \) and \( q_2(1) = q_2^o \). Moreover, the solution is strictly increasing in \( \alpha \), and it satisfies \( q_2(\alpha) < q_2^h(\alpha) \) for each \( \alpha \in (0,1) \). Note also that this solution is unique since the saddle path is unique.

Next consider Eq. (41) which characterizes the interest rate function, \( r_1(\alpha) \). Note that \( \frac{dr_1(\alpha)}{d\alpha} > 0 \) since \( \frac{dq_2(\alpha)}{d\alpha} > 0 \) (recall that \( \alpha' = \alpha\lambda_1^2/\lambda_2(\alpha) \)). Note also that \( r_1'(\alpha) > r_1'(0) > 0 \), where the latter inequality follows since Assumptions 1-3 holds for the pessimistic belief. Thus, the interest rate in state 1 is always positive, which verifies our conjecture and completes the proof. 

Figure 12: The phase diagram that describes the equilibrium with heterogeneous beliefs.
A.5. Omitted derivations in Section 6.1 on equilibrium values

This subsection derives the HJB equation that describes the normalized value function in equilibrium. It
then characterizes this equation further for various cases analyzed in Section 6.1.

Characterizing the normalized value function in equilibrium. Consider the recursive version of
the portfolio problem in (A.6). Recall that the value function has the functional form in Eq. (12). Our
goal is to characterize the value function per unit of capital, \( v^i_{t,s} \) (corresponding to \( a^i_{t,s} = Q_{t,s} \)). To facilitate
the analysis, we define,

\[
\xi^i_{t,s} = v^i_{t,s} - \frac{\log Q_{t,s}}{\rho}.
\]  

(A.25)

Note that \( \xi^i_{t,s} \) is the value function per unit wealth (corresponding to \( a^i_{t,s} = 1 \)), and that the value function
also satisfies \( V^i_{t,s} (a^i_{t,s}) = \frac{\log(a^i_{t,s})}{\rho} + \xi^i_{t,s} \). We first characterize \( \xi^i_{t,s} \). We then combine this with Eq. (A.25) to
characterize our main object of interest, \( v^i_{t,s} \).

Consider the HJB equation (A.6). We substitute the optimal consumption rule from Eq. (17), the
contingent allocation rule from Eq. (19), and \( a^i_{t,s} = 1 \) (to characterize the value per unit wealth) to obtain,

\[
\rho \xi^i_{t,s} = \log \rho + \frac{1}{\rho} \left( r^f_{t,s} + \omega^k_{t,s} (r^k_{t,s} - r^f_{t,s}) - \frac{1}{2} \left( \omega^k_{t,s} \right)^2 \sigma^2_{s} - \rho - \omega^s_{t,s} \right)
\]

\[+ \frac{\partial \xi^i_{t,s}}{\partial t} + \lambda^i_s \left( \frac{1}{\rho} \log \left( \frac{\lambda^i_s}{\lambda^i_s} \right) + \xi^i_{t,s'} - \xi^i_{t,s} \right). \]

(A.26)

As we describe in Section 5, the market clearing conditions imply the optimal investment in capital and
contingent securities satisfies, \( \omega^k = 1 \) and \( \omega^s_{t,s'} = \lambda^i_s - \overline{\lambda}_{t,s} \), and the price of the contingent security is given
by, \( \frac{1}{\rho} = \overline{\lambda}_{t,s}. \) Here, \( \overline{\lambda}_{t,s} \) denotes the weighted average belief defined in (33). Using these conditions, the
HJB equation becomes,

\[
\rho \xi^i_{t,s} = \log \rho + \frac{1}{\rho} \left( r^k_{t,s} - \rho - \frac{1}{2} \sigma^2_{s} \right.
\]

\[+ \frac{\partial \xi^i_{t,s}}{\partial t} + \lambda^i_s \left( \frac{1}{\rho} \log \left( \frac{Q_{t,s'}}{Q_{t,s}} \right) + \xi^i_{t,s'} - \xi^i_{t,s} \right). \]

(A.27)

After substituting the return to capital from (24), the HJB equation can be further simplified as,

\[
\rho \xi^i_{t,s} = \log \rho + \frac{1}{\rho} \left( \psi \log \left( Q_{t,s} \right) - \delta + \left( \rho - \frac{1}{2} \sigma^2_{s} \right) \right.
\]

\[+ \frac{\partial \xi^i_{t,s}}{\partial t} + \lambda^i_s \left( \frac{1}{\rho} \log \left( \frac{Q_{t,s'}}{Q_{t,s}} \right) + \xi^i_{t,s'} - \xi^i_{t,s} \right) \].

Here, the term inside the summation on the second line, \( - (\lambda^i_s - \overline{\lambda}_{t,s}) + \lambda^i_s \log \left( \frac{\lambda^i_s}{\overline{\lambda}_{t,s}} \right) \), is zero when there are
no disagreements, and it is strictly positive when there are disagreements. This illustrates that speculation
increases the expected value for optimists as well as pessimists.

We finally substitute \( v^i_{t,s} = \xi^i_{t,s} + \frac{\log Q_{t,s}}{\rho} \) (cf. (A.25)) into the HJB equation to obtain the differential
equation,
\[ \rho v^i_{t,s} = \log \rho + \log \langle Q_{t,s} \rangle + \frac{1}{\rho} \left( \psi \log (Q_{t,s}) - \delta - \frac{1}{2} \sigma_s^2 \right) + \lambda_s^i (v_{t,s'} - v_{t,s}) \]
\[ + \partial v^i_{t,s} + \lambda_s^i (vt_s - v_{t,s}) \]

Here, we have canceled terms by using the observation that \( \frac{\partial v^i_{t,s}}{\partial t} = \frac{\partial v^i_{t,s}}{\partial t} - \frac{1}{\rho} \frac{\partial \log Q_{t,s}}{\partial t} = \frac{\partial v^i_{t,s}}{\partial t} - \frac{1}{\rho} \mu_{t,s}^i \). We have thus obtained Eq. (43) in the main text.

Solving for the value function in the common beliefs benchmark. Next consider the benchmark with common beliefs. In this case, the price level is stationary, \( q_{t,s} = q_s \) for each \( s \) (see Section 4). Then, the HJB equation (43) implies the value functions are also stationary, \( v_{t,s} = v_s \), with values that satisfy,
\[ \rho v_s = \log \rho + q_s + \frac{1}{\rho} \left( \psi q_s - \delta - \frac{1}{2} \sigma_s^2 \right) + \lambda_s (v_{s'} - v_s). \]

Consider the same equation for \( s' \neq s \). Multiplying that equation with \( \lambda_s \) and the above equation with \( (\rho + \lambda_{s'}) \), and adding up, we obtain a closed form solution,
\[ \rho v_s = \log \rho + q_s + \frac{1}{\rho} \left( \psi q_s - \delta - \frac{1}{2} \sigma_s^2 \right), \quad (A.28) \]
\[ \text{where } \overline{q}_s = \beta_s q_s + (1 - \beta_s) q_{s'} \text{ and } \sigma_s^2 = \beta_s \sigma_s^2 + (1 - \beta_s) \sigma_{s'}^2, \]
\[ \text{and } \beta_s = \frac{\rho + \lambda_{s'}}{\rho + \lambda_{s'} + \lambda_s}. \]

Here, the weights \( \beta_s \) and \( 1 - \beta_s \) can be thought of as capturing the “discounted expected time” the economy spends in each state (note that the economy starts in state \( s \) and the investors discount the future at rate \( \rho \)). The value in a state is the sum of the utility from (the discounted average of) current consumption and the present value of the risk-adjusted growth rate. All else equal, the value is decreasing in the weighted average risk, \( \sigma_s \), but it is increasing in the weighted-average price level, \( \overline{q}_s \).

Note also that the weights (the discounted expected times) satisfy the following property,
\[ \beta_s = \frac{\rho + \lambda_{s'}}{\rho + \lambda_{s'} + \lambda_s} > 1 - \beta_s = \frac{\lambda_{s'}}{\rho + \lambda_s + \lambda_{s'}}. \]

Here, \( \beta_s \) (resp. \( 1 - \beta_s \)) is the discounted time the investor spends in state \( s \) when she starts in state \( s \) (resp. in the other state \( s' \)). Thus, \( \beta_s > 1 - \beta_s \) implies that the economy spends more discounted time in the state it starts with. Combining this observation with \( q_2 < q_1 = q^* \) and \( \sigma_2^2 > \sigma_1^2 \), Eq. (A.28) implies \( v_2 < v_1 \). Intuitively, investors have a lower expected value when they are in the high-risk state since they expect asset prices to be lower and the risk to be higher.

Next note that \( \{v^*_{s}\}_s \) is defined as the solution to the same equation system with \( q_{s} = q^* \) for each \( s \). The gap value, \( w_s = v_s - v^*_s \), can be calculated by subtracting the corresponding equations for \( v_s \) an \( v^*_s \). With some algebra, we obtain,
\[ \rho w_s = (\overline{q}_s - q^*) \left( 1 + \frac{\psi}{\rho} \right). \quad (A.29) \]

That is, the gap value is proportional to the weighted-average price gap relative to the first best. Note also that we have \( q_1 - q^* = 0 \) and \( q_2 - q^* < 0 \). Since \( \beta_s \in (0,1) \), this implies \( w_s < 0 \) for each \( s \in \{1,2\} \). Since \( \beta_2 > 1 - \beta_1 \), we further obtain \( w_2 < w_1 < 0 \).
Solving the value function with belief disagreements. With belief disagreements, the value function and its components, \( \{v^i_{t,s}, v^i_{t,s}^*, w^i_{t,s}\}_{s,i} \), can be written as functions of optimists’ wealth share, \( \{v^i_s (\alpha), v^i_s^* (\alpha), w_s (\alpha)\}_{s,i} \), that solve appropriate ordinary differential equations.

Recall that the price level in each state can be written as a function of optimists’ wealth shares, \( q_{t,s} = q_s (\alpha) \) (where we also have, \( q_1 (\alpha) = q^* \)). Plugging in these price functions, and using the evolution of \( \alpha_{t,s} \) from Eq. \( (30) \), the HJB equation \( (43) \) can be written as,

\[
\rho v^i_s (\alpha) = \log \rho + q_s (\alpha) + \frac{1}{\rho} \left( \psi q_s (\alpha) - \delta - \frac{1}{2} \sigma^2 q_s - \left( \lambda^i_s - \lambda_s (\alpha) \right) + \lambda^i_s \log \left( \frac{\lambda^i_s}{\lambda_s (\alpha)} \right) \right). 
\]

For each \( i \in \{o, p\} \), the value functions, \( \{v^i_s (\alpha)\}_{s \in \{1,2\}} \), are found by solving this system of ODEs. For \( i = 0 \), the boundary conditions are that the values, \( \{v^o_s (1)\}_{s} \), are the same as the values in the common belief benchmark characterized in Section \( 4 \) when all investors have the optimistic beliefs. For \( i = p \), the boundary conditions are that the values, \( \{v^p_s (0)\}_{s} \), are the same as the values in the common belief benchmark when all investors have the pessimistic beliefs.

Likewise, the first-best value functions, \( \{v^i_s^* (\alpha)\}_{s \in \{1,2\}} \), are found by solving the analogous system after replacing \( q_s (\alpha) \) with \( q^* \) (and changing the boundary conditions appropriately). Finally, after substituting the price functions into Eq. \( (45) \), the gap-value functions, \( \{w^i_s (\alpha)\}_{s,i} \), are found by solving the following system (with appropriate boundary conditions),

\[
\rho w^i_s (\alpha) = \left( 1 + \frac{\psi}{\rho} \right) (q_s (\alpha) - q^*) - \frac{\partial v^i_s (\alpha)}{\partial \alpha} \left( \lambda^o_s - \lambda^p_s \right) \alpha (1 - \alpha) + \lambda^i_s \left( w^i_s (\alpha) \frac{\lambda^o_s}{\lambda_s (\alpha)} \right). 
\]

Figure \( 7 \) in the main text plots the solution to these differential equations for a particular parameterization.

A.6. Omitted derivations in Section 6.2 on macroprudential policy

Recall that macroprudential policy induces optimists to choose allocations as if they have more pessimistic beliefs, \( \lambda^{a,pl} = (\lambda^{o,pl}_1, \lambda^{o,pl}_2) \), that satisfy, \( \lambda^{a,pl}_1 \geq \lambda^1_s \) and \( \lambda^{a,pl}_2 \leq \lambda^2_s \). We next show that this allocation can be implemented with portfolio restrictions on optimists. We then show that the planner’s Pareto problem reduces to solving problem \( (46) \) in the main text. Finally, we derive the equilibrium value functions that result form macroprudential policy and present the proofs of Propositions \( 3 \) and \( 4 \).

Implementing the policy with risk limits. Consider the equilibrium that would obtain if optimists had the planner-induced beliefs, \( \lambda^{a,pl}_s \). Using our analysis in Section \( 5 \) optimists’ equilibrium portfolios are given by,

\[
\omega^{k,a,pl}_{t,s} = 1 \quad \text{and} \quad \omega^{s',a,pl}_{t,s} = \lambda^{a,pl}_s - \lambda^{pl}_{t,s} \quad \text{for each} \ t, s. \tag{A.30}
\]

We first show that the planner can implement the policy by requiring optimists to hold exactly these portfolio weights. We will then relax these portfolio constraints into inequality restrictions (see Eq. \( (A.32) \)).

Formally, an optimist solves the HJB problem \( (A.6) \) with the additional constraint \( (A.30) \). In view of log utility, we conjecture that the value function has the same functional form \( (12) \) with potentially different normalized values, \( \xi^i_{t,s} \), \( v^i_{t,s} \), that reflect the constraints. Using this functional form, the optimality condition for consumption remains unchanged, \( c_t = \rho w^o_{t,s} \) [cf. Eq. \( (17) \)]. Plugging this equation and the portfolio holdings in \( (A.30) \) into the objective function in \( (A.6) \) verifies that the value function has the conjectured
Combining these expressions implies, subject to the more relaxed restrictions in constraints optimality condition with complementary slackness. Note also that, choose the portfolio weights in Eq. In particular, we will establish that all inequality constraints bind, which implies that optimists optimally value function has a similar characterization as before [cf. Eq. (A.26)] with the difference that optimists’ portfolio holdings reflect the constraints.

Since pessimists are unconstrained, their optimality conditions are unchanged. It follows that the equilibrium takes the form in Section 5 with the difference that investors’ beliefs are replaced by their as-if beliefs, $\lambda_s^{i,pl}$. This verifies that the planner can implement the policy using the portfolio restrictions in (A.30). We next show that these restrictions can be relaxed to the following inequality constraints,

$$
\begin{align*}
\omega^{k,op}_{t,s} &\leq 1 \text{ for each } s, \\
\omega^{2,op}_{t,1} &\geq \omega^{2,op}_{t,1} \equiv \lambda^{op}_{1} - \lambda^{pl}_{t,1} \text{ and } \omega^{1,op}_{t,2} \leq \omega^{1,op}_{t,2} \equiv \lambda^{op}_{2} - \lambda^{pl}_{t,2}.
\end{align*}
$$

(A.32)

In particular, we will establish that all inequality constraints bind, which implies that optimists optimally choose the portfolio weights in Eq. (A.30). Thus, our earlier analysis continues to apply when optimists are subject to the more relaxed restrictions in (A.32).

The result follows from the assumption that the planner-induced beliefs are more pessimistic than optimists’ actual beliefs, $\lambda^{op}_{1} \geq \lambda^{o}_{1}$ and $\lambda^{op}_{2} \leq \lambda^{o}_{2}$. To see this formally, note that the optimality condition for capital is given by the following generalization of Eq. (18),

$$
\omega^{k,op}_{t,s} \sigma_s \leq \frac{1}{\sigma_s} \left( r^f_{t,s} - r^k_{t,s} + \lambda^{o}_{s} \frac{a^{o}_{t,s} a^{o}_{t,s'}}{a^{o}_{t,s'}} Q_{t,s'} - Q_{t,s} \right) \text{ and } \omega^{k,op}_{t,s} \leq 1,
$$

(A.33)

with complementary slackness. Note also that,

$$
\lambda^{o}_{s} \frac{a^{o}_{t,s} Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} = \lambda^{op}_{s} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \geq \lambda^{pl}_{s} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \text{ for each } s.
$$

Here, the equality follows from Eq. (A.36) and the inequality follows by considering separately the two cases, $s \in \{1, 2\}$. For $s = 2$, the inequality holds since $Q_{t,s'} - Q_{t,s} > 0$ and the beliefs satisfy, $\lambda^{o}_{s} \geq \lambda^{op}_{s}$. For $s = 1$, the inequality holds since $Q_{t,s'} - Q_{t,s} < 0$ and the beliefs satisfy, $\lambda^{op}_{s} \geq \lambda^{o}_{s}$. Note also that in equilibrium the return to capital satisfies the risk balance condition [cf. Eq. (34)],

$$
\sigma_s = \frac{1}{\sigma_s} \left( r^k_{t,s} - r^f_{t,s} + \lambda^{pl}_{t,s} \left( 1 - \frac{Q_{t,s}}{Q_{t,s'}} \right) \right).
$$

Combining these expressions implies, $\sigma_s \leq \frac{1}{\sigma_s} \left( r^k_{t,s} - r^f_{t,s} + \lambda^{o}_{s} \frac{a^{o}_{t,s} Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \right)$, which in turn implies the optimality condition (A.33) is satisfied with $\omega^{k,op}_{t,s} = 1$. A similar analysis shows that optimists also choose the corner allocations in contingent securities, $\omega^{2,op}_{t,1} = \omega^{2,o}_{t,1}$ and $\omega^{1,op}_{t,2} = \omega^{1,o}_{t,2}$, verifying that the portfolio constraints (A.30) can be relaxed to the inequality constraints in (A.32).
Simplifying the planner’s problem. Recall that, to trace the Pareto frontier, we allow the planner to do a one-time wealth transfer among the investors at time 0. Let \( V_{t,s}^i \left( a_{t,s}^i \right) \) denote type \( i \) investors’ expected value in equilibrium when she starts with wealth \( a_{t,s}^i \) and the planner commits to implement the policy, \( \left\{ \lambda_{t,s}^{o,pl} \right\} \). Then, the planner’s Pareto problem can be written as,

\[
\max_{\lambda_{t,s}^{o,pl}, \hat{\alpha}_{t,s}^o} \gamma^o V_{0,s}^o \left( \hat{\alpha}_{0,s} Q_{0,s} k_{0,s} \lambda_{t,s}^{o,pl} \right) + \gamma^p V_{0,s}^p \left( (1 - \hat{\alpha}_{0,s}) Q_{0,s} k_{0,s} \lambda_{t,s}^{o,pl} \right),
\]

(A.34)

Here, \( \gamma^o, \gamma^p \geq 0 \) (with at least one strict inequality) denote the Pareto weights, and \( Q_{0,s} \) denotes the endogenous equilibrium price that obtains under the planner’s policy.

Next recall that the investors’ value function with macroprudential policy has the same functional form in (42) (with potentially different \( \xi_{t,s}^o, v_{t,s}^o \) for optimists that reflect the constraints). After substituting \( a_{t,s}^i = \alpha_{t,s}^i k_{t,s} Q_{t,s} \), the functional form implies,

\[
V_{t,s}^i = v_{t,s}^i + \frac{\log (\alpha_{t,s}^i) + \log (k_{t,s})}{\rho}.
\]

Using this expression, the planner’s problem (A.34) can be rewritten as,

\[
\max_{\lambda_{t,s}^{o,pl}, \hat{\alpha}_{t,s}^o} \left( \gamma^o v_{0,s}^o + \gamma^p v_{0,s}^p \right) + \frac{\gamma^o \log (\hat{\alpha}_{0,s}^o) + \gamma^p \log (1 - \hat{\alpha}_{0,s}^o)}{\rho} + \frac{\gamma^o + \gamma^p}{\rho} \log (k_{0,s}).
\]

Here, the last term (that features capital) is a constant that doesn’t affect optimization. The second term links the planner’s choice of wealth redistribution, \( \alpha_{t,s}^o, \alpha_{t,s}^p \), to her Pareto weights, \( \gamma^o, \gamma^p \). Specifically, the first order condition with respect to optimists’ wealth share implies \( \gamma^o = \frac{\alpha_{0,s}^o}{1 - \alpha_{0,s}} \). Thus, the planner effectively maximizes the first term after substituting \( \gamma^o \) and \( \gamma^p \) respectively with the optimal choice of \( \alpha_{0,s}^o \) and \( 1 - \alpha_{0,s}^o \). This leads to the simplified problem (46) in the main text.

Characterizing the value functions with macroprudential policy. We first show that the normalized value functions, \( v_{t,s}^i \), are characterized as the solution to the following differential equation system,

\[
\rho v_{t,s}^i - \frac{\partial v_{t,s}^i}{\partial t} = \log \rho + q_{t,s} + \frac{1}{\rho} \left( -\left( \psi_{t,s}^i - \delta - \frac{1}{2} \sigma_{t,s}^2 \right) + \lambda_{t,s}^i \log \left( \frac{\lambda_{t,s}^{o,pl}}{\lambda_{t,s}^{o,pl}} \right) \right) + \lambda_{t,s}^i (v_{t,s'}^i - v_{t,s}^i).
\]

(A.35)

This is a generalization of Eq. (43) in which investors’ positions are calculated according to their as-if beliefs, \( \lambda_{t,s}^{o,pl} \), but the transition probabilities are calculated according to their actual beliefs, \( \lambda_{t,s}^i \).

First consider the pessimists. Since they are unconstrained, their value function is characterized by solving the earlier equation system (A.31). In this case, equation (A.35) also holds since it is the same as the earlier equation.

Next consider the optimists. In this case, the analysis in Section 5 and Appendix A.4 applies with as-if beliefs. In particular, we have,

\[
\frac{a_{t,s'}^o}{a_{t,s}^o} = \frac{\alpha_{t,s'} Q_{t,s'}}{\alpha_{t,s} Q_{t,s}} = \frac{\lambda_{t,s}^{o,pl} Q_{t,s'}}{\lambda_{t,s}^{o,pl} Q_{t,s}}.
\]

(A.36)
Plugging this expression as well as Eq. (A.30) into Eq. (A.31), optimists’ unit-wealth value function satisfies,

$$
\xi_{t,s}^o = \log \rho + \frac{1}{\rho} \left( \frac{v_{t,s}^k - \rho - \frac{1}{2} \sigma_s^2}{\xi_{t,s}^o} \right) + \left( \lambda_{s,i}^o - \frac{1}{\lambda_{t,s}^o} \right) + \lambda_s^o \log \left( \frac{\lambda_{s,i}^o - \xi_{t,s}^o}{\lambda_{t,s}^o} \right) + \frac{\partial \xi_{t,s}^o}{\partial t} + \lambda_s^o \left( \frac{1}{\rho} \left( \frac{Q_{t,s'}}{Q_{t,s}} \right) + \xi_{t,s'}^o - \xi_{t,s}^o \right),
$$

This is the same as Eq. (A.31) with the difference that the as-if beliefs, \( \lambda_{s,i}^o \), are used to calculate their positions on (and the payoffs from) the contingent securities, whereas the actual beliefs, \( \lambda_{s,i}^o \), are used to calculate the transition probabilities. Using the same steps after Eq. (A.31), we also obtain (A.35) with \( i = o \).

We next characterize the first-best and the gap value functions, \( v_{t,s}^{ib} \) and \( w_{t,s}^i \), that we use in the main text. By definition, the first-best value function solves the same differential equation (A.35) after substituting \( q_{t,s} = q^* \). It follows that the gap value function \( w_{t,s}^i = v_{t,s}^{ib} - v_{t,s}^i \), solves,

$$
\rho w_{t,s}^i - \frac{\partial w_{t,s}^i}{\partial t} = \left( 1 + \frac{\psi}{\rho} \right) (q_{t,s} - q^*) + \lambda_s^i \left( w_{t,s'}^i - w_{t,s}^i \right),
$$

which is the same as the differential equation (45) without macroprudential policy. The latter affects the path of prices, \( q_{t,s} \), but it does not affect how these prices translate into gap values.

Note also that, as before, the value functions can be written as functions of optimists’ wealth share, \( \{v_{s}^{i} (\alpha), v_{s}^{ib} (\alpha), w_{s} (\alpha)\} \). For completeness, we also characterize the differential equations that these functions satisfy in equilibrium with macroprudential policy. Combining Eq. (A.35) with the evolution of optimists’ wealth share conditional on no transition, \( \dot{\alpha}_{t,s} = - \left( \lambda_{s,i}^o - \lambda_{s,i}^o \right) \alpha_{t,s} (1 - \alpha_{t,s}) \), the value functions, \( \{v_{s}^{i} (\alpha)\} \), are found by solving,

$$
\rho v_{s}^{i} (\alpha) = \left[ \begin{array}{c}
\log \rho + q_s (\alpha) + \frac{1}{\rho} \left( \psi q_s (\alpha) - \delta - \frac{1}{2} \sigma_s^2 \right) + \lambda_s^i \log \left( \frac{\lambda_{s,i}^o - \xi_{t,s}^o}{\lambda_{t,s}^o} \right) \\
- \frac{\partial v_{s}^{i}}{\partial \alpha} \left( \lambda_{s,i}^o - \lambda_{s,i}^o \right) \alpha (1 - \alpha) + \lambda_s^i \left( \alpha \lambda_{s,i}^o - \xi_{t,s}^o \right) - v_{s}^{i} (\alpha)
\end{array} \right],
$$

with appropriate boundary conditions. Likewise, the first-best value functions, \( \{v_{s}^{ib} (\alpha)\} \), are found by solving the analogous system after replacing \( q_s (\alpha) \) with \( q^* \). Finally, combining Eq. (45) with the evolution of optimists’ wealth share, the gap-value functions, \( \{w_{s}^{i} (\alpha)\} \), are found by solving Eq. (48) in the main text.

**Proof of Proposition 3** For this and the next proof, we find it useful to work with the transformed state variable,

$$
b_{t,s} = \log \left( \frac{\alpha_{t,s}}{1 - \alpha_{t,s}} \right), \text{ which implies } \alpha_{t,s} = \frac{1}{1 + \exp (-b_{t,s})}. \quad (A.37)
$$

The variable, \( b_{t,s} \), varies between \((-\infty, \infty)\) and provides a different measure of optimism, which we refer to as “bullishness.” Note that there is a one-to-one relation between optimists’ wealth share, \( \alpha_{t,s} \in (0, 1) \), and the bullishness, \( b_{t,s} \in \mathbb{R} = (-\infty, +\infty) \). Optimists’ wealth dynamics in (36) become particularly simple when expressed in terms of bullishness,
\[
\begin{aligned}
\begin{cases}
\dot{b}_{t,s} = -\left(\lambda_{s}^{o,pl} - \lambda_{s}^{p}\right), & \text{if there is no state change}, \\
\dot{b}_{t,s'} = b_{t,s} + \log \lambda_{s}^{o,pl} - \log \lambda_{s}^{p}, & \text{if there is a state change}.
\end{cases}
\end{aligned}
\]

(A.38)

With a slight abuse of notation, we also let \( q_{s}(b) \) and \( w_{s}(b) \) denote, respectively, the price function and the gap value function in terms of bullishness.

Note also that, since \( \frac{db}{d\alpha} = \frac{1}{\alpha(1-\alpha)} \), we have the identities,

\[
\frac{\partial q_{2}(b)}{\partial b} = \alpha(1-\alpha) \frac{\partial q_{2}(\alpha)}{\partial b} \quad \text{and} \quad \frac{\partial w_{i}(b)}{\partial b} = \alpha(1-\alpha) \frac{\partial w_{i}(\alpha)}{\partial \alpha}.
\]

(A.39)

Using this observation, the differential equation for the price function, Eq. (A.39), can be written in terms of bullishness as,

\[
\frac{\partial q_{2}(b)}{\partial b} \left(\lambda_{s}^{o,pl} - \lambda_{s}^{p}\right) = \rho + \psi q_{2} - \delta + \overline{\lambda}_{2}(\alpha) \left(1 - \frac{Q_{2}}{Q'_{2}}\right) - \sigma_{2}^{2}.
\]

(A.40)

Likewise, the differential equation for the gap value function, Eq. (A.48) can be written in terms of bullishness as,

\[
\rho w_{s}^{i}(b) = \left(1 + \frac{\psi}{\rho}\right) (q_{s}(b) - q^{\ast}) - \left(\lambda_{s}^{o,pl} - \lambda_{s}^{p}\right) \frac{\partial w_{s}^{i}(b)}{\partial b} + \lambda_{s}^{i} \left(w_{s}^{i}(b') - w_{s}^{i}(b)\right).
\]

(A.41)

We next turn to the proof. To establish the comparative statics of the gap value function, we first describe it as a fixed point of a contraction mapping. Recall that, in the time domain, the gap value function solves the HJB equation (45). Integrating this equation forward, we obtain,

\[
w_{s}^{i}(b_{0,s}) = \int_{0}^{\infty} e^{-\left(\rho + \lambda_{s}^{i}\right)t} \left(1 + \frac{\psi}{\rho}\right) (q_{s}(b_{t,s}) - q^{\ast}) + \lambda_{s}^{i} w_{s}^{i}(b_{t,s'}) \right) dt,
\]

(A.42)

for each \( s \in \{1, 2\} \) and \( b_{0,s} \in \mathbb{R} \). Here, \( b_{t,s} \) denotes bullishness conditional on there not being a transition before time \( t \), whereas \( b_{t,s'} \) denotes the bullishness if there is a transition at time \( t \). Solving Eq. (A.38) (given as-if beliefs, \( \lambda_{s}^{o,pl} \)) we further obtain,

\[
\begin{aligned}
\dot{b}_{t,s} &= b_{t,s} - t \left(\lambda_{s}^{o,pl} - \lambda_{s}^{p}\right), \\
\dot{b}_{t,s'} &= b_{t,s} - t \left(\lambda_{s}^{o,pl} - \lambda_{s}^{p}\right) + \log \lambda_{s}^{o,pl} - \log \lambda_{s}^{p}.
\end{aligned}
\]

(A.43)

Hence, Eq. (A.42) describes the value function as a solution to an integral equation given the closed form solution for bullishness in (A.43).

Let \( B(\mathbb{R}^{2}) \) denote the set of bounded value functions over \( \mathbb{R}^{2} \). Given some continuation value function, \((\tilde{w}^{i}_{s}(b))_{s} \in B(\mathbb{R}^{2})\), we define the function, \((T\tilde{w}^{i}_{s}(b))_{s} \in B(\mathbb{R}^{2})\), so that

\[
T\tilde{w}^{i}_{s}(b_{0,s}) = \int_{0}^{\infty} e^{-\left(\rho + \lambda_{s}^{i}\right)t} \left(1 + \frac{\psi}{\rho}\right) (q_{s}(b_{t,s}) - q^{\ast}) + \lambda_{s}^{i} \tilde{w}^{i}_{s}(b_{t,s'}) \right) dt,
\]

(A.44)

for each \( s \) and \( b_{0,s} \in \mathbb{R} \). Note that the resulting value function is bounded since the price function, \( q_{s}(b_{t,s}) \), is bounded (in particular, it lies between \( q_{p} \) and \( q^{\ast} \)). It can be checked that operator \( T \) is a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the gap value function, \((w^{i}_{s}(b))_{s}\).

We next show that the value function has strictly positive derivative with respect to bullishness as well as optimists’ wealth share. To this end, we first note that the value function is differentiable since it solves
the differential equation \((48)\). Next, we implicitly differentiate the integral equation \((4.42)\) with respect to \(b_{0,s}\), and use Eq. \((4.43)\), to obtain,

\[
\frac{\partial w^s_i}{\partial b} (b_{0,s}) = \int_0^\infty e^{-(\rho + \lambda^i_s) t} \left( \left( 1 + \frac{\psi}{\rho} \right) \frac{\partial q_2 (b_{t,s})}{\partial b} + \lambda^i_s \frac{\partial w^s_i (b_{t,s})}{\partial b} \right) dt. \tag{4.45}
\]

Note from Eq. \((4.40)\) that the derivative of the price function, \(\frac{\partial q_2 (b_{0,s})}{\partial b}\), is bounded. Thus, Eq. \((4.45)\) describes the derivative of the value function, \(\frac{\partial w^s_i (b_{0,s})}{\partial b}\), as a fixed point of a corresponding operator \(T^{\partial b}\) over bounded functions (which is related to but different than the earlier operator, \(T\)). This operator is also a contraction mapping with respect to the sup norm. Since \(\frac{\partial q_2 (b_{0,s})}{\partial b} > 0\) for each \(b\) and \(\lambda^i_s > 0\) for each \(s\), it can further be seen that the fixed point satisfies, \(\frac{\partial w^s_i (b_{0,s})}{\partial b} > 0\) for each \(b\) and \(s \in \{1, 2\}\). Using Eq. \((4.39)\), we also obtain \(\frac{\partial w^i_j (\alpha)}{\partial \alpha} > 0\) for each \(\alpha \in (0, 1)\) and \(s \in \{1, 2\}\).

Next consider the comparative statics of the fixed point with respect to macroprudential policy. We implicitly differentiate the integral equation \((4.42)\) with respect to \(\lambda^{o,p}_{1}\), and use Eq. \((4.43)\), to obtain,

\[
\frac{\partial w^1_i}{\partial \lambda^{o,p}_{1}} (b_{0,1}) = \int_0^\infty e^{-(\rho + \lambda^i_1) t} \frac{\partial w^2_i}{\partial \lambda^{o,p}_{1}} (b_{t,2}) \frac{\partial w^2_i (b_{t,2})}{\partial b} \left( \frac{\partial q_2 (b_{t,2})}{\partial b} + \lambda^i_2 \frac{\partial w^2_i (b_{t,2})}{\partial b} \right) dt, \tag{4.46}
\]

\[
\frac{\partial w^2_i}{\partial \lambda^{o,p}_{1}} (b_{0,2}) = \int_0^\infty e^{-(\rho + \lambda^i_1) t} \frac{\partial w^1_i}{\partial \lambda^{o,p}_{1}} (b_{t,1}) \frac{\partial w^1_i (b_{t,1})}{\partial b} \left( \frac{\partial q_2 (b_{t,2})}{\partial b} + \lambda^i_2 \frac{\partial w^2_i (b_{t,2})}{\partial b} \right) dt.
\]

Note also that, using Eq. \((4.43)\) implies \(\frac{\partial b_{i,2}}{\partial \lambda^{o,p}_{1}} = -t + \frac{1}{\lambda^i_2}\). Plugging this into the previous system, and evaluating the partial derivatives at \(\lambda^{o,p}_{1} = \lambda_1\), we obtain,

\[
\frac{\partial w^1_i}{\partial \lambda^{o,p}_{1}} (b_{0,1}) = h (b_{0,1}) + \int_0^\infty e^{-(\rho + \lambda^i_1) t} \lambda^i_1 \frac{\partial w^2_i (b_{t,2})}{\partial \lambda^{o,p}_{1}} \frac{\partial q_2 (b_{t,2})}{\partial b} \left( \frac{\partial q_2 (b_{t,2})}{\partial b} + \lambda^i_2 \frac{\partial w^2_i (b_{t,2})}{\partial b} \right) dt, \tag{4.46}
\]

\[
\frac{\partial w^2_i}{\partial \lambda^{o,p}_{1}} (b_{0,2}) = \int_0^\infty e^{-(\rho + \lambda^i_1) t} \lambda^i_2 \frac{\partial w^1_i (b_{t,1})}{\partial \lambda^{o,p}_{1}} \frac{\partial q_2 (b_{t,2})}{\partial b} \left( \frac{\partial q_2 (b_{t,2})}{\partial b} + \lambda^i_2 \frac{\partial w^2_i (b_{t,2})}{\partial b} \right) dt,
\]

where \(h (b_{0,1}) = \int_0^\infty e^{-(\rho + \lambda^i_1) t} \lambda^i_1 \frac{\partial w^2_i (b_{t,2})}{\partial b} \left( -t + \frac{1}{\lambda^i_2} \right) dt\).

Note that the function, \(h (b)\), is bounded since the derivative function, \(\frac{\partial h (b)}{\partial b}\), is bounded (see \((4.45)\)). Hence, Eq. \((4.46)\) describes the partial derivative functions, \(\frac{\partial w^i_j (b_{0,s})}{\partial \lambda^{o,p}_{1}} |_{\lambda^{o,p}_{1} = \lambda^i_s}\), as a fixed point of a corresponding operator \(T^{\partial \lambda}\) over bounded functions (which is related to but different than the earlier operator, \(T\)). Since \(h (b)\) is bounded, it can be checked that the operator \(T^{\partial \lambda}\) is also a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the partial derivative functions.

The analysis so far applies generally. We next consider the special case, \(\lambda^o_1 = \lambda^p_1\), and show that it implies the partial derivatives are strictly positive. In this case, \(\lambda^1_1 = \lambda^i_1\) for each \(i \in \{o, p\}\). In addition, Eq. \((4.43)\) implies \(b_{t,2} = b_{0,2}\). Using these observations, for each \(b_{0,1}\), we have,

\[
h (b_{0,1}) = \frac{\partial w^1_i (b_{0,2})}{\partial b} \int_0^\infty e^{-(\rho + \lambda^i_1) t} \left( -t + \frac{1}{\lambda^i_1} \right) dt = \frac{\partial w^1_i (b_{0,2})}{\partial b} \left( -\frac{\lambda^i_1}{\rho + \lambda^i_1} \frac{1}{\rho + \lambda^i_1} + \frac{1}{\rho + \lambda^i_1} \right) > 0.
\]

Here, the inequality follows from our earlier result that \(\frac{\partial w^i_j (b_{0,s})}{\partial b} > 0\). Since \(h (b) > 0\) for each \(b\) and \(\lambda^i_s > 0\),
it can further be seen that the fixed point that solves \((A.46)\) satisfies \(\frac{\partial w^i_s(b)}{\partial \lambda_s} > 0\) for each \(b\) and \(s \in \{1, 2\}\). Using Eq. \((A.39)\), we also obtain \(\frac{\partial w^i_s(\alpha)}{\partial \lambda_{s}^{o, pl}} > 0\) for each \(\alpha \in (0, 1)\) and \(s \in \{1, 2\}\).

**Proof of Proposition 4.** A similar analysis as in the proof of Proposition 3 implies that the partial derivative function, \(\frac{\partial w^i_s(b)}{\partial \lambda_s} \), is characterized as the fixed point of a contraction mapping over bounded functions (the analogue of Eq. \((A.46)\) for state 2). In particular, the partial derivative exists and it is bounded. Moreover, since the corresponding contraction mapping takes continuous functions into continuous functions, the partial derivative is also continuous over \(b \in \mathbb{R}\). Using Eq. \((A.39)\), we further obtain that the partial derivative, \(\frac{\partial w^i_s(\alpha)}{\partial \lambda_{s}^{o, pl}} \), is continuous over \(\alpha \in (0, 1)\).

Next note that \(w^i_s(1) = \lim_{\alpha \to 1} w^i_s(\alpha)\) exists and is equal to the value function according to type \(i\) beliefs when all investors are optimistic. In particular, the asset prices are given by \(q_1 = q^*\) and \(q_2 = q^o\), and the transition probabilities are evaluated according to type \(i\) beliefs. Then, following the same steps as in our analysis of value functions in Appendix \(A.5\), we obtain,

\[
 w^i_s(1) = \left(1 + \frac{\psi}{\rho}\right) \left(\beta^i_s q^o - q^*\right)
\]

where \(\beta^i_s = \frac{\rho + \lambda^i_s}{\rho + \lambda^o_s + \lambda^i_s} \).

Here, \(\beta^i_s\) denotes the expected discount time the investor spends in state \(s\) according to type \(i\) beliefs. We consider this equation for \(s = 2\) and take the derivative with respect to \((-\lambda_{2}^{o, pl})\) to obtain,

\[
\frac{\partial w^i_s(1)}{\partial \left(-\lambda_{2}^{o, pl}\right)} = \left(1 + \frac{\psi}{\rho}\right) \beta^2_s \frac{dq^o_s}{d \left(-\lambda_{2}^{o, pl}\right)} < 0.
\]

Here, the inequality follows since reducing optimists’ optimism reduces the price level in the common belief benchmark (see Section \(4\)).

Note that the inequality, \(\frac{\partial w^i_s(1)}{\partial \left(-\lambda_{2}^{o, pl}\right)} < 0\), holds for each state \(s\) and each belief type \(i\). Using the continuity of the partial derivative function, \(\frac{\partial w^i_s(\alpha)}{\partial \lambda_{s}^{o, pl}}\), we conclude that there exists \(\overline{\alpha}\) such that \(\frac{\partial w^i_s(\alpha)}{\partial \lambda_{s}^{o, pl}} \bigg|_{\lambda_{s}^{o, pl}=\lambda^o} < 0\) for each \(i, s\) and \(\alpha \in (\overline{\alpha}, 1)\), completing the proof.