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A search-based model of the interbank money market and monetary policy implementation*

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Abstract

We present a search-based model of the interbank money market and monetary policy implementation. Banks are subject to reserve requirements and the central bank tenders reserves. Interbank payments redistribute holdings and banks trade with each other in a decentralized (over-the-counter) market. The central bank provides standing facilities where banks can either deposit surpluses or borrow to cover shortfalls of reserves overnight. The model provides insights on liquidity, trading volume, and rate dispersion in the interbank market - features largely absent from the canonical models in the tradition of Poole (1968) - and fits a number of stylized facts for the Eurosystem observed during the recent period of unconventional monetary policies. Moreover, it provides insights on the implications of different market structures.

1 Introduction

We present a model of the interbank money market and monetary policy implementation in a corridor system like that used by the European Central Bank (ECB). The model is in line with

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a number of stylized facts concerning interbank market functioning in the euro area during the recent period of unconventional monetary policies. The facts are that a surge in excess reserves: (i) drives the overnight rate to the rate at which the central bank remunerates reserves; (ii) decreases trading volume; and (iii) reduces overnight rate volatility; but (iv) does not impact the end-of-day lending by the central bank.

To replicate these stylized facts, we model the over-the-counter (OTC) structure of the interbank market. In particular, we study the search for counter-parties using the sorting approach of Matsui and Shimizu (2005). Our set-up works as follows. First the central bank tenders reserves. Then during the normal course of business, interbank payments shuffle these reserves across banks. So banks may wish to trade with each other to satisfy reserves requirements. Prior to trading, banks endogenously sort themselves into borrowers and lenders. In equilibrium, banks with a shortfall of reserves opt to become borrowers while those with a surplus choose to become lenders. Borrowers and lenders are then matched at random but the matching can be more or less efficient - reflecting differences in market structure. Once matched, borrowers and lenders bargain over both the amount to trade and the rate but due to settlement risk they may abstain from trading. Banks trade heterogenous amounts and rates, depending on their reserve holdings. Hence, the model allows us to compute - among other things - volume-weighted money markets statistics including the "effective" (average) rate, which is the operational target preferred by many central banks.

In addition to studying pricing, trading, and volatility in the interbank market, our model can be used to analyze monetary policy implementation, as we explicitly model how the central bank supply reserves to the banking system. We assume that the central bank uses a full-allotment tender, so that any bid is met at a fixed pre-specified tender rate. The tender rate disseminates through the system as follows. When the tender rate is low, banks bid aggressively and the aggregate amount of reserves in the system is large. Therefore, after the payment shocks, most banks will still be long in reserves and, as those banks choose to become lenders, there will be many lenders and few borrowers as a consequence of sorting. The matching process thus implies that there will be few trades and so market volume will tend to be low.[1] Moreover, as most lenders will tend to hold a lot of excess reserves while most borrower banks' deficit will tend to be small, the interbank rate will tend to be low and the lending rate will not display much dispersion. Finally, even though there is a large amount of aggregate excess reserves, some borrower banks will still be unmatched and will have to use the central bank end-of-day lending facility.

[1]With purely random search, it is possible that two banks both holding positive (or negative) excess reserves meet. In cases where they choose to trade, aggregate trading volume can be independent of the amount of reserves, which contradicts the second stylized fact.

The model closest to ours is Afonso and Lagos (2014).[2] They study intraday trading dynamics in the federal funds market using multiple trading rounds and random search. They find equilibria with endogenous intermediation or speculative trading. That is some banks buy reserves early in expectation of being able to sell them later for a higher price. Our paper is also related to the models of banks' reserves management in the tradition of Poole (1968).[3] However, they focus solely on price, i.e., interest rate, and have little to say about the properties of trading volume and liquidity.

We structure the paper as follows. Section 2 presents the basic stylized facts for the Eurosystem. Section 3 details the environment. We then solve the model and define the equilibrium in Section 4. Section 5 characterizes the equilibrium for different market structures and analyze bank behavior both at the central bank tender and in the interbank market. In particular, we analyze the dynamics of the overnight rate, trading volumes, interbank rate dispersion, and use of the central bank lending facility. We conclude in Section 6.

2 Some stylized facts for the Eurosystem

The European Central Bank (ECB) relies on a so-called corridor system for the implementation of monetary policy. The rate at the central bank's deposit facility serves as a "floor" for overnight interbank rates and the rate at its (marginal) lending facility serves as a "ceiling" (ECB, 2002). Prior to October 2008, refinancing operations of the ECB aimed at keeping excess reserves[4] close to zero. This allowed the ECB to steer the Eonia®[5] rate close to the minimum bid rate in its weekly variable rate tenders - the so-called main refinancing operations (MROs). The MRO rate was set at the midpoint of the corridor (see Figure 1). As excess reserves were remunerated at below

[2]Afonso and Lagos (forthcoming) and Bech and Monnet (2013) present small-scale versions of the dynamic equilibrium over-the-counter theory of trade in the federal funds market developed in Afonso and Lagos (2014). Afonso and Lagos (forthcoming) restricts feasible reserve holdings to a set with three values whereas Bech and Monnet (2013) present a version with two trading rounds.

[3]The standard models include e.g. Woodford (2001), Bindseil (2004), Whitesell (2006) and Ennis and Keister (2008). Also see Walsh (2010) for an introduction.

[4]Excess reserves is here defined as deposits at the ECB's deposit facility net of the recourse to the ECB's marginal lending facility, plus current account holdings of banks in excess of those contributing to minimum reserve requirements. In ECB parlance this quantity is known as "excess liquidity" as the ECB uses the notion of "excess reserves" to describe the narrower concept of current account holdings in excess of reserve requirements. With a view to be consistent with the existing literature, e.g., Poole (1968) and Afonso and Lagos (2014), we refer to excess reserves as all central bank overnight deposits beyond required reserves and hence do not make a distinction between whether they are held in a current account or at a deposit facility.

[5]Eonia® (Euro OverNight Index Average) is the effective overnight reference rate for the euro. It is computed as a weighted average of overnight unsecured lending transactions in the interbank market, undertaken in the European Union and European Free Trade Association (EFTA) countries. It is provided by the European Money Markets Institute. There are about 35 banks in the panel and it is computed with the help of the ECB.

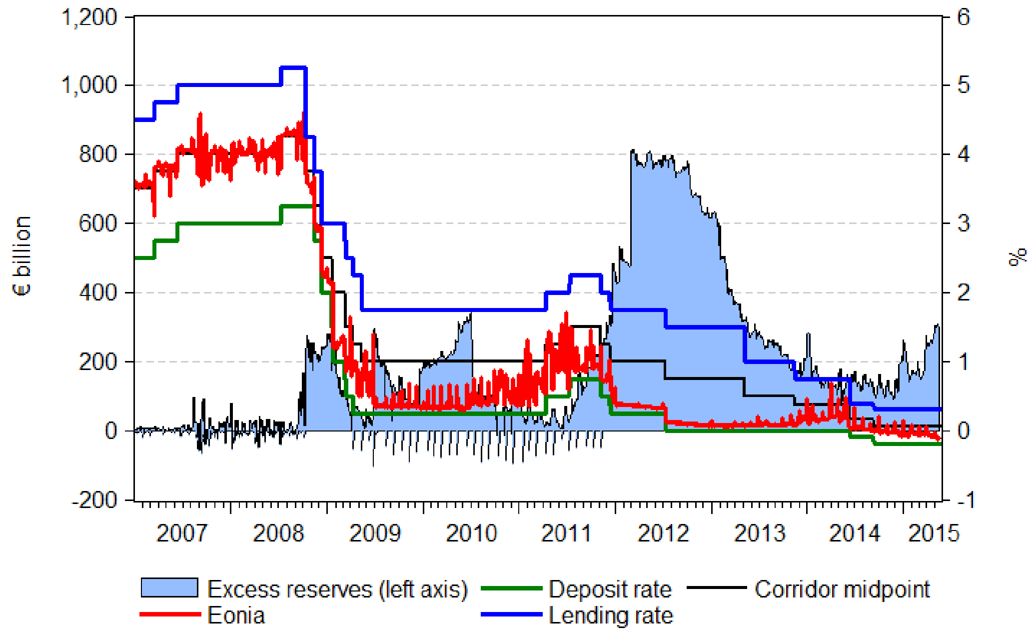


Figure 1: Excess reserves, ECB policy rates and Eonia

market rates banks held only small amounts. The interbank money market smoothly redistributed reserves from "the haves" to "the have nots."

In October 2008, the operational framework changed significantly (ECB, 2014). Facing elevated demands for liquidity and strained money markets, the ECB introduced tender operations with full allotment at a fixed rate. Since then euro area banks have operated in an environment with excess reserves - reaching a peak of over €800 billion in March 2012 - following allotment of two three-year, so-called longer term, refinancing operations (LTROs). However, in early 2013 – as financial conditions improved - some euro area banks began repaying their borrowed funds and excess reserves fell to below €200 billion in mid-2014. In 2015, ECB's purchases of private and public sector securities have pushed excess reserves up again.

The swings in excess reserves have had a significant impact on the Eonia rate and money market functioning in the euro area as illustrated in Figure 2. Using daily data for the period from January 2009 to May 2015, we plot four key statistics with respect to the overnight interbank market against the amount of excess reserves. The statistics cover interbank pricing, trading volume and volatility as well as end-of-day lending by the central bank. The price statistics is the spread between the Eonia rate and the rate at the ECB's deposit facility - measured in basis points. Trading volume is measured by aggregate value of the transactions reported by the banks that participate in the calculation of the Eonia and volatility is measured as a 20-day moving standard deviation of the Eonia to ECB deposit rate spread. The last statistics is the recourse to the ECB's marginal lending facility.

The scatter plots based on daily observations are "noisy" - reflecting in part calendar effects and other idiosyncratic factors. Hence, in order to distill stylized facts, we overlay each plot with a non-parametric regression fit.[6] The advantage is that we do not a-priori impose a specific model or functional form on the data. The downside is that the non-parametric methods are data intensive and thus may not provide robust fits over regions with sparse data. As evident in Figure 1, negative excess reserves are a rare occurrence and thus the stylized facts only pertain to positive levels of excess reserves.

The scatter plot and non-parametric fit in Figure 2a suggest an inverse relationship between the amount of excess reserves and the spread between the Eonia and the deposit rate.[7] The spread is on average just below 50 basis points when excess reserves are close to zero and converges to about 10 basis points as excess reserves goes above €300 billion. We articulate the following stylized fact:

Stylized Fact 1. *Increasing the amount of excess reserves drives the overnight interbank rate towards the floor of an interest rate corridor.*

Similarly, Figure 2b plots excess reserves against Eonia volume. Again, the non-parametric fit suggests an inverse relationship. Eonia volume is highest when excess reserves are close to zero. It averages around €40 billion per day but falls to about €25 billion per day on average once excess reserves exceeds €300 billion.[8] We note the following stylized fact:

Stylized Fact 2. *Increasing the amount of excess reserves reduces overnight interbank volume.*

Figure 2c plots excess reserves against a 20-day moving standard deviation of the spread between the Eonia and the deposit rate. It ranges between one and 35 basis points. Again, the scatter plot and the non-parametric fit are suggestive of an inverse relationship for positive levels

[6]Specifically, we use the kernel fit procedure in Eviews 8. We choose the normal kernel and the default "rule of thumb" band width. The Eviews procedure fits Y at each value of x by choosing β to minimize the weighted sum-of-squared residuals: $SSR(x) = \sum_{i=1}^{N}(Y_i - \beta_0 - \beta_1(x - X_i))^2 K\left(\frac{x-X_i}{h}\right)$ where N is the number of observations, h is the bandwidth parameter and K is a kernel function that integrates to one. The kernel assigns higher weights to observations closer to x, each x results in a different estimates of β and $\hat{Y}_i\,|x = \beta_0$.

[7]An alternative price measure is to divide the Eonia - deposit rate spread with the width of the ECB corridor. This measure controls for the fact the the width of the corridor changed over the sample period. ECB (2014) fits a generalized logistic curve to this measure and the result is similar to ours.

[8]Like Afonso, Kovner and Schoar (2011) for the federal funds market, the figure illustrates that while trading volumes fell, the overnight Eonia market did not disappear during the financial turmoil.

of excess reserves. Volatility tends to be higher when liquidity conditions are balanced and lowest when liquidity is plentiful.

Stylized Fact 3. *Increasing the amount of excess reserves lowers overnight rate volatility.*

The last scatterplot in Figure 2 shows the use of the marginal lending facility against excess reserves. Lending goes as high as €15 billion on a few days, but are below €5 billion on most days. In this case, the non-parametric regression does not suggest any relationship between the two quantities.

Stylized Fact 4. *Increasing the amount of excess reserves does not impact the recourse to the central bank lending facility.*

3 The environment

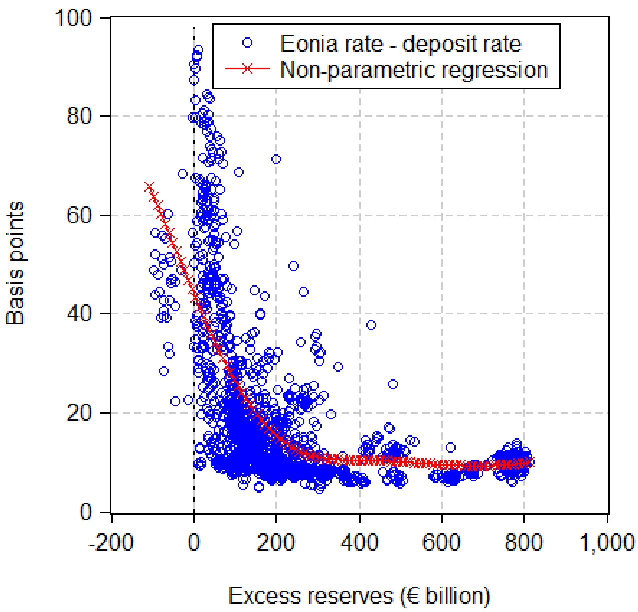
We consider the following representation of the daily reserves management problem of banks. A day consists of three periods $t = 0, 1, 2$. Period 0 is the beginning of the day whereas period 2 is the end. In period 0 there is a *centralized* market for reserves in which the central bank supplies reserves to banks while period 1 contains a *decentralized* market, where banks trade reserves with each other bilaterally. At the end of periods 0 and 1, the reserves holdings of banks are hit by random shocks as explained in further detail below.

There is a unit measure of risk neutral banks indexed by $k \in \mathcal{K}$, and we let $m_k(t) \in \mathbb{R}$ denote the reserve holdings of bank k at the start of period t. Banks are subject to reserve requirements under which they have to hold at least $\bar{m}_k \in \mathbb{R}_0^+$ at the end of the day. That is $m_k(2) \geq \bar{m}_k$.

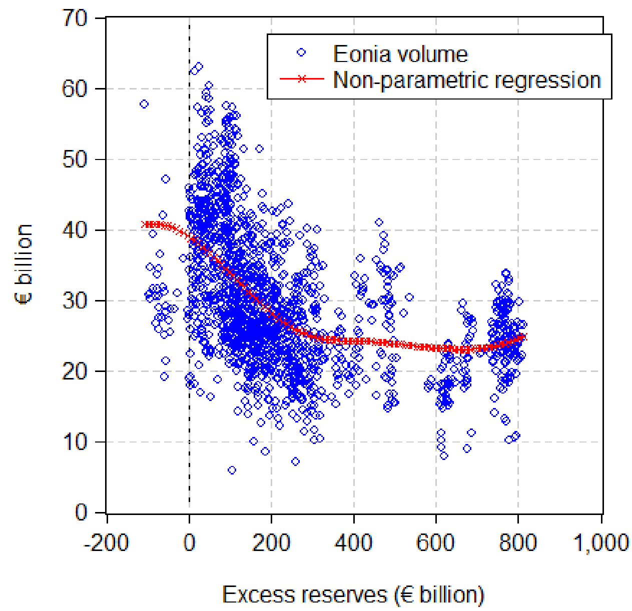
In the morning ($t = 0$), banks hold $m_k(0)$ units of reserves when the central bank tenders $M \in \mathbb{R}$ units of reserves. Bank k bids for $y'_k \in \mathbb{R}$ units of reserves and receives $y_k \in \mathbb{R}$ at the central bank tender and hence it exits the tender with reserve holdings of $m_k(0) + y_k$. The central bank uses a full allotment tender that we describe precisely below.

Following the tender banks receive an idiosyncratic payment shock $v_k \in \mathbb{R}$ to their reserve holdings.[9] We assume that, the shock is symmetrically distributed with mean zero according to a cumulative distribution function F. Hence, the reserve holdings of bank k at the start of period 1 is $m_k(1) = m_k(0) + y_k + v_k$.

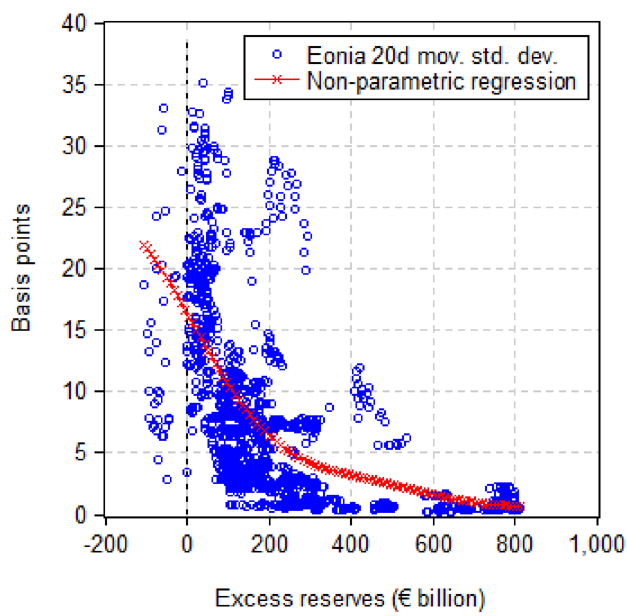
[9]This shock captures the fact that interbank payments redistribute reserves holdings among banks during the day. Central banks usually operate an electronic interbank payment system. The Euro-system runs TARGET2 system. See e.g. Bech and Hobijn (2007) for a discussion.



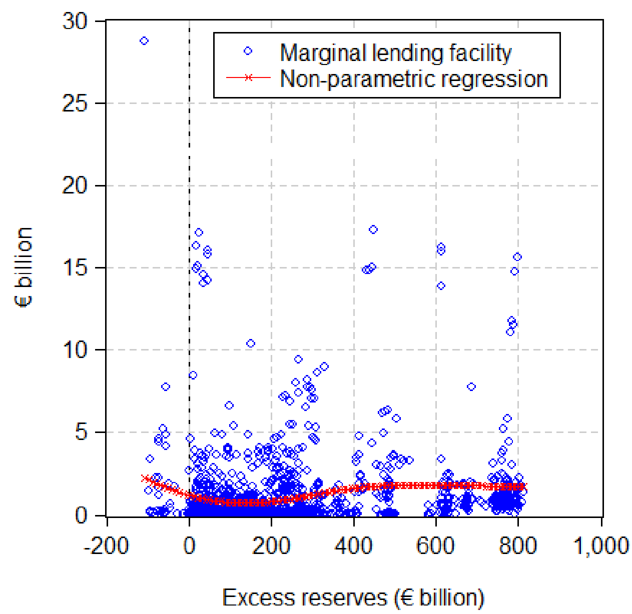
(a) EONIA rate.



(b) EONIA volume



(c) EONIA dispersion



(d) Volume borrowed at the lending facility

Figure 2: EONIA statistics vs Excess reserves

Later in the day ($t = 1$), banks can access a bilateral OTC market for reserves. We assume that banks first irrevocably commit to whether they want to lend or borrow reserves. Second, borrowers and lenders are matched with each other and - if matched - then the two counterparties - in a third step - ascertain whether there are gains from trade, and if so, bargain over the terms of the loan.[10] Let $q_k \in \mathbb{R}$ denote the amount borrowed (lent) by bank k in the OTC market and let i_q denote the rate paid (received).[11] The reserve holdings of bank k exiting the decentralized market are $m_k(1) + q_k$.

While trading allows banks to get closer to their reserve requirement it also exposes them to settlement risk.[12] We capture this risk by an additional late shock $\varepsilon_k \in \mathbb{R}$ to banks that have traded. We assume that, the shock is symmetrically distributed with zero mean according to a cumulative distribution function H with full support over $\mathbb{R}$.[13] Thus, bank k holds $m_k(2) = m_k(1) + q_k + \varepsilon_k$ at the end of period 1, where $q_k = \varepsilon_k = 0$ for banks that were not matched or that were matched but did not trade. We will say that the settlement risk becomes negligible whenever the settlement shock distribution weakly converges to the Dirac distribution which is centered at zero.

At the end of the day, if a bank holds more reserves than required ($m_k(2) > \bar{m}_k$), then it deposits this surplus overnight with the central bank, earning i_d per unit of funds. On the other hand, if bank k has a shortfall of reserves at the end-of-day ($m_k(2) < \bar{m}_k$), then it borrows the deficiency from the central bank paying a penalty lending rate $i_p \geq i_d$. Required reserves are remunerated at $i_{\bar{m}}$. Figure 3 summarizes the reserve holdings of bank k over the day.

[10]This set up can e.g. be thought of as reflecting the use of brokers in interbank money market. Banks tell brokers whether they want to buy or sell funds together with at least an indication of agreeable terms. The broker then seeks to match the bank with another bank that wants to be on the opposite side of the trade.

[11]Unlike Afonso and Lagos (2014), intraday intermediation of reserves does not occur as there for simplicity is only one round of trading.

[12]Many interbank payment systems reserve the last part of the business for settling interbank transaction such money market transactions. For example, TARGET2 – the Eurosystem real-time gross settlement (RTGS) system for payments in euros – closes for customer payment at 17:00 whereas interbank transfers can occur until 18:00. Settlement risk is the risk that banks fail to deliver the agreed amount of funds to counterparties before the close of business. Such failures can be due to miscommunications, operational outages, fat finger errors, etc. For a discussion of settlement risk in the context of interbank payment systems see e.g. Mills and Nesmith (2008) and Klee (2010).

[13]The full support assumption implies that the settlement shock can be arbitrarily large. While we will be mostly interested in limit results when the settlement shock becomes unlikely, there is evidence that "fat finger" errors can indeed be large. For example, on October 1, 2014, a "fat finger" caused over \$600 billion worth of orders in the Japanese OTC market (See http://www.bbc.com/news/business-29454265).

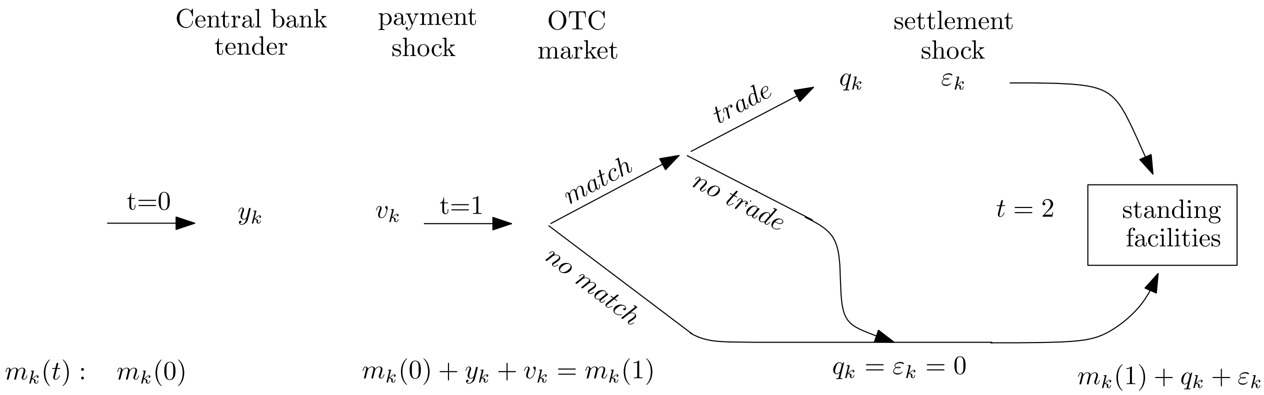


Figure 3: Time line - Intraday reserve holdings

4 Solving the model

We solve the model backwards. First, we compute the end-of-day ($t = 2$) payoffs of each bank depending on their history of trade. Second, given those payoffs we derive the rates and quantities banks trade in the OTC interbank market at $t = 1$, if they choose to trade. We then untangle the decision of banks to become borrowers or lenders. Finally, we derive a bank's willingness to pay for reserves at the time of the central bank's reserves tender at $t = 0$.

4.1 End-of-day payoff ($t = 2$)

The end-of-day payoff of bank k is

$$\hat{P}(i_q, q_k, m_k(2), \bar{m}_k) = (m_k(2) - \bar{m}_k)\, i[m_k(2) - \bar{m}_k] - i_q q_k + \bar{m}_k i_{\bar{m}} \tag{1}$$

where $i[z]$ is a function which equals i_p whenever $z < 0$ and i_d whenever $z \geq 0$. The first term in (1) reflects bank k's use of the central bank's standing facilities. That is, the penalty the bank pays on a shortfall of reserves, or the remuneration it obtains on reserves in excess of the required amount. The second term is the bank's borrowing cost ($q_k > 0$) or lending income ($q_k < 0$) from trading in the interbank market when it agreed to the interest rate i_q. The last term reflects remuneration of required reserves. This term is known to the bank at the beginning of the day, and hence it does not affect any of its decisions. Consequently, going forward we use the following rescaled version of (1) :

$$P(i_q, q_k, x_k(2)) = x_k(2) i[x_k(2)] - i_q q_k \tag{2}$$

where $x_k(t) \equiv m_k(t) - \bar{m}_k$ denote the amount of "excess reserves" at the start of period t.

4.2 Over-the-counter market ($t = 1$)

As indicated above, the OTC market involves three sub-periods: a search and matching stage, a trade or no trade stage, and a bargaining stage. We start by describing the latter.

4.2.1 Bargaining - rates and volumes

Once two banks are matched and establish that trade is beneficial, they bargain over the terms of trade (the quantity borrowed q and the interest rate i_q). Let b and ℓ denote two banks holding excess reserves of x_b and x_ℓ, respectively, at the start of period 1. Assume - without loss of generality - that bank b decided to borrow reserves and bank ℓ decided to lend reserves. So bank b borrows the amount q from bank ℓ ($q_b = -q_\ell = q > 0$) for a payment d. The surpluses of the two banks from trading are

$$S_b = \int (x_b + q + \varepsilon) i[x_b + q + \varepsilon] dH(\varepsilon) - d - x_b i[x_b], \tag{3}$$

and

$$S_\ell = \int (x_\ell - q + \varepsilon) i[x_\ell - q + \varepsilon] dH(\varepsilon) + d - x_\ell i[x_\ell]. \tag{4}$$

We assume Nash bargaining with equal bargaining power. Hence, the two banks split the gains from trade equally and it suffices to focus on maximizing the joint surplus $S = S_b + S_\ell$. The joint surplus is

$$S = \int (x_b + q + \varepsilon) i[x_b + q + \varepsilon] dH(\varepsilon) + \int (x_\ell - q + \varepsilon) i[x_\ell - q + \varepsilon] dH(\varepsilon) - x_\ell i[x_\ell] - x_b i[x_b]. \tag{5}$$

The following Proposition summarizes the unique bargaining solution conditional on trading.[14]

Proposition 1. *A borrower bank holding x_b trades with a lending bank holding x_ℓ if $x_\ell > x_b$. The unique terms of trade are*

$$q(x_\ell, x_b) = \frac{x_\ell - x_b}{2}, \tag{6}$$

and the OTC rate is

$$i_q(x_\ell, x_b) = \frac{x_\ell i[x_\ell] - x_b i[x_b]}{(x_\ell - x_b)}. \tag{7}$$

[14] With linear payoffs, there can be multiple solutions when banks bargain over both the amount to be traded and the rate (see Tapking, 2006). To deal with this indeterminacy, Afonso and Lagos (2014) assumes that banks display a degree of risk aversion. Here, the settlement risk effectively makes banks' payoff concave. Notice that (7) assumes a simple form, as banks are not subject to settlement risk if they do not trade. Otherwise, the OTC rate would be a function of the settlement risk which would complicate the analysis somewhat.

Proof. The joint surplus in (5) is a concave function of q as long as $i_p > i_d$. The first order condition implies

$$i_p \int_{-\infty}^{-(x_b+q)} dH(\varepsilon) + i_d \int_{-(x_b+q)}^{\infty} dH(\varepsilon) = i_p \int_{-\infty}^{-(x_\ell-q)} dH(\varepsilon) + i_d \int_{-(x_\ell-q)}^{\infty} dH(\varepsilon).$$

Solving for q yields (6). This solution is unique when H has full support over the real line which we assumed. By assumption, b should borrow from ℓ and hence $q_b = q(x_\ell, x_b) > 0$. By (6) this happens if and only if $x_\ell > x_b$. Therefore, as banks commit to lend or borrow, a borrower bank with x_b meeting a lender bank with $x_\ell < x_b$ will not trade, as (6) implies that the borrower bank would end up lending. Given (6), we can solve for the payment $d = i_q q(x_\ell, x_b)$ by equalizing the borrower's and lender's surplus in (3) and (4). Using (6) yields the OTC rate (7). □

Proposition 1 implies that banks equate their reserve holdings if they trade. When both banks have enough reserves to satisfy the reserves requirement, they trade at the deposit rate, as in this case $i[x_\ell] = i[x_b] = i_d$. On the other hand, if both banks have too few reserves to satisfy their requirement, then they trade at the lending rate, as in this case $i[x_\ell] = i[x_b] = i_p$. Finally, if $x_\ell > 0$ but $x_b < 0$, then banks trade at a rate which is a weighted average of the central bank facility rates. The weights are given by the relative reserves position of each bank. If the lender has a lot of excess reserves, then the rate tends to the deposit rate. Otherwise, the rate tends to the central bank lending rate. Note, that we can compute the rates and trades in each match irrespective of how banks are matched.

4.2.2 To trade or not to trade

Before banks engage in bargaining, they assess whether trading is worthwhile. Substituting (6) and (7) in (2), we obtain the expected end of day payoffs of the borrower and the lender banks, respectively, *would they trade*. Denoting the expected payoff of bank $k = b, \ell$ when the borrower bank holds x_b and the lender bank holds x_ℓ by $\tilde{P}_k(x_\ell, x_b)$ we have

$$\tilde{P}_k(x_\ell, x_b) = x_k i[x_k] + \int \left(\frac{x_\ell + x_b}{2} + \varepsilon \right) i \left[\frac{x_\ell + x_b}{2} + \varepsilon \right] dH(\varepsilon) - \frac{x_\ell i[x_\ell] + x_b i[x_b]}{2}, \qquad (8)$$

The payoffs are intuitive. The no-trade payoff is augmented by the surplus from trade which banks split equally. However, settlement risk implies that banks will sometimes prefer not to trade. Indeed, a quick inspection of (8) reveals that banks will trade if and only if the surplus from trade is large enough to compensate for the settlement risk, or

$$\int \left(\frac{x_\ell + x_b}{2} + \varepsilon\right) i\left[\frac{x_\ell + x_b}{2} + \varepsilon\right] dH(\varepsilon) \geq \frac{x_\ell i[x_\ell] + x_b i[x_b]}{2}. \tag{9}$$

Recall that $xi[x]$ is the benefit of holding excess reserves x. So condition (9) says that banks trade if and only if the benefit from holding the average reserves (post-trade) is greater than the average benefit of both banks (pre-trade). Given that the benefit function is piecewise concave, (9) would always hold absent settlement risk. However, the next result shows that the risk of settlement failure implies that sometimes banks will not trade.

Proposition 2. *There is a threshold function $\bar{x}_b(x_\ell)$ such that for any pair (x_ℓ, x_b) banks trade $q(x_\ell, x_b) > 0$ if and only if $x_\ell > 0 > \bar{x}_b(x_\ell) \geq x_b$. Otherwise banks do not trade. The threshold $\bar{x}_b(x_\ell)$ is increasing with x_ℓ and converges to zero from below as settlement risk becomes negligible.*

Proof. See Appendix A. □

This result is intuitive. If both banks in the pair have enough reserves to satisfy their reserve requirement, then there is no gain from trade, as banks trade at the central bank deposit rate. But trading exposes them to settlement risk which they prefer not to take. Similarly, when both banks do not have enough reserves to satisfy their reserve requirements, then there is no gain from trade. As they would become exposed to the settlement risk, in this case again banks prefer not to trade. Gains from trade occur when the lender can offset the shortfall of the borrower, at least partially. The borrower saves $i_p - i(x_\ell, x_p)$ on each unit he borrows, while the lender earns $i(x_\ell, x_p) - i_d$ per unit he lends. Therefore, the per unit gain from trade is proportional to the width of the corridor $i_p - i_d$. The gains from trade have to be relatively large to overcome the settlement risk. Hence, $x_\ell - x_b$ has to be large relative to $i_p - i_d$. Finally, when banks do not trade, they are not subject to settlement risk so that $P_k(x_\ell, x_b) = i[x_k]x_k$. To summarize, the payoff of bank $k = b, \ell$ when the borrower holds x_b and meets a lender holding x_ℓ at the start of $t = 1$ is

$$P_k(x_\ell, x_b) = \begin{cases} \tilde{P}_k(x_\ell, x_b) & \text{if } x_\ell > 0 \text{ and } \bar{x}_b(x_\ell) \geq x_b, \\ x_k i[x_k] & \text{otherwise.} \end{cases}$$

None of the results below require settlement risk to be large. In fact, as we show, all our results holds for negligible settlement risk.[15]

[15] Proposition 2 makes it clear that the settlement risk is akin to a fixed trading cost. Assuming a fixed trading cost would however not imply that (6) and (7) are the unique solution to the bargaining problem.

4.2.3 Search and matching

Given the solution to the bargaining problem, we can now analyze the search strategy of banks at the start of $t = 1$. Following their payment shocks, banks decide whether they wish to lend or to borrow and they commit to this choice before accessing the OTC market. We follow the sorting approach in Matsui and Shimizu (2005) and we assume that banks sort themselves in two bins. For simplicity we refer to those bins as "borrowers" and "lenders" respectively. Subsequently, a matching function randomly pairs a bank from one bin with a bank from the other. Banks from the same bin are not matched together.[16]

The matching function works as follows. Let b denote the measure of banks in the borrowers' bin and ℓ denote the measure of banks in the lenders' bin. We assume that the matching function $\psi\mathcal{M}(b,\ell)$ has two components, a matching technology $\mathcal{M}(b,\ell)$ and a matching efficiency parameter $\psi \in [0,1]$. The matching technology displays constant return to scale and satisfies $\frac{\partial\mathcal{M}(b,\ell)}{\partial b} > 0$, $\frac{\partial\mathcal{M}(b,\ell)}{\partial \ell} > 0$, $\psi\mathcal{M}(b,\ell) \leq \min\{b,\ell\}$. The probability that a borrower meets a lender is $\psi_b = \frac{\psi\mathcal{M}(b,\ell)}{b}$ and similarly the probability that a lender meets a borrower is $\psi_\ell = \frac{\psi\mathcal{M}(b,\ell)}{\ell}$.[17] A useful benchmark is $\psi = 0$. Then banks are never matched as if there was no interbank market.

Given the probabilities of matches, we can compute the expected value of being a borrower or a lender, respectively. This in turn lets us characterize the choice of each bank to become a borrower or a lender given the above equilibrium OTC rates. Let G_ℓ (G_b) be the distribution of excess reserve holdings across banks that choose to become lenders (borrowers) following the idiosyncratic payment shock.

We denote by $V_b(\tilde{x})$ ($V_\ell(\tilde{x})$) the expected value of being a borrower (a lender) for a bank holding $\tilde{x}$ units of reserves when the distribution of excess reserves of lender (borrower) banks is G_ℓ (G_b). Then for the borrower

$$V_b(\tilde{x}) = \psi_b \int P_b(x_\ell, \tilde{x})dG_\ell(x_\ell) + (1-\psi_b)\tilde{x}i[\tilde{x}] \tag{10}$$

and similarly for the lender bank. Two remarks are worthwhile making. First observe that $V_k(\tilde{x}) \geq \tilde{x}i[\tilde{x}]$ for $k = b, \ell$, as banks always have the option to reject a trade. Second, notice that the expected value of being a borrower or a lender only depends on the bank's holdings of excess reserves and not on the bank itself.

The expected payoff of a bank with $\tilde{x}$ at the beginning of the trading stage is just determined

[16]While the matching function matches a bank in the borrowers bin with a bank in the lenders bin and vice versa, which lender (borrower) a borrower (lender) bank meets, if any, is random.

[17]As is usual, we sometimes refer to $\theta = \frac{\ell}{b}$ as the market tightness. Then $\psi_\ell(\theta) = \psi\mathcal{M}(\frac{1}{\theta}, 1) \in [0,1]$ is decreasing in θ. In particular, as $\theta \to \infty$ then $\psi_\ell(\theta) \to 0$, while as $\theta \to 0$ then $\psi_\ell(\theta) \to \psi$.

by the value of being a borrower or a lender,

$$V(\tilde{x}) = \max\{V_b(\tilde{x}), V_\ell(\tilde{x})\}, \tag{11}$$

and the bank becomes a lender if $V_\ell(\tilde{x})$ is greater than $V_b(\tilde{x})$ and a borrower otherwise. So (11) implies a sorting strategy for each bank k that we summarize by $s_k \in [0,1]$ where $s_k = 0$ if bank k strictly prefers to become a lender and $s_k = 1$ if bank k strictly prefers to be a borrower. Therefore, the sorting strategy of bank k is described by

$$s_k(x) = \begin{cases} 1 & \text{if } V_b(x) > V_\ell(x), \\ (0,1) & \text{if } V_b(x) = V_\ell(x), \\ 0 & \text{if } V_b(x) < V_\ell(x). \end{cases} \tag{12}$$

4.3 Central bank tender ($t = 0$)

In the morning, banks acquire reserves from the central bank at rate r. The central bank seeks to allocate an amount M of reserves to the banking system. It uses a full allotment tender, so that it satisfies all bids, $y_k = y_k' \ \forall\ k \in \mathcal{K}$. The bids are functions of the interest rate set by the central bank and the central bank understands this relationship. Hence, it sets the rate r such that $\int_{k\in\mathcal{K}} y_k'(r)dk = M = \int_{k\in\mathcal{K}} y_k dk$.

The amount of aggregate excess reserves in the system is $X = \int_{k\in\mathcal{K}}(m_k(0) + y_k - \bar{m}_k)dk = \int_{k\in\mathcal{K}} x_k(0)dk + M$. Therefore, X is an exogenous policy variable that is decided by the central bank and the rate r has to adjust to "clear" the tender. The liquidity position of the banking system is said to be balanced whenever there is no excess reserves in the aggregate, that is $X = 0$.[18]

Given the structure of our economy, we can compute the willingness to pay for reserves of each bank when the tender rate is r. To do so, we define $W_k(x_k)$ as the value for bank k of exiting the tender stage with excess reserves x_k given the sorting strategy of bank k,

$$W_k(x_k) \quad = \quad \int_{-\infty}^{\infty} s_k(x_k + v)V_b(x_k + v) + (1 - s_k(x_k + v))\, V_\ell(x_k + v)dF(v). \tag{13}$$

Our generic bank k, holding $m_k(0)$ reserves and required to hold $\bar{m}_k$ units of reserves, will then bid y_k to solve[19]

$$\max_{b_k} W_k(y_k + x_k(0)) - ry_k.$$

[18] An alternative specification for the central bank tender is that the central bank fix the amount of tendered reserves and banks would bid a rate.

[19] The interest rate ry_k is paid indifferently during the tender stage, or when settlement takes place.

Bank k will bid an amount y_k such that its excess reserves $y_k + x_k(0)$ satisfies

$$W_k'(y_k + x_k(0)) = r. \tag{14}$$

Notice, that while the marginal cost is the same for all banks, the marginal benefit may vary, as the marginal benefit can possibly depend on a bank type and its sorting strategy. However, below we show this is not the case.

4.4 Equilibrium

We are now ready to define an equilibrium.

Definition 1. (Equilibrium) *Given the policy variables X, i_d, and i_p, an equilibrium is*

- *a bid amount y_k for each bank $k \in \mathcal{K}$,*
- *a sorting strategy s_k for each bank $k \in \mathcal{K}$,*
- *distributions of reserves G_b and G_ℓ, and*
- *the interest rate r,*

such that given r, y_k solves (14), s_k solves (12), the distribution of reserves are consistent with the bidding and sorting strategies, and $\int_k (y_k + x_k(0))\, dk = X$.

The remainder of the paper shows that a unique equilibrium exists and characterizes its properties. We first show that banks adopt a simple and intuitive sorting strategy.

Lemma 1. (Sorting strategies) *All banks holding excess reserves $x \geq 0$ choose to become lenders. Otherwise they choose to become borrowers.*

Proof. Suppose bank k becomes a lender whenever $x < 0$. Then the only possible trades where bank k would indeed lend to a bank is one where the potential borrower would have excess reserves $x_b < x$. However, we showed that in this case banks do not trade, as there are no gains from trade. Hence, $xi_\ell = V_\ell(x) \leq V_b(x)$ whenever $x < 0$. Similarly, suppose bank k becomes a borrower whenever $x > 0$. Again the only possible trades are ones where bank k meets a lender bank with $x_\ell > x$. But banks do not trade in this case, so that $xi_d = V_b(x) \leq V_\ell(x)$. Therefore it is a (weakly) dominating strategy for banks to become a lender whenever $x > 0$ and borrower otherwise. Furthermore, this strategy is strictly preferred when banks expect other banks to choose the same strategy and mixed strategy eliminates the indifference. □

Looking back at (13), the fact that all banks adopt the same sorting strategies implies that all banks have the same continuation payoff function, that is $W_k(x) = W(x)$ for all k. Then we obtain the following Corollary to Proposition 1.

Corollary 1. *All banks exit the tender stage with the same amount of excess reserves, X.*

Proof. Since $W_k(x) = W(x)$ for all $k \in \mathcal{K}$, (14) implies that all banks exit the tender stage with the same amount of *excess* reserves, $y_k(r) + x_k(0) = y_j(r) + x_j(0)$ for all $k, j \in \mathcal{K}$, and the tender clearing condition implies $y_k(r) + x_k(0) = X$ for all $k \in \mathcal{K}$. □

While all banks will hold X excess reserves following the tender, notice that banks can bid very different amounts. For example, if all banks start with zero reserves, $m_k(0) = 0\ \forall\ k$, then banks with higher reserve requirements will bid for larger amounts of reserves than banks with smaller requirements. In sum, banks place bids in order to exit the central bank's tender with the same amount of excess reserves, and the tender rate r adjusts so that this amount is X.

Then we can easily solve for the measure of borrowers and lenders as well as for the distributions of reserves among each group. Since all banks exit the tender with excess reserves X, all banks that receive a payment shock smaller than $-X$ will hold negative excess reserves and by Lemma 1 all these banks will choose to become borrowers. Hence there will be a measure $F(-X)$ of borrowers. And by symmetry there is a measure $1 - F(-X) = F(X)$ of lenders. We have the following Corollary (without proof).

Corollary 2. *The measure of borrowers and lenders in equilibrium are $b^* = F(-X)$ and $\ell^* = 1 - F(-X) = F(X)$. The distribution of reserves among borrowers $G_b(x)$ and among lenders $G_\ell(x)$ are:*

$$
\begin{aligned}
G_b(x) &= \frac{F(x-X)}{F(-X)}, \qquad \textit{for all } x \leq 0 \\
G_\ell(x) &= \frac{F(x-X) - F(-X)}{F(X)}, \qquad \textit{for all } x > 0.
\end{aligned}
$$

Given the bidding behavior, the sorting strategies, the distributions of excess reserves among borrowers and lenders, and the policy variables X, i_d and i_p, the payoff functions $W(x)$, $V_b(x)$ and $V_\ell(x)$ are all well defined, so that a unique equilibrium exists.

Proposition 3. (Existence) *A unique equilibrium exists where banks exit the tender stage with the same amount of excess reserves X, and following the liquidity shock, banks with negative (positive) excess reserves choose to become borrowers (lenders).*

Proof. This is a direct consequence of Proposition 1, Proposition 2 and Lemma 1. □

5 Characterization of equilibrium

In this section, we characterize the equilibrium of the model as it relates to a number of key aspects of monetary policy implementation and functioning of the overnight interbank market. By making specific assumptions about the payment shock and the matching function, we can derive exact solutions when settlement risk is negligible. We start by looking at banks' willingness to pay for excess reserves at the central bank tender operation before turning to the model equivalent of the four stylized facts. That is the aggregate trading volume, the effective overnight rate, the interbank rate dispersion, and the use of the central bank lending facility.

5.1 Bidding behavior in central bank tender operation

To understand the bidding behavior at the central bank tender, it is useful to compute the marginal values of reserves. The marginal value of holding an additional unit of reserves for a borrower is the rate that he will not have to pay at the lending facility i_p, minus the expected benefit of acquiring this unit in the OTC market. The expected benefits naturally depends on market tightness, as well as on the probability to meet a lender who has enough reserves so that both banks can satisfy their reserve requirements. This is similar for lenders. Precisely, the marginal values of reserves for a borrower and a lender are, respectively,

$$\frac{\partial V_b(\tilde{x})}{\partial \tilde{x}} = i_p - \psi_b^* \frac{(i_p - i_d)}{2} \int_{\bar{x}_\ell(\tilde{x})}^{\infty} \left[1 - H\left(-\frac{x_\ell + \tilde{x}}{2}\right)\right] dG_\ell(x_\ell) \leq i_p \quad \text{for any } \tilde{x} < 0, \tag{15}$$

and

$$\frac{\partial V_\ell(\tilde{x})}{\partial \tilde{x}} = i_d + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{\bar{x}_b(\tilde{x})} H\left(-\frac{\tilde{x} + x_b}{2}\right) dG_b(x_b) \geq i_\ell \quad \text{for any } \tilde{x} \geq 0, \tag{16}$$

where ψ_b^* and ψ_ℓ^* are the equilibrium meeting probability for borrowers and lenders respectively.

Relative to a model without an OTC market (i.e. setting the matching efficiency to $\psi = 0$) and where banks only have access to the central bank's end-of-day facilities, the OTC market is making reserves less valuable for borrowers and more valuable for lenders. The reason is that the OTC market allows banks to smooth their reserves holdings, and this opportunity is just not present when there is no interbank market. Notice that when settlement risk becomes negligible, the marginal values for reserves converge to

$$\begin{aligned}\frac{\partial V_b(\tilde{x})}{\partial \tilde{x}} &\xrightarrow{\varepsilon} i_p - \psi_b^* \frac{(i_p - i_d)}{2} \frac{1 - F(-X - \tilde{x})}{F(X)} \text{ for any } \tilde{x} < 0, \\ \frac{\partial V_\ell(\tilde{x})}{\partial \tilde{x}} &\xrightarrow{\varepsilon} i_d + \psi_\ell^* \frac{(i_p - i_d)}{2} \frac{F(-X - \tilde{x})}{F(-X)} \text{ for any } \tilde{x} \geq 0,\end{aligned}$$

where we use $\xrightarrow{\varepsilon}$ to denote "convergence when settlement risk becomes negligible."

From (15)-(16) observe that banks' demand at the tender will become more elastic as the OTC market becomes better at allocating reserves from lenders to borrowers. But to determine banks' bidding behavior we need to define the marginal value of reserves at the end of the tender stage. Using Leibniz's rule on (13), as well as (15) and (16) we find that

$$\begin{aligned}W'(X) &= i_d\left[1 - F(-X)\right] + i_p F(-X) \\ &\quad + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-a}^{\infty} \int_{-\infty}^{\bar{x}_b(X+v)} H\left(-\frac{x_b + X + v}{2}\right) dG_b(x_b) dF(v) \\ &\quad - \psi_b^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{-a} \int_{\bar{x}_\ell(X+v)}^{\infty} \left[1 - H\left(-\frac{x_\ell + X + v}{2}\right)\right] dG_\ell(x_\ell) dF(v),\end{aligned}$$

where $\bar{x}_b(x_\ell)$ is defined in proposition 2 and $\bar{x}_\ell$ is defined as follows: a borrower holding x_b trades with a lender holding x_ℓ if and only if $x_\ell \geq \bar{x}_\ell(x_b)$. Using $\mathcal{H}(X)$ as a short-hand for the last two terms, we obtain

Proposition 4. (Bank bidding behavior) *The marginal values of excess reserves at the central bank tender operation is*

$$W'(X) = F(-X) i_p + F(X) i_d - \frac{i_p - i_d}{2} \mathcal{H}(X), \tag{17}$$

where $\mathcal{H}(X) \xrightarrow{\varepsilon} \psi_b^* \frac{F(-X)}{F(X)} - \frac{\psi_b^*}{F(X)} \int_{-\infty}^{-X} F(-2X - v) dF(v) - \frac{\psi_\ell^*}{F(-X)} \int_{-X}^{\infty} F(-2X - v) dF(v)$. *We have*

$$\begin{aligned}W'(0) &\xrightarrow{\varepsilon} \frac{1}{2}(i_p + i_d) - 2\left(\psi_\ell^* - \psi_b^*\right) \int_0^\infty F(-v) dF(v), \\ W'(X) &\xrightarrow{\varepsilon} i_d \textit{ as } X \to \infty, \\ W'(X) &\xrightarrow{\varepsilon} i_p \textit{ as } X \to -\infty.\end{aligned}$$

Proof. See Appendix B for $W'(X)$. □

In the model without interbank market, the value of an additional unit of reserves after the payment shock is the probability of being short reserves, $F(-X)$, times the saved cost of not having to borrow from the central bank, i_p, plus the probability of being long reserves times the interest rate earned on deposit at the central bank, i_d.

The additional terms capture the marginal value of being able to adjust reserve holdings in the OTC market. Basically, banks are willing to pay less for reserves at the tender than the rate with "no market" when there is a surplus of reserves, $X > 0$, as there is a chance to cover one's shortfall in the OTC market. Similarly, banks are willing to pay more than the "no market" rate when there is a deficit of reserves, as there is a chance they can lend out their excess reserves in the OTC market. The amount banks are willing to pay relative to the "no market" rate depends on how efficient the OTC market is in matching banks with offsetting reserves positions and the distribution of the payment and settlement shocks.

Example. (Gaussian payment shock) Assume that the payment shock is a mean zero Gaussian, $v_k \sim N(0, \sigma_v^2)$, and let $i_d = 2\%$, and $i_p = 4\%$. Moreover, consider two matching technologies

$$\mathcal{M}_{\min}(b, \ell) = \min(b, \ell),$$

and

$$\mathcal{M}_{CD}(b, \ell) = \frac{\min(b, \ell)}{\max(b, \ell)} b^{\alpha} \ell^{1-\alpha}.$$

We use "min" to denote the first matching technology and "Cobb Douglas" or "CD" to denote the second. Figure 4 shows the willingness to pay, $W'(X)$, for the different matching efficiencies, matching functions and volatilities of the payment shock. The blue and the teal lines show $W'(X)$ for the model without an interbank market ($\psi = 0$) for $\sigma_v = 2$ or 3, respectively. Higher volatility decreases the sensitivity of the willingness to pay with respect to the level of excess reserves. The green and red lines are versions of the model with full matching efficiency ($\psi = 1$) and $\sigma_v = 2$ but different matching technologies. In case of the green line the matching function is the min whereas in case of the red line the matching function is the Cobb-Douglas for $\alpha = 0.9$. Notice that the OTC market offers insurance against the payment shock. Hence, banks shade their bids when the central bank offers reserves in excess of the reserve requirement and bid more aggressively when there is a deficit of reserves. The effects are more pronounced as ψ increases, as then the OTC market offers a higher degree of insurance.

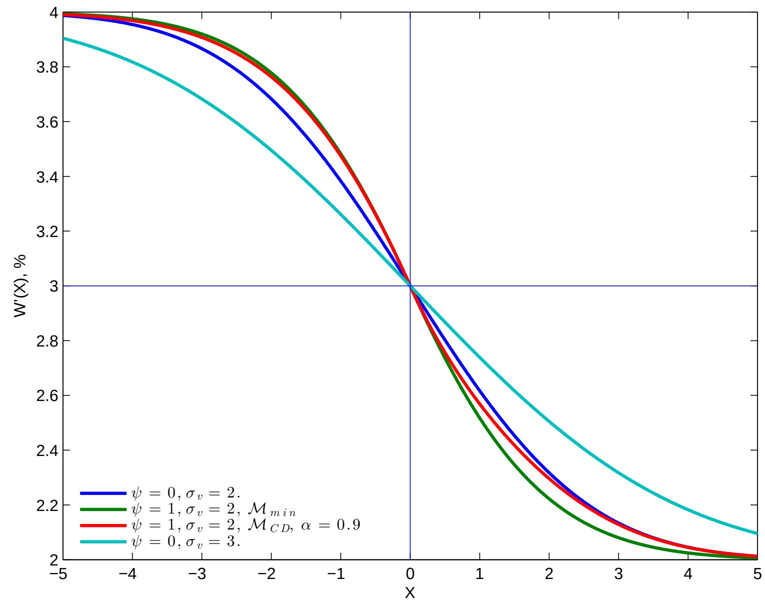


Figure 4: Willingness to pay for reserves, $W'(X)$.

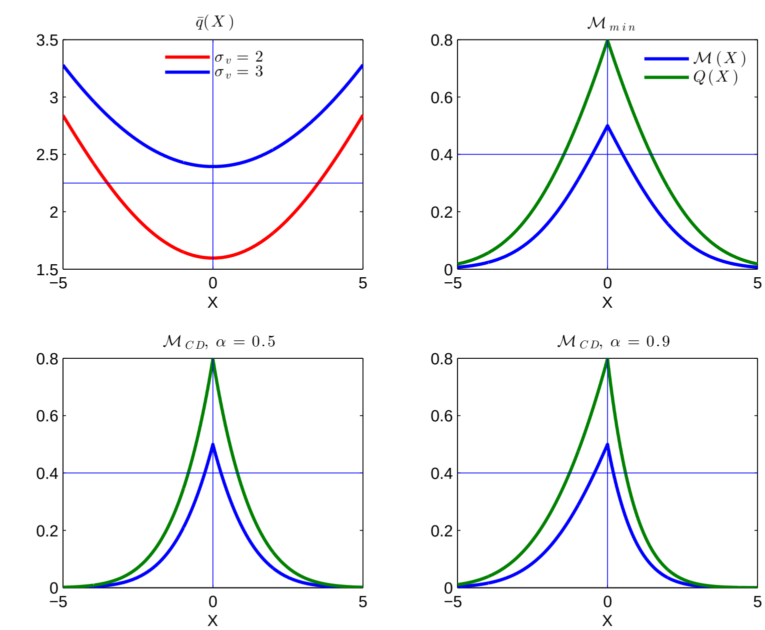


Figure 5: Matches, trading volume, and average traded size.

5.2 Model equivalents of stylized facts

In this section, we compute a number of descriptive statistics for the overnight interbank market. We focus on those that tend to be publicly available and are closely watched by both central banks and market participants. The statistics are the traded volume, the volume-weighted average or "effective" rate, the intraday rate dispersion, and the use of the lending facility. We leave most of details of the derivations to the Appendix.

5.2.1 Trading volume

We compute the aggregate trading volume (Q) as the measure of matches (the extensive margin) times the average trade size $\bar{q}$ *when a trade occurs* (the intensive margin). We have

$$Q(X) = \psi \mathcal{M}(X)\bar{q}(X), \tag{18}$$

where $\mathcal{M}(X) = \mathcal{M}(F(-X), F(X))$ and

$$\bar{q}(X) \quad = \quad \int_0^\infty \int_{-\infty}^{\bar{x}_b(x_\ell)} q(x_\ell, x_b) dG_b(x_b) dG_\ell(x_\ell).$$

The following proposition characterizes the average trade size and the average trading volume in equilibrium when the settlement shock becomes negligible.

Proposition 5. (Trading volume) *The (limit) average trade size is*

$$\bar{q}(X) \xrightarrow{\varepsilon} -\frac{1}{2}\frac{E\left[v\,|v < -X\right]}{1-b^*}, \tag{19}$$

which is independent of the matching function. The (limit) aggregate trading volume is

$$Q(X) \quad \xrightarrow{\varepsilon} \quad -\psi_\ell^* \frac{1}{2} E\left[v\,|v < -X\right].$$

With no settlement risk, $Q(X)$ is single-peaked and attains its maximum at $\bar{X} \leq 0$. Moreover, $\lim_{X\to\infty} Q(X) = \lim_{X\to-\infty} Q(X) = 0$, i.e., the aggregate volume decreases towards zero as the amount of excess reserves is large (negative or positive).

Proof. See Appendix C for the derivation and the properties of $\bar{q}(X)$ and $Q(X)$. □

Example. (Gaussian payment shock - continued) Figure 5 shows the average trade size and aggregate trading volumes as functions of X for different configurations of the model with $\psi = 1$.

The upper left-hand panel shows $\bar{q}(X)$ for $\sigma_v = 2$ or 3. Trade sizes are the smallest on average when $X = 0$ and are increasing in dispersion of the payment shock.

The remaining three panels look at the measure of matches and aggregate trading volume for different matching functions. For the min and Cobb-Douglas matching technologies, the measure of matches peaks when $X = 0$ and as the extensive margin effect dominates the smaller average trade size, aggregate trading volume also peaks. Moreover, $\mathcal{M}_{\min}(0) = \mathcal{M}_{CD}(0)$ which implies that $Q_{\min}(0) = Q_{CD}(0)$. However for any $X \neq 0$, $\mathcal{M}_{CD}(X) < \mathcal{M}_{\min}(X)$ so that $Q_{CD}(X) < Q_{\min}(X)$. Looking at the Cobb-Douglas matching function, as α increases from 0.5 to 0.9, borrowers contribute relatively more to matches, which creates an asymmetry in trading volume as a function of X, as shown in the bottom right hand panel of Figure 5. Finally, we note that the inverse relationship between trading volume and excess reserves in Figure 5 is consistent with the second stylized fact regarding volumes shown in Figure 2b.

5.2.2 Volume-weighted average interest rate

With the interbank market volumes in hand, we can compute the "effective" or volume-weighted average rate in the OTC market. It is given by

$$\bar{i}_q(X) = \int_0^\infty \int_{-\infty}^{\bar{x}_b(x_\ell)} \psi \mathcal{M}(X) i(x_\ell, x_b) \frac{q(x_\ell, x_b)}{Q(X)} dG_b(x_b) dG_\ell(x_\ell). \tag{20}$$

The following proposition describes the effective rate.

Proposition 6. (Effective rate) *The volume-weighted average interest rate for trades in the interbank market is a weighted average of the policy rates i_d and i_p. The (limit) effective rate is given by*

$$\bar{i}_q(X) \xrightarrow{\varepsilon} i_d F(-X) + i_p F(X) - \frac{X}{\bar{q}(X)} \frac{(i_p - i_d)}{2} \tag{21}$$

Proof. See Appendix D. □

The first two terms of $\bar{i}_q(X)$ is the Poole (1968) interbank rate, which in our model is the willingness to pay for reserves at the central bank tender when there is no interbank market. The third term is the per bank gain from trade times the ratio of excess reserves to the average trade in the OTC market.

Notice that the effective rate does not depend on the matching technology or the matching efficiency. The reason is simple: The matching function does not depend on the distribution of reserves across borrowers and lenders, but rather just on the measure of borrowers and lenders. Therefore a "better" matching function will match more banks, but not in a better way. We return to this issue in Section 5.3. Moreover, we have the following corollary (without proof) on the location of the effective rate within the corridor.

Corollary 3. *When the surplus of reserves is large then the volume-weighted average rate tends to the deposit facility rate and when the deficit of reserves is large then the volume-weighted average rate tends to the lending facility rate. When excess reserves are zero the volume-weighted average rate is at the mid-point of the corridor.*

Example. (Gaussian payment shock - continued) The volume-weighted interest rate is plotted in Figure 6 for $\sigma_v = 2$ or 3 along with the equivalent of the rate in Poole (1968). This Figure shows a pattern for the interbank market rate that is consistent with the first stylized fact shown in Figure 2a. The effective rate in our model is much less sensitive than the Poole rate to changes in the amount of excess reserves.[20] So, as the interbank market becomes more efficient, the central bank has to inject more reserves to affect the effective rate by the same magnitude.

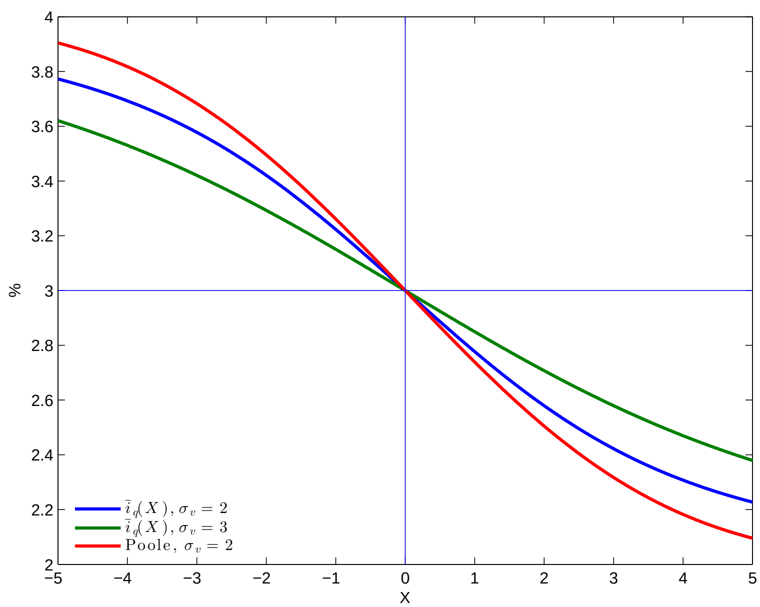


Figure 6: The effective interbank rate.

[20]In the real world, the supply of reserves cannot be perfectly controlled by the central bank due to so-called autonomous factors, e.g. government outlays and tax receipts. Hence, this sensitivity (or elasticity) matters as lower sensitivity makes it is easier for the central bank to hit its target for the overnight interbank rate.

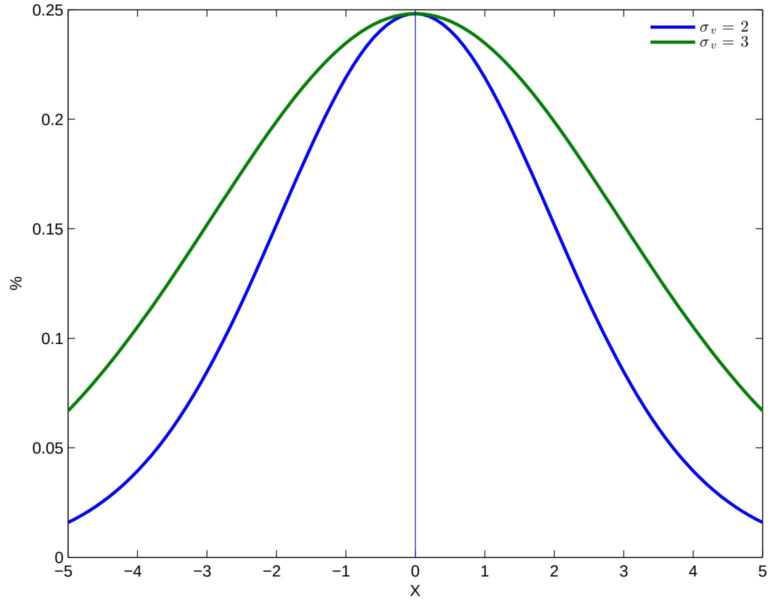


Figure 7: Interbank rate dispersion

5.2.3 Interbank rate dispersion

We can compute the interbank rate dispersion, using the volume-weighted variance,

$$\sigma_i^2(X) = \int_{-\infty}^{0} \int_{\bar{x}_\ell(x_b)}^{\infty} \frac{q(x_\ell, x_b)}{\bar{q}(X)} [i_q(x_\ell, x_b) - \bar{i}_q(X)]^2 dG_\ell(x_\ell) dG_b(x_b), \tag{22}$$

Example. (Gaussian payment shock- continued) Figure 7 plots the standard deviation of effective rate for $\sigma_v = 2$ or 3. Intuitively, larger payment shocks leads to higher interbank rate dispersion, i.e., across pairs of banks that are matched. Moreover dispersion falls as more reserves are added to, or drained from, the banking system. Notably, Figure 7 is consistent with the third stylized fact that dispersion falls as excess reserves increases as shown in Figure 2c.

Notice that (22), like (21), is independent of the matching function. To understand why rate dispersion is falling with $|X|$, let us take the case where $X > 0$ is large. When there is a large surplus of excess reserves, the negative position of most borrowers tends to be small, while the positive position of most lenders tends to be large. So most matches will involve a borrower with a small deficit, and a lender with a large surplus. Hence, the rate dispersion is low. A similar intuition holds for $X < 0$ and large.

5.2.4 Borrowing from the central bank lending facility

In this section we describe the banks' use of the lending facility $L(X)$ as a function of the aggregate amount of excess reserves in the economy. Recall that any bank that cannot satisfy its reserve requirement at the end of the day has to borrow from the central bank.

When $\psi = 0$, there is no interbank trades and banks borrow from the central bank if the payment shock exhausts their reserve holdings, i.e, $x_k < 0 \Rightarrow v_k < -X$. When $\psi > 0$, there is an active interbank market, so that borrowing from the central bank is less prevalent, as to some degree, banks redistribute reserves among each other. Banks that do need to borrow from the central bank are one of two types. The first type are banks that wished to borrow in the interbank market but that either were not matched or matched with a lender that did not generate positive gains from trade, i.e. borrowers meeting lenders holding $x_\ell < \bar{x}_\ell(x_b)$. The second type is banks in a trading pair who will not satisfy their reserve requirement in spite of trading. Combining the borrowing of these two types of banks yields the following proposition

Proposition 7. *The (limit) use of the central bank lending facility is*

$$\begin{aligned} L(X) \;\; &\overset{\varepsilon}{\to} \;\; -b^*(1-\psi_b^*)(X + E[v \mid v < -X]) - \\ & b^*\psi_b^* \int_{-\infty}^{-a} \int_{-X}^{-2X-v_b} (2X + v_\ell + v_b)\, \frac{dF(v_\ell)}{F(X)} \frac{dF(v_b)}{F(-X)}, \end{aligned} \tag{23}$$

where $b^* = F(-X)$ and $\psi_b^* = \psi \mathcal{M}(X)/b^*$.

Proof. See Appendix E. ∎

Corollary 4. *If there is no interbank market ($\psi = 0$), then the (limit) use of the central bank lending facility is $L_{\psi=0}(X) \overset{\varepsilon}{\to} -F(-X)(X + E[v \mid v < -X])$. and $L_{\psi=0}(X) \geq L(X)$* for all X.

Example. (Gaussian payment shock - continued) Figure 8 shows borrowing from the central bank lending facility as a function of the excess reserves supplied by the central bank for different configurations of the model. Intuitively, if reserves are in short supply then borrowing is equal to the deficit and if reserves are plentiful then there is little to no borrowing from the central bank at the end-of-day. Moreover, borrowing is highest when there is no interbank market and lowest when banks are matched according to the minimum matching function. Figure 8 suggests that with an interbank market the use of the central bank lending facility should fall as excess reserves increase but also that the relationship is relatively "flat". It is probably in the eye of the beholder whether this is (broadly) consistent with our last stylized facts regarding the use of the lending facility shown in Figure 2d. But we should mention that for the data range at hand, banks could have been unwilling to trade with one another because of counterparty risk, thus increasing the use of the marginal lending facility in spite of the large amount of excess reserves.

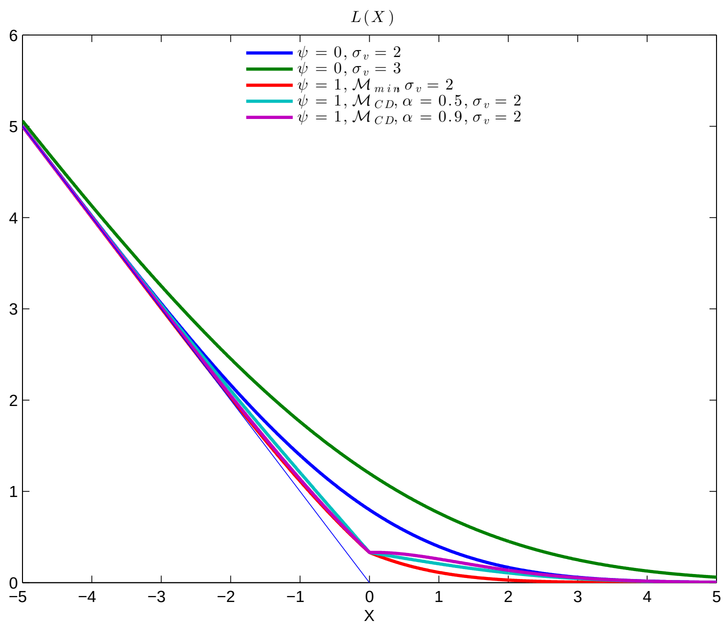


Figure 8: End-of-day borrowing from the central bank.

5.3 Market structure

In our model, the market structure is captured by the two elements of the matching function, the matching technology $\mathcal{M}$ and the matching efficiency ψ. How does the market structure affect the efficacy of the interbank market? By interbank market efficacy, we understand the extent to which banks are able to reallocate funds between each other.

A straightforward way to quantify the interbank market efficacy given a matching function $\psi\mathcal{M}$ is to look at the use of the lending facility $L_{\psi\mathcal{M}}(X)$ when $X \geq 0$, or of the deposit facility (net of required reserves) $D_{\psi\mathcal{M}}(X) = X + L_{\psi\mathcal{M}}(X)$ when $X < 0$.[21] To see this, suppose the interbank market is either frictionless, or banks are matched perfectly – i.e. borrowers holding $X + v_b$ are matched with lenders holdings $X - v_b$. With $X \geq 0$ banks would only borrow from the lending facility because of the late settlement shock. If it vanishes, no banks would use it. Symmetrically, when $X < 0$ and with no late settlement shock, banks borrow $-X$ at the lending facility and no banks use the deposit facility.

[21] The amount deposited is $\overline{M} + X + L_{\psi\mathcal{M}}(X)$ where $\bar{M} = \int_k \bar{m}_k dk$ is aggregate reserves requirement.

We can measure efficacy in either absolute or relative terms, using as a benchmark the recourse to the lending facility in the case with no interbank market, $L_{\psi=0}(X)$. Given X, the absolute measure of efficacy is

$$\mathcal{E}_A(X;\psi\mathcal{M}) = L_{\psi=0}(X) - L_{\psi\mathcal{M}}(X), \text{ for all } X.$$

This measure gives the reduction in the recourse to the lending facility due to the interbank market. It gives a sense as to the currency amount that banks can save thanks to introducing a more effective interbank market, and therefore to the willingness of banks to pay for a better market. Figure 9a shows $\mathcal{E}_A$ for our two matching functions. As the figure makes clear, $\mathcal{E}_A$ is largest when $X = 0$ and tends to zero as $|X|$ becomes large. Therefore, for monetary policy stances that imply large $|X|$, we should not expect banks to be willing to invest in better interbank market structures.

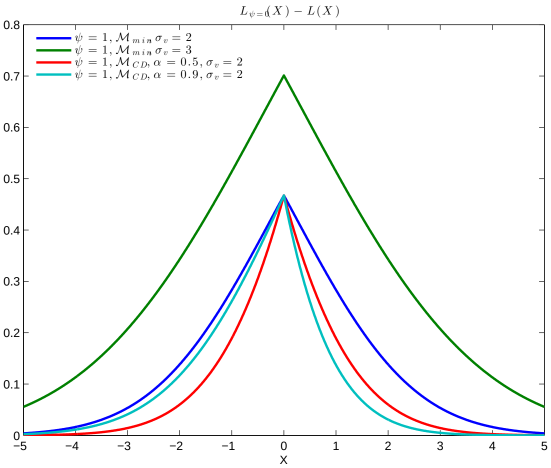
However, this last property is driven by the fact that $L_{\psi=0}(X)$ also goes to zero as $|X|$ becomes large. So the absolute measure is not very useful to get a sense for the efficacy of an existing interbank market. For this purpose a relative measure of efficacy does better. Figure 9b shows the relative measure of efficacy,

$$\mathcal{E}_R(X;\psi\mathcal{M}) = \begin{cases} 1 - \frac{L_{\psi\mathcal{M}}(X)}{L_{\psi=0}(X)} & \text{if } X \geq 0, \\ 1 - \frac{X+L_{\psi\mathcal{M}}(X)}{X+L_{\psi=0}(X)} & \text{if } X < 0. \end{cases}$$

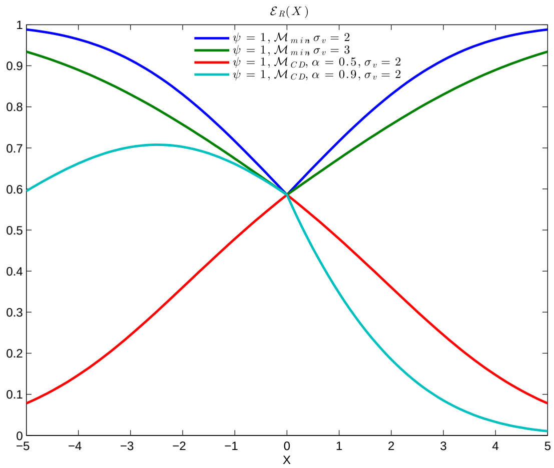
This measure gives the lending reduction as a proportion of the maximum possible reduction. Naturally $\mathcal{E}_R(X;\psi\mathcal{M}) \in [0,1]$ for all X, it is equal to 0 when the market does not re-allocate reserves, and 1 when the interbank market re-allocates reserves perfectly.

To understand Figure 9b, it is useful to examine how the Cobb-Douglas matching technology works. When $X > 0$, we know $b < \ell$ so that $\psi\mathcal{M}(b,\ell) = b^{1+\alpha}\ell^{-\alpha}$. Then $1+\alpha$ is the share of borrowers' contribution to the number of matches, that is $1+\alpha = b\frac{\partial\mathcal{M}(b,\ell)}{\partial b}/\mathcal{M}(b,\ell)$ when $X > 0$. Therefore as α becomes close to 1, each additional borrower "creates" a lot of matches. However, when $X > 0$ there are few borrowers, and as lenders do not contribute much to matches, we should expect the interbank market to be less effective as X increases. Figure 9b confirms this as the interbank market is much worse when $X > 0$ with $\alpha = 0.9$ than with $\alpha = 0.5$. The argument is similar when $X < 0$. Finally, the figure also makes clear that maximum relative efficacy of an existing interbank market is not necessarily reached for neutral liquidity provision.[22]

[22]Let us note that α could be related to the un-modeled search intensity of banks. If banks are reluctant to use the lending facility, maybe due to a stigma effect, then they will tend to search more intensively. Investigating this aspect seriously is worthwhile but beyond the scope of this paper.



(a) $\mathcal{E}_A(X; \psi\mathcal{M})$.



(b) $\mathcal{E}_R(X; \psi\mathcal{M})$, $\bar{M} = 0$

Figure 9: Efficacy measures

We should stress that it is difficult to disentangle the efficacy of the interbank market from the central bank's provision of excess reserves. For example, in a floor system where the central bank flood the market with reserves and control the interest rate by moving its deposit facility rate, the interbank market could be very ineffective in re-allocating reserves across banks as Figure 9b shows. Also, a floor system would not encourage banks to either trade among each other or to invest in a better infrastructure (such as introducing a central counterparty for unsecured interbank trades), as Figure 9a shows that the absolute gains would be small.

Usually, central banks prefer that banks reallocate reserves among each other and they show reluctance in being the main "point of access" for reserves. If they set too high excess reserves then our model predicts that unsecured interbank trades will disappear. As our model shows, it disappears not because the market is "broken" but merely because there are no gains from trade. This discussion implies that the unsecured segment of the money market remains a candidate to be the prime market in which central bank implements monetary policy going forward.

Finally, Proposition 4 shows that a more efficient market structure increases the elasticity of the demand for reserves. Therefore as the market structure becomes more efficient, say through the introduction of brokers/dealers, the central bank will typically have to increase the main refinancing operation rate r by less when it wants to increase the interbank rate. This effect may however be tampered by new regulatory measures like the introduction of a liquidity coverage ratio, see Bech and Keister (2013).

6 Conclusion

We presented a search-based model of the interbank market and monetary policy implementation. Following a central bank tender and a payment shock, banks sort themselves into two types: borrowers and lenders. This "borrower or lender be" decision allows us to replicate a number of stylized facts about liquidity, trading volumes and rate dispersion in the interbank market that cannot be explained by models in the tradition of Poole (1968). With an increasing supply of reserves most banks desire to be a lender but with relatively few borrowers around, market liquidity deteriorates and trading volume falls. Moreover, rate dispersion is curtailed as most lenders find themselves in a "buyers market" with little room for bargaining. In terms of pricing, our framework suggests a relationship between (excess) reserves and the overnight rate that is qualitatively similar to Poole (1968) but we have a smaller liquidity effect. We find that improvements in market efficiency are beneficial for banks in the decentralized market but affect the central bank in terms of implementing monetary policy and in terms of reducing the use of the marginal lending facility lending.

Of course, we make a number of simplifying assumptions for tractability and the framework can be improved in some areas. While one can interpret the sorting approach as a short-hand version of banks seeking the service of a broker (or having information about the likely reserve holdings of certain counter-parties), it would be necessary to model brokers/dealers explicitly to understand how this would change the model predictions. Also, rising counter-party risk was a key feature of the interbank market during the financial crisis and trading ceased in certain segments. In future research, we are working on introducing bank default. Another worthwhile extension is to introduce a secured market into the framework, given some central banks are considering to implement monetary policy through the secured money market instead.

References

[1] Afonso, Gara and Ricardo Lagos (2014). "Trade Dynamics in the Market for Federal Funds," *Econometrica,* forthcoming.

[2] Afonso, Gara and Ricardo Lagos (2015). "The Over-the-Counter Theory of the Fed Funds Market: A Primer," *Journal of Money, Credit, and Banking,* forthcoming.

[3] Afonso, Gara, Anna Kovner and Antoinette Schoar (2011). "Stressed, Not Frozen: The Federal Funds Market in the Financial Crisis," *Journal of Finance* 66(4), 1109-1139.

[4] Bech, Morten and Todd Keister (2013). "Liquidity regulation and the implementation of monetary policy," BIS Working Paper 432.

[5] Bech, Morten and Elizabeth Klee (2011). "The mechanics of a graceful exit: interest on reserves and segmentation in the federal funds market", *Journal of Monetary Economics*, 58(5).

[16] Bech, Morten and Bart Hobijn (2007). "Technology Diffusion within Central Banking: The Case of Real-Time Gross Settlement", *International Journal of Central Banking*, September.

[6] Bech, Morten and Cyril Monnet (2013). "The Impact of Unconventional Monetary Policy on the Overnight Interbank Market," in "Liquidity and Funding Markets" - Reserve Bank of Australia conference volume. Eds. Alexandra Heath, Matthew Lilley and Mark Manning.

[7] Bindseil, Ulrich (2004). "Monetary Policy Implementation: Theory, Past, and Present," Oxford University Press, Oxford.

[8] ECB (2002). "The liquidity management of the ECB", *ECB Monthly Bulletin*, May, 41-53.

[9] ECB (2014). "Recent developments in excess liquidity and money market rates", *ECB Monthly Bulletin*, January, 69-82.

[10] Ennis, Huberto and Todd Keister (2008). "Understanding Monetary Policy Implementation," Federal Reserve Bank of Richmond Economic Quarterly, 94(3), 235–263.

[11] Matsui, Akihiko, and Takashi Shimizu (2005) "A Theory of money and marketplaces," *International Economic Review* 46(1), 35-59.

[12] Owen, D. B. (1980). "A table of normal integrals," *Communications in Statistics - Simulation and Computation* 9(4), 389-419.

[13] Poole, William (1968), "Commercial Bank Reserve Management in a Stochastic Model: Implications for Monetary Policy," *Journal of Finance* 23(5), 769–791.

[14] Ng, Edward W. and Murray Geller (1968). "A Table of Integrals of the Error Functions," *Journal of Research of the National Bureau of Standards - B. Mathematical Sciences*, 738(1).

[15] Tapking, Jens (2006). "Multiple equilibrium overnight rates in a dynamic interbank market game," *Games and Economic Behavior* 56(2), 350-370.

[16] Walsh, Carl E., (2010). "Monetary Theory and Policy," third edition, The MIT Press, Cambridge, MA.

[17] Whitesell, William (2006). "Interest Rate Corridors and Reserves," *Journal of Monetary Economics* 53(6), 1177–1195.

[18] Woodford, Michael (2001). "Monetary Policy in the Information Economy", Paper presented at the Federal Reserve Bank of Kansas City Economic Symposium 'Economic Policy for the Information Economy', Jackson Hole, 30 August–1 September.

Appendix

A Proof of Proposition 2

In the sequel, we use $\bar{x} = \frac{x_\ell + x_b}{2}$ as the average excess reserves of two banks considering to trade. These two banks trade iff their surplus is positive, that is

$$\begin{aligned}\int (\bar{x}+\varepsilon)\, i\left[\bar{x}+\varepsilon\right] dH(\varepsilon) &\geq \frac{x_\ell i[x_\ell] + x_b i[x_b]}{2} \\ i_d \bar{x} + (i_p - i_d)\int_{-\infty}^{-\bar{x}} (\bar{x}+\varepsilon) dH(\varepsilon) &\geq \frac{x_\ell i[x_\ell] + x_b i[x_b]}{2}\end{aligned}$$

Suppose that $x_\ell > x_b > 0$ so that $\bar{x} > 0$. Then the two banks trade iff

$$i_d \bar{x} + (i_p - i_d)\int_{-\infty}^{-\bar{x}} (\bar{x}+\varepsilon) dH(\varepsilon) \geq i_d \bar{x}$$

which cannot be the case, as $i_p > i_d$ and the integral is negative. Then suppose $x_b < x_\ell < 0$. Then the two banks trade iff

$$i_d \bar{x} + (i_p - i_d)\int_{-\infty}^{-\bar{x}} (\bar{x}+\varepsilon) dH(\varepsilon) \geq \bar{x} i_p$$

However, arranging terms, this implies

$$\int_{-\infty}^{-\bar{x}} (\bar{x}+\varepsilon) dH(\varepsilon) - \bar{x} \geq 0$$

This inequality is not satisfied at $\bar{x} = 0$, as then the left-hand-side (LHS) is negative. Also as $\bar{x} \to -\infty$ the inequality just holds with equality. Finally, the derivative of the LHS with respect to any $\bar{x}$ is $LHS'(\bar{x}) = H(\bar{x}) - 1 < 0$. Therefore from $\bar{x} \to -\infty$, the LHS decreases from 0. Hence the inequality can never hold and banks with $x_b < 0$ and $x_\ell < 0$ will not trade.

Therefore, the only pairs of banks that will trade will have $x_\ell > 0 > x_b$ so that the condition for trade is

$$\begin{aligned}i_d \frac{x_\ell + x_b}{2} + (i_p - i_d)\int_{-\infty}^{-\bar{x}} (\bar{x}+\varepsilon) dH(\varepsilon) &\geq \frac{x_\ell i_d + x_b i_p}{2} \\ \int_{-\infty}^{-\frac{x_\ell + x_b}{2}} \left(\frac{x_\ell + x_b}{2} + \varepsilon\right) dH(\varepsilon) - \frac{x_b}{2} &\geq 0 \qquad (24)\end{aligned}$$

Let $\tau_\ell(x)$ be the set of borrowers' reserves that permit trades when the lender is holding x excess reserves, i.e.

$$\tau_\ell(x) = \left\{ x_b : \int_{-\infty}^{-\frac{x+x_b}{2}} \left(\frac{x + x_b}{2} + \varepsilon\right) dH(\varepsilon) \geq \frac{x_b}{2} \right\}.$$

Given $x_\ell > 0$ define $\bar{x}_b(x_\ell)$ as the value of x_b that satisfies (24) with equality. Notice that the derivate of the LHS of (24) with respect to x_b is $\frac{1}{2}(H(-\bar{x}) - 1) < 0$. Therefore, any $x_b > \bar{x}_b(x_\ell)$ would violate the inequality. Finally, the derivative of the LHS of (24) with respect to x_ℓ is $\frac{1}{2}H(-\bar{x}) > 0$, therefore $\bar{x}_b'(x_\ell) > 0$. As $H(\varepsilon) \to \delta_0(\varepsilon)$, the Dirac function at 0, the integral in (24) is zero if $\bar{x} > 0$ and any $x_b < 0$ satisfies (24). Similarly if $\bar{x} < 0$ the integral in (24) is $\bar{x}$ and so x_b is necessarily negative.

B Willingness to pay in the tender.

A bank with $x_b < 0$ becomes a borrower and its expected payoff is

$$V_b(x_b) = \psi_b^* \int_{\bar{x}_\ell(x_b)}^{\infty} P_b(x_\ell, x_b) dG(x_\ell) + (1 - \psi_b^*) i_p x_b$$

where $\bar{x}_\ell(x_b)$ is given by the trading condition (9) fixing x_b and solving for (9) at equality. Therefore, we can write

$$V_b(x_b) = \psi_b^* \left[\int_{\bar{x}_\ell(x_b)}^{\infty} P_b(x_\ell, x_b) dG(x_\ell) + \int_0^{\bar{x}_\ell(x_b)} i_p x_b dG(x_\ell) \right] + (1 - \psi_b^*) i_p x_b$$

The payoff function of a borrower (a bank with $x_b < 0$) is (using $\bar{x} = (x_b + x_\ell)/2$):

$$P_b(x_\ell, x_b) = \begin{cases} i_d \bar{x} + (i_p - i_d) \int_{-\infty}^{-\bar{x}} (\bar{x} + \varepsilon) dH(\varepsilon) - \frac{x_\ell i_d - x_b i_p}{2} & \text{if } x_\ell \geq \bar{x}_\ell(x_b) \\ i_p x_b & \text{if } x_\ell < \bar{x}_\ell(x_b) \end{cases} \tag{25}$$

Therefore, we have

$$\frac{\partial P_b(x_\ell, x_b)}{\partial x_b} = \begin{cases} \frac{i_p + i_d}{2} + \frac{(i_p - i_d)}{2} H(-\bar{x}) & \text{if } x_\ell \geq \bar{x}_\ell(x_b) \\ i_p & \text{if } x_\ell < \bar{x}_\ell(x_b) \end{cases} \tag{26}$$

Replacing the expressions for the derivatives, we obtain[23]

$$\begin{aligned} V_b'(x_b) &= \psi_b^* \left[\int_{\bar{x}_\ell(x_b)}^{\infty} \frac{\partial P_b(x_\ell, x_b)}{\partial x_b} dG_\ell(x_\ell) + \int_0^{\bar{x}_\ell(x_b)} i_p dF(v_b) \right] + (1 - \psi_b^*) i_p \\ &= \psi_b^* \left[\int_{\bar{x}_\ell(x_b)}^{\infty} \left[\frac{i_p + i_d}{2} + \frac{(i_p - i_d)}{2} H\left(-\frac{x_\ell + x_b}{2}\right) \right] dG_\ell(x_\ell) + \int_0^{\bar{x}_\ell(x_b)} i_p dG_\ell(x_\ell) \right] + (1 - \psi_b^*) i_p \\ &= \psi_b^* \left[\frac{i_p + i_d}{2} [1 - G_\ell(\bar{x}_\ell(x_b))] + \frac{(i_p - i_d)}{2} \int_{\bar{x}_\ell(x_b)}^{\infty} H\left(-\frac{x_\ell + x_b}{2}\right) dG_\ell(x_\ell) + i_p G_\ell(\bar{x}_\ell(x_b)) \right] + (1 - \psi_b^*) i_p \\ &= i_p - \psi_b^* \frac{(i_p - i_d)}{2} [1 - G_\ell(\bar{x}_\ell(x_b))] + \psi_b^* \frac{(i_p - i_d)}{2} \int_{\bar{x}_\ell(x_b)}^{\infty} H\left(-\frac{x_\ell + x_b}{2}\right) dG_\ell(x_\ell) \end{aligned}$$

and so

$$V_b'(x_b) = i_p - \psi_b^* \frac{(i_p - i_d)}{2} \int_{\bar{x}_\ell(x_b)}^{\infty} \left[1 - H\left(-\frac{x_\ell + x_b}{2}\right) \right] dG_\ell(x_\ell)$$

Let us now turn to lenders. A bank with $x_\ell > 0$ becomes a borrower and its expected payoff is

$$V_\ell(x_\ell) = \psi_\ell^* \int_{-\infty}^{\bar{x}_b(x_\ell)} P_\ell(x_\ell, x_b) dG(x_b) + (1 - \psi_\ell^*) i_d x_\ell$$

[23] The derivatives with respect to the limits cancel out.

$$V_\ell(x_\ell) = \psi_\ell^* \left[\int_{-\infty}^{\bar{x}_b(x_\ell)} P_\ell(x_\ell, x_b) dG(x_b) + \int_{\bar{x}_b(x_\ell)}^{0} i_d x_\ell dG(x_b) \right] + (1-\psi_\ell^*) i_d x_\ell$$

The payoff function of a lender (a bank with $x_\ell > 0$) is (using again $\bar{x} = (x_b + x_\ell)/2$):

$$P_\ell(x_\ell, x_b) = \begin{cases} i_d \bar{x} + (i_p - i_d) \int_{-\infty}^{-\bar{x}} (\bar{x} + \varepsilon) dH(\varepsilon) + \frac{x_\ell i_d - x_b i_p}{2} & \text{if } x_b \leq \bar{x}_b(x_\ell) \\ i_d x_\ell & \text{if } x_b > \bar{x}_b(x_\ell) \end{cases} \tag{27}$$

Therefore, we have

$$\frac{\partial P_\ell(x_\ell, x_b)}{\partial x_\ell} = \begin{cases} i_d + \frac{(i_p - i_d)}{2} H(-\bar{x}) & \text{if } x_b \leq \bar{x}_b(x_\ell) \\ i_d & \text{if } x_b > \bar{x}_b(x_\ell) \end{cases} \tag{28}$$

Then

$$\begin{aligned} V_\ell'(x_\ell) &= \psi_\ell^* \left[\int_{-\infty}^{\bar{x}_b(x_\ell)} \frac{\partial P_\ell(x_\ell, x_b)}{\partial x_\ell} dG_b(x_b) \right] + (1-\psi_\ell^*) i_d x_\ell \\ &= \psi_\ell^* \left[\int_{-\infty}^{\bar{x}_b(x_\ell)} \left[i_d + \frac{(i_p - i_d)}{2} H\left(-\frac{x_\ell + x_b}{2}\right)\right] dG_b(x_b) + \int_{\bar{x}_b(x_\ell)}^{0} i_d dG_b(x_b) \right] + (1-\psi_\ell^*) i_d x_\ell \\ &= \psi_\ell^* \left[i_d G_b(\bar{x}_b(x_\ell)) + \frac{(i_p - i_d)}{2} \int_{-\infty}^{\bar{x}_b(x_\ell)} H\left(-\frac{x_\ell + x_b}{2}\right) dG_b(x_b) + i_d \left[1 - G_b(\bar{x}_b(x_\ell))\right] \right] + (1-\psi_\ell^*) i_d x_\ell \end{aligned}$$

and so

$$V_\ell'(x_\ell) = i_d + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{\bar{x}_b(x_\ell)} H(-\frac{x_\ell + x_b}{2}) dG_b(x_b).$$

Now, we can compute the willingness to pay, as for some x,

$$W_S'(x) = \int_{-\infty}^{-x} V_b'(x+v) dF(v) + \int_{-x}^{\infty} V_\ell'(x+v) dF(v).$$

We can replace the derivatives to find,

$$\int_{-\infty}^{-x} V_b'(x+v) dF(v) = i_p F(-x) - \psi_b^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{-x} \int_{\tau_b(x+v)} \left[1 - H(-\frac{x_\ell + x + v}{2}) \right] dG_\ell(x_\ell) dF(v)$$

and

$$\int_{-x}^{-\infty} V_\ell'(x+v) dF(v) = i_d \left[1 - F(-x)\right] + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-x}^{\infty} \int_{\tau_\ell(x+v)} H(-\frac{x_b + x + v}{2}) dG_b(x_b) dF(v)$$

Adding both terms, we obtain

$$\begin{aligned} W'(x) &= i_p F(-x) - \psi_b^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{-x} \int_{\bar{x}_\ell(x+v)}^{\infty} \left[1 - H\left(-\frac{x_\ell + x + v}{2}\right)\right] dG_\ell(x_\ell) dF(v) \\ &\quad + i_d \left[1 - F(-x)\right] + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-x}^{\infty} \int_{-\infty}^{\bar{x}_b(x+v)} H\left(-\frac{x_b + x + v}{2}\right) dG_b(x_b) dF(v). \end{aligned}$$

In particular, whenever $H(\varepsilon) \to \delta_0(\varepsilon)$ we obtain that $\bar{x}_\ell(x_b) \to 0$ and $\bar{x}_b(x_\ell) \to 0$. Therefore $H(-\frac{x_\ell + x + v_b}{2}) = 0$ whenever $-\frac{x_\ell + x + v_b}{2} < 0$ and 1 otherwise. Hence, for $x = X$

$$\begin{aligned} W'(X) &= i_p F(-X) - \psi_b^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{-X} \int_{-X}^{\infty} \left[1 - H(-\frac{X + v_\ell + X + v_b}{2})\right] \frac{dF(v_\ell)}{1-b} dF(v_b) \\ &\quad + i_d F(X) + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-X}^{\infty} \int_{-\infty}^{-X} H(-\frac{X + v_b + X + v_\ell}{2}) \frac{dF(v_b)}{b} dF(v_\ell). \\ &= i_p F(-X) - \psi_b^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{-X} \int_{-2X - v_b}^{\infty} \frac{dF(v_\ell)}{1-b} dF(v_b) \\ &\quad + i_d F(X) + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-X}^{\infty} \int_{-\infty}^{-2X - v_\ell} \frac{dF(v_b)}{b} dF(v_\ell). \\ &= i_p F(-X) + i_d F(X) - \psi_b^* \frac{(i_p - i_d)}{2} \int_{-\infty}^{-X} \frac{1 - F(-2X - v)}{1-b} dF(v) \\ &\quad + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-X}^{\infty} \frac{F(-2X - v)}{b} dF(v). \\ &= i_p F(-X) + i_d F(X) - \psi_b^* \frac{(i_p - i_d)}{2} \frac{F(-X) - \int_{-\infty}^{-X} F(-2X - v) dF(v)}{1-n} \\ &\quad + \psi_\ell^* \frac{(i_p - i_d)}{2} \int_{-X}^{\infty} \frac{F(-2X - v)}{n} dF(v). \\ &= i_p F(-X) + i_d F(X) \\ &\quad - \frac{(i_p - i_d)}{2} \left[\psi_b^* \frac{F(-X)}{F(X)} - \frac{\psi_b^*}{1-b} \int_{-\infty}^{-X} F(-2X - v) dF(v) - \frac{\psi_\ell^*}{b} \int_{-X}^{\infty} F(-2X - v) dF(v)\right] \end{aligned}$$

And using $F(-X) = b^*$,

$$W'(X) = i_p b^* + i_d (1 - b^*) - \frac{(i_p - i_d)}{2(1 - b^*)} \left[\psi_b^* b^* - \psi_b^* \int_{-\infty}^{-X} F(-2X - v) dF(v) - \frac{(1 - b^*)\psi_\ell^*}{b^*} \int_{-a}^{\infty} F(-2X - v) dF(v)\right]$$

Also, whenever $\psi \mathcal{M}(b, 1-b) = \min\{b, 1-b\}$, we have $\psi_b^* = \min\{1, \frac{1-b}{b}\}$ and $\psi_b^*/(1 - b^*) = \psi_\ell^*/b^*$ so that,

$$W'(X) = i_p b^* + i_d (1 - b^*) - \frac{(i_p - i_d)}{2} \frac{\psi_b^*}{1 - b^*} \left[b^* - \int_{-\infty}^{\infty} F(-2X - v) dF(v)\right]$$

C Proof of Proposition 5

We compute the aggregate trading volume (Q) as the measure of matches times the average trade size when a trade occurs ($\bar{q}$). We have

$$Q(X) = \psi\mathcal{M}(F(-X), F(X))\bar{q}(X)$$

where

$$\begin{aligned}\bar{q}(X) &= \int_{x_\ell}\int_{-\infty}^{\bar{x}_b(x_\ell)} q(x_\ell, x_b)dG_b(x_b)dG_\ell(x_\ell) \\ &= \int_{-X}^{\infty}\int_{-\infty}^{-\bar{v}_b(v_\ell)} \frac{v_\ell - v_b}{2}\frac{dF(v_b)}{F(-X)}\frac{dF(v_\ell)}{F(X)} \end{aligned} \tag{29}$$

and $\bar{v}_b(v_\ell) = \bar{x}_b(x_\ell) - X$. Hence we can rewrite $\bar{q}(X)$ as

$$\begin{aligned}\bar{q}(X) &= \frac{\int_{-X}^{\infty} v_\ell F(-\bar{v}_b(v_\ell))dF(v_\ell) - \int_{-\infty}^{-X} v_b\left[1 - F(-\bar{v}_\ell(v_b))\right]dF(v_b)}{2F(-X)F(X)} &(30)\\ &= \frac{E[vF(-\bar{v}_b(v)) \mid v > -X] - E[v\left[1 - F(-\bar{v}_\ell(v))\right] \mid v < -X]}{2F(-X)F(X)} &(31)\end{aligned}$$

Note that as $E\left[v\right] = F(-X)E\left[v\left|v < -X\right.\right] + (1 - F(-X))E\left[v\left|v > -X\right.\right] = 0$, it follows that $E\left[v\left|v > -X\right.\right] = -F(-X)E\left[v\left|v < -X\right.\right] / (1 - F(-X))$. Substituting back into (30) and assuming that the payment system shock vanishes yields

$$\bar{q}(X) \xrightarrow{\varepsilon} -\frac{1}{2}\frac{E\left[v\left|v < -X\right.\right]}{1 - b^*} \tag{32}$$

Returning to (30), given v_ℓ we want to know how $\bar{v}_b(v_\ell)$ as defined by (9) changes with X. Using the implicit function theorem, we get

$$\frac{\partial\bar{v}_b(v_\ell, X)}{\partial X} = 2\frac{H(-\frac{2X+v_\ell+v_b}{2}) - \frac{1}{2}}{1 - H(-\frac{2X+v_\ell+v_b}{2})}$$

As $H(0) = 1/2$, this is positive iff $-\frac{v_\ell+v_b}{2} > X$ and negative otherwise. Then

$$\begin{aligned}\bar{q}'(X) &= \frac{f(-X)}{F(-X)F(X)}\bar{q}(X) \\ &\quad + \frac{1}{F(-X)F(X)}\left[\begin{array}{c} -\frac{\partial\bar{v}_b(-X,X)}{\partial X} \times \frac{-X-\bar{v}_b(-X)}{2} f(-\bar{v}_b(-X))f(-X) \\ +\int_{-X}^{\infty}\left(\frac{v_\ell-\bar{v}_b(v_\ell)}{2}\right) f(-\bar{v}_b(v_\ell))\left(\left(\frac{\partial\bar{v}_b(v_\ell)}{\partial X}\right)^2 - \frac{\partial^2\bar{v}_b(v_\ell)}{(\partial X)^2}\right) dF(v_\ell) \\ +\int_{-X}^{\infty}\left(\frac{\partial\bar{v}_b(v_\ell)}{\partial X}\right)^2 \frac{1}{2} f(-\bar{v}_b(v_\ell))dF(v_\ell) \end{array}\right] \\ &= \frac{1}{F(-X)F(-X)}\left[f(-X)\bar{q}(X) + \xi(X)\right]\end{aligned}$$

where $\xi(X)$ is the term in brackets. Notice that $\frac{\partial\bar{v}_b(-X,X)}{\partial X} = 2\frac{H(-\frac{X+v_b}{2})-\frac{1}{2}}{1-H(-\frac{X+v_b}{2})} > 0$ as $v_b < -X$. As the settlement shock becomes negligible $\bar{v}_b(x_\ell) \to -X$, so that $q(0, X + \bar{v}_b(-X)) = \frac{-X-\bar{v}_b(-X)}{2} \to 0$ and $\left(\frac{\partial\bar{v}_b(v_\ell)}{\partial X}\right)^2 = \frac{\partial^2\bar{v}_b(v_\ell)}{(\partial X)^2} = 0$.

Hence with a negligible settlement shock $\xi(X) \to 0$. In this case,

$$\begin{aligned} Q(X) &= \psi\mathcal{M}(F(-X), F(X)) \times \bar{q}_S(X) \\ &\to \psi\mathcal{M}(F(-X), F(X)) \times \frac{1}{2}\frac{-E\left[v\,|v < -X\right]}{F(X)} \\ &= \psi\mathcal{M}(\frac{F(-X)}{F(X)}, 1) \times \frac{1}{2}\left\{-E\left[v\,|v < -X\right]\right\} \\ &= -\psi_\ell^*\frac{1}{2}E\left[v\,|v < -X\right] \end{aligned}$$

$F(-X) \to 0$ as $X \to \infty$ and $\mathcal{M}(0, 1) = 0$ and so $Q_S(X) \to 0$ as $X \to \infty$. As the mean shock is zero while $\psi_\ell^* \to \psi$ as $\frac{F(-X)}{F(X)} \to \infty$, we obtain $Q(X) \to 0$ as $X \to -\infty$. At $X = 0$, $\mathcal{M}(1, 1) = 1$ and $Q(X) = \frac{\psi}{2}\left\{-E\left[v\,|v < 0\right]\right\}$.

And

$$\begin{aligned} Q'(X) &\to \theta'(X)\psi_\ell^{*\prime}(\theta(X))\bar{q}(X) - \psi_\ell^*(\theta(X))\frac{X}{2}f(-X) \\ &= \psi\frac{-f(-X)}{F(X)^2}\mathcal{M}_b\left(\frac{F(-X)}{F(X)}, 1\right)\bar{q}_S(X) - \psi\mathcal{M}(\frac{F(-X)}{F(X)}, 1)\frac{X}{2}f(-X) \\ &= -\frac{\psi}{2}f(-X)\mathcal{M}(\frac{F(-X)}{F(X)}, 1)\left[\frac{1}{F(X)}\frac{\mathcal{M}_b\left(\frac{F(-X)}{F(X)}, 1\right)}{\mathcal{M}(\frac{F(-X)}{F(X)}, 1)}\left\{-E\left[v\,|v < -X\right]\right\} + X\right] \\ &= -\frac{\psi}{2}f(-X)\mathcal{M}(\frac{F(-X)}{F(X)}, 1)\left[\frac{F(-X)}{F(X))}\frac{\mathcal{M}_b\left(\frac{F(-X)}{F(X)}, 1\right)}{\mathcal{M}(\frac{F(-X)}{F(X)}, 1)}\left\{-E\left[v\,|v < -X\right]\right\} + F(-X)X\right] \\ &= -\frac{\psi}{2}\frac{f(-X)}{F(-X)}\mathcal{M}(\frac{1}{\theta(X)}, 1)\left[\frac{\frac{1}{\theta(X)}\mathcal{M}_b\left(\frac{1}{\theta(X)}, 1\right)}{\mathcal{M}(\frac{1}{\theta(X)}, 1)}\left\{-E\left[v\,|v < -X\right]\right\} + F(-X)X\right] \end{aligned}$$

Since $\mathcal{M}$ displays constant return to scale, $\frac{\frac{1}{\theta(X)}\mathcal{M}_b\left(\frac{1}{\theta(X)},1\right)}{\mathcal{M}(\frac{1}{\theta(X)},1)}$ is a constant say α. Then

$$\begin{aligned} Q'(X) &\to -\frac{\psi}{2}\frac{f(-X)}{F(-X)}\mathcal{M}(\frac{1}{\theta(X)}, 1)\left[-E\left[\alpha v\,|v < -X\right] + F(-X)X\right] \\ &= -\frac{\psi}{2}\frac{f(-X)}{F(-X)}\mathcal{M}(\frac{1}{\theta(X)}, 1)\left[\int_{-\infty}^{-X}(X - \alpha v)\,dF(v)\right] \end{aligned}$$

Notice that $-\int_{-\infty}^{-X} vdF(v) \geq F(-X)X$. Hence, $Q'(X) < 0$ whenever $X > 0$. Now define $\bar{X}$ as solving

$$\int_{-\infty}^{-X}(X - \alpha v)\,dF(v) = 0.$$

This LHS expression is positive for all $X > 0$ and it is increasing in X for all $X < 0$. Therefore, $\bar{X} < 0$. And the LHS is negative for all $X < \bar{X}$ and positive otherwise. Hence, $Q'(X)$ is positive for all $X < \bar{X}$, and negative for all $X > \bar{X}$. Since $Q(X) \to 0$ as $X \to \infty$ or as $X \to -\infty$, it must be that $Q(X)$ reaches a maximum at $\bar{X}$.

When $\psi\mathcal{M}(b, 1 - b) = \min\{b, 1 - b\}$, $Q(X)$ is differentiable almost everywhere but it has a kink at $X = 0$. Assuming that the distribution function $F(v)$ is symmetric, we have $F(X) < 1 - F(X)$ whenever $X < 0$, while

$F(X) > 1 - F(X)$ whenever $X > 0$. Therefore, for all $X < 0$ we obtain

$$Q(X)\,|_{X<0} = \frac{-\int_{-\infty}^{-X} vdF(v)}{2F(X)},$$

with derivative

$$2Q'(X)\,|_{X<0} \quad = \quad \frac{-F(X)Xf(X) + f(X)\left[-\int_{-\infty}^{-X} vdF(v)\right]}{F(X)^2}.$$

Hence, $Q'(X) > 0$ whenever $X < 0$. Also, $\lim_{X\to-\infty} Q(X) = 0$. Now, for all $X > 0$ we obtain

$$Q(X)\,|_{X>0} = \frac{-\int_{-\infty}^{-X} vdF(v)}{2F(-X)},$$

with derivative

$$2Q'(X)\,|_{X>0} \quad = \quad \frac{F(-X)\left[-Xf(X)\right] - f(X)\left[-\int_{-\infty}^{-X} vdF(v)\right]}{F(-X)^2}$$

and since $X > 0$ we obtain that $Q'(X) < 0$ whenever $X > 0$. Also, $\lim_{X\to\infty} Q(X) = 0$, as the mean shock is zero.

D Volume-weighted average rate in OTC market

$$\bar{i}_q(X) = \int_{x_\ell}\int_{x_b\in\tau(x_\ell)} \psi\mathcal{M}(n^*, 1-n^*)i(x_\ell, x_b)\frac{q(x_\ell, x_b)}{Q(X)}dG(x_b)dG(x_\ell) \tag{33}$$

$$\bar{q}(X) = \int_{-X}^{\infty}\int_{-\infty}^{-\bar{v}_b(v_\ell)} \frac{v_\ell - v_b}{2}\frac{dF(v_b)}{F(-X)}\frac{dF(v_\ell)}{F(X)}$$

Using (6), (7), and (18) yields

$$\begin{aligned}
\bar{i}_q(X) &= \int_{x_\ell}\int_{x_b\in\tau(x_\ell)} i(x_\ell, x_b)\frac{q(x_\ell, x_b)}{\bar{q}(a)}dG_b(x_b)dG_\ell(x_\ell) \\
&= \frac{1}{\bar{q}(X)}\int_0^{\infty}\int_{-\infty}^{\bar{x}_b(x_\ell)}\left[\frac{x_\ell - x_b}{2}\left(\frac{x_\ell}{x_\ell - x_b}i_d - \frac{x_b}{x_\ell - x_b}i_p\right)\right]dG_b(x_b)dG_\ell(x_\ell) \\
&= \frac{1}{2\bar{q}(X)}\left[\int_{-X}^{\infty}\int_{-\infty}^{-\bar{v}_b(v_\ell)}\left[(X+v_\ell)i_d - (X+v_b)i_p\right]\frac{dF(v_b)}{b^*}\frac{dF(v_\ell)}{1-b*}\right] \\
&= \frac{1}{2\bar{q}(X)}\left[\int_{-X}^{\infty}\int_{-\infty}^{-X}\left[(X+v_\ell)i_d - (X+v_b)i_p\right]\frac{dF(v_b)}{b^*}\frac{dF(v_\ell)}{1-b*} - \int_{-X}^{\infty}\int_{-\bar{v}_b(v_\ell)}^{-X}(X+v_\ell)i_d - (X+v_b)i_p\frac{dF(v_b)}{b*}\frac{dF(v_\ell)}{1-b*}\right] \\
&= \frac{1}{2\bar{q}(X)}\left[i_dE\left[v \mid v > -X\right] - i_pE\left[v \mid v < -X\right] - X(i_p - i_d)\right] - \xi(\tau(H))
\end{aligned}$$

where $\xi(\tau(H)) \geq 0$ and converges to zero as the settlement shock vanishes. Notice that where $\bar{v}_b(v_\ell) \to 0$ as the settlement shock becomes negligible. Hence, the average OTC rate is equal to the rate in Poole (1968) whenever

$X = 0$.

Example 1. From Owen (1980) p. 396, we have $\int z\phi(z)dz = -\phi(z)$. Hence, it follows that $\int_{-X}^{\infty} vdF(v) = \int_{-X}^{\infty} \frac{v}{\sigma}\phi(\frac{v}{\sigma})dv = \sigma \int_{-X/\sigma}^{\infty} z\phi(z)dz = \sigma\phi(-\hat{X})$ and $\int_{-\infty}^{-X} vdF(v) = -\sigma\phi(-\hat{X})$. Moreover, from Owen (1980) p. 398, we have that $\int z\Phi(\alpha z)\phi(z)dz = \alpha\Phi\left(z\sqrt{1+\alpha^2}\right)/\sqrt{\pi}\sqrt{1+\alpha^2} - \Phi(\alpha z)\phi(z)$. Hence, it follows that $\int_{-X}^{\infty} v2F(v)dF(v) = 2\sigma \int_{-X/\sigma}^{\infty} z\Phi(z)\phi(z)dz = \frac{\sigma}{\sqrt{\pi}}\Phi\left(\sqrt{2}\hat{X}\right) + \Phi(-\hat{X})\phi(-\hat{X})$ where we have used the fact that $\Phi(z) = 1 - \Phi(-z)$. Under sorting, we have from (21) that

$$\bar{i}_S(X, \sigma) = i_d\Phi(-\hat{X}) + i_p\Phi(\hat{X}) - (i_p - i_d)\frac{\hat{X}\Phi(-\hat{X})\Phi(\hat{X})}{\phi(-\hat{X})}$$

Note, that for the effective rates it is the ratio of excess reserves injected by the central bank over the dispersion of the payment shock that matters, i.e., $\hat{X}$, and not X and σ individually.

E Use of central bank lending facility

Recall that banks that need to borrow from the central bank are a combination of two types. First, are those banks that wished to borrow in the interbank market but were not matched with a lender, or those banks that were matched with the wrong lender, that is those borrowers holding x_b that are matched with lenders that do not generate positive gains from trade, i.e. borrowers meeting lenders holding $x_\ell < \bar{x}_\ell(x_b)$. Second are those banks in a trading pair who won't satisfy their reserve requirement despite trading. On average each bank of the first type borrows

$$-E\left[x_b \mid x_b < 0\right] = -(X + E\left[v \mid v < -X\right]) \tag{34}$$

and there is a measure $b^*\left[1 - \psi_b^* + \psi_b^* \int_{-\infty}^{-X}\int_{-X}^{\bar{v}_\ell(v_b)} \frac{dF(v_\ell)}{1-b^*}\frac{dF(v_b)}{b^*}\right]$ of these banks in equilibrium. As the settlement shock becomes negligible, again $\bar{v}_\ell(v_b) \xrightarrow{\varepsilon} -X$ and this measure converges to $b^*(1 - \psi_b^*)$.

Second are those banks in a trading pair who won't satisfy their reserve requirement in spite of trading. That is, matched borrowers holding $x_b < 0$ and lenders holding $x_\ell \geq \bar{x}_\ell(x_b)$ that sustained a large enough (negative) settlement shock. On average each bank in those pairs borrows

$$-E\left[\frac{x_\ell + x_b}{2} + \varepsilon \mid x_b \leq \bar{x}_b(x_\ell), \varepsilon < -\frac{x_\ell + x_b}{2}\right] = \tag{35}$$

$$-E\left[X + \frac{v_\ell + v_b}{2} + \varepsilon \mid v_b \leq \bar{v}_b(v_\ell), \varepsilon < -X - \frac{v_\ell + v_b}{2}\right] = \tag{36}$$

$$-\int_{-X}^{\infty}\int_{-\infty}^{\bar{v}_b(v_\ell)}\int_{-\infty}^{-X-\frac{v_\ell+v_b}{2}}\left(X + \frac{v_\ell + v_b}{2} + \varepsilon\right)dH(\varepsilon)\frac{dF(v_b)}{F(-X)}\frac{dF(v_\ell)}{F(X)}$$

and there is a measure of $b^*\psi_b^*$ of such pairs. Notice that as the settlement shock becomes negligible, again $\bar{v}_b(v_\ell) \rightarrow -X$ and the integral will be positive iff $X + \frac{v_l+v_b}{2} \leq 0$, that iff $x_\ell + x_b < 0$. Therefore, when the settlement shock becomes negligible, the pair of banks borrows from the central bank:

$$\begin{aligned} -E\left[x_\ell + x_b \mid 0 < x_\ell < -x_b\right] &= -E\left[2X + v_\ell + v_b \mid -X < v_\ell < -2X - v_b\right] \\ &= -\int_{-\infty}^{-X}\int_{-X}^{-2X-v_b}(2X + v_\ell + v_b)\frac{dF(v_\ell)}{F(X)}\frac{dF(v_b)}{F(-X)} \end{aligned} \tag{37}$$

Hence, the use of the central bank lending facility is

$$\begin{aligned} L(X) &= -b^*\left[1-\psi_b^*+\psi_b^*\int_{-\infty}^{-X}\int_{-X}^{\bar{v}_\ell(v_b)}\frac{dF(v_\ell)}{1-b^*}\frac{dF(v_b)}{b^*}\right](X+E[v\mid v<-X]) \\ &\quad -n^*\psi_b^*\int_{-X}^{\infty}\int_{-\infty}^{\bar{v}_b(v_\ell)}\int_{-\infty}^{-X-\frac{v_\ell+v_b}{2}} 2\left(X+\frac{v_\ell+v_b}{2}+\varepsilon\right)dH(\varepsilon)\frac{dF(v_\ell)}{F(X)}\frac{dF(v_b)}{F(-X)}, \end{aligned}$$

and as the settlement risk is negligible,

$$\begin{aligned} L(X) &\overset{\varepsilon}{\to} -b*(1-\psi_b^*)(X+E[v\mid v<-X])- \\ &\quad b^*\psi_b^*\int_{-\infty}^{-X}\int_{-X}^{-2X-v_b}(2X+v_\ell+v_b)\frac{dF(v_\ell)}{F(X)}\frac{dF(v_b)}{F(-X)}. \end{aligned}$$

where $b^*=F(-X)$ and $\psi_b^*=\psi\mathcal{M}\left(b^*,1-b^*\right)/b^*$.

In case of sorting when the settlement risk becomes negligible, we need to determine the following integral

$$I(X)=-\frac{1}{F(X)F(-X)}\int_{-\infty}^{-X}\int_{-X}^{-2X+v_b}(2X+v_\ell+v_b)dF(v_\ell)dF(v_b)$$

which consists of three part. The first part is

$$\begin{aligned} C_1 &= 2X\int_{-\infty}^{-X}\int_{-X}^{-2X-v_b}dF(v_\ell)dF(v_b)=2X\left[\int_{-\infty}^{-X}(F(-2X-v)dF(v)-F(-X)^2\right] \\ &= 2X\left[\int_{-\infty}^{-\hat{X}}\Phi\left(-2\hat{X}-z\right)\phi(z)dz-\Phi(-\hat{X})^2\right] \end{aligned}$$

From Owen (1980) p. 402, we have that

$$\int_{-\infty}^{-h}\Phi\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)\phi(x)dx=\mathrm{BvN}\left[h,k;\rho\right]$$

where BvN denotes the CDF of the bivariate standard normal distribution with correlation parameter ρ. Hence, letting $\rho=\frac{1}{\sqrt{2}}$, $h=-\hat{X}$, $k=-\sqrt{2}\hat{X}$, we have that

$$C_1=2X\left[\mathrm{BvN}\left(-\hat{X},\sqrt{2}\hat{X};\frac{1}{\sqrt{2}}\right)-\Phi(-\hat{X})^2\right]$$

The second component is

$$\begin{aligned} C_2 &= \int_{-\infty}^{-X}\int_{-X}^{-2X-v_b}v_\ell dF(v_\ell)dF(v_b)=\int_{-X}^{\infty}v_\ell\int_{-\infty}^{-2X-v_\ell}dF(v_b)dF(v_\ell) \\ &= \int_{-X}^{\infty}vF(-2X-v)dF(v) \end{aligned}$$

The third component is

$$\begin{aligned} C_3 &= \int_{-\infty}^{-X} v_b \int_{-X}^{-2X-v_b} dF(v_\ell)dF(v_b) = \int_{-\infty}^{-X} v\left[F(-2X-v) - F(-X)\right]dF(v) \\ &= \int_{-\infty}^{-X} vF(-2X-v)dF(v) - F(-X)^2 E[v \mid v < -X] \end{aligned}$$

Combining parts two and three yields and using Owen (1980) p. 404 yields

$$\begin{aligned} C_2 + C_3 &= \int_{-\infty}^{\infty} vF(-2X-v)dF(v) - F(-X)^2 E[v \mid v < -X] \\ &= \sigma \int_{-\infty}^{\infty - 2X} z\Phi\left(-2\hat{X} - z\right)\phi(z)dz + \sigma\phi(-\hat{X})\Phi\left(-\hat{X}\right) \\ &= -\frac{\sigma}{\sqrt{2}}\phi\left(-\sqrt{2}\hat{X}\right) + \sigma\phi(-\hat{X})\Phi\left(-\hat{X}\right) \end{aligned}$$

Collecting all the parts yields

$$\begin{aligned} L(X) &= -b^*(1-\psi_b^*)\left[X - \sigma\frac{\phi(-\hat{X})}{\Phi(\hat{X})}\right] + \\ &\quad -\frac{b^*\psi_b^*}{\Phi(\hat{X})\Phi(-\hat{X})}\left[\begin{array}{c} 2a\left[\mathrm{BvN}\left(-\hat{X}, \sqrt{2}\hat{X}; \frac{1}{\sqrt{2}}\right) - \Phi(-\hat{X})^2\right] \\ -\frac{\sigma}{\sqrt{2}}\phi\left(-\sqrt{2}\hat{X}\right) + \sigma\phi(-\hat{X})\Phi\left(-\hat{X}\right) \end{array}\right] \end{aligned}$$

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