# Model notes: The impact of artificial intelligence on output and inflation.


**Abstract**

These notes describe the model underlying the paper "The impact of artificial intelligence on output and inflation".


# 1 The Model

The economy is made up of $\mathcal{F}$ industries. I use the letter $j$ to describe individual industries and the letter $\iota$ to describe individual firms within each industry. With some abuse of notation, I also use $\iota$ to denote household-level variables.

## 1.1 Nonlinear model

### 1.1.1 Households

The economy features two types of households: Ricardian households, who have access to financial markets and non-Ricardian households, who don't. The share of the two household types is $\omega_r$ and $1-\omega_r$.

**Ricardian Households**

There is a continuum of identical households indexed by $\iota$ (which I suppress when not important). The household's problem is to choose aggregate and industry-level consumption, investment and capital, household-by-industry level wages and aggregate bond holdings to maximise utility:

$$\sum_{t=0}^{\infty} \beta^t \left[ e^{\xi_{c,t}} \log(C_t^r - hC_{t-1}^r) - \frac{A_N}{1+\nu} N_t^r(\iota)^{1+\nu} \right] \tag{1}$$

subject to the budget constraint:

$$P_{C,t}C_t^r + P_{I,t}I_t^r + \frac{B_{t+1}}{R_t} \leq B_t + \sum_{j=1}^{\mathcal{F}} \left( P_{C,t} \frac{r_{j,t}^K k_{j,t} u_{j,t}}{\mathcal{M}} + w_{j,t}(\iota) n_{j,t}^r(\iota) - a(u_{j,t}) k_{j,t} \right) + T_t^r \tag{2}$$

and capital accumulation constraints for each industry:

$$k_{j,t+1} = (1-\delta)k_{j,t} + \left( 1 - \mathcal{S}\left( \frac{z_{j,t}}{z_{j,t-1}} \right) \right) z_{j,t} \tag{3}$$

where $C_t$ is aggregate consumption, $\xi_{c,t}$ is a consumption preference shifter, $I_t$ is aggregate investment, $k_{j,t}$ is the capital stock of industry $j$, $u_{j,t}$ is the utilisation of capital in industry $j$ and $z_{j,t}$ is gross investment in industry $j$. $w_{j,t}$ is the wage in industry $j$, which is distinct from the wage paid to household $\iota$ in industry $j$, $w_{j,t}(\iota)$. Similarly $n_{j,t}(\iota)$ is hours worked by household $\iota$ in industry $j$, while $n_{j,t}$ is total hours worked in industry $j$. The household takes industry-level wages and hours worked as given in making its decisions. $T_t$ are lump sum transfers to the government. $\mathcal{M}$ is a wedge between the return on capital paid by firms and the amount received by households. It can be viewed as a reduced form for firm defaults or other factors that cause investors to demand a risk premium on lending to corporates.

Aggregate consumption and investment consist of bundles of consumption and investment goods sourced from each industry:

$$C_t^r = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{cj}^{\frac{1}{\eta}} c_{j,t}^{r\,\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{4}$$

$$I_t^r = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{ij}^{\frac{1}{\eta}} i_{j,t}^{r\,\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{5}$$

The price indices accompanying the consumption and investment aggregates are:

$$P_{C,t} = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{c,j} p_{j,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{6}$$

$$P_{I,t} = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{i,j} p_{j,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{7}$$

where $p_{j,t}$ is the price of the good produced by industry $j$.
It follows that the demand functions for the output of individual industries are:

$$c_{j,t} = \omega_{cj} \left( \frac{p_{j,t}}{P_{C,t}} \right)^{-\eta} C_t \tag{8}$$

$$i_{j,t} = \omega_{ij} \left( \frac{p_{j,t}}{P_{I,t}} \right)^{-\eta} I_t \tag{9}$$

Similarly, total labour supply, $N_t(\iota)$ is a bundle of labour supplied to each sector:

$$N_t^r(\iota) = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{nj}^{-\frac{1}{\xi}} n_{j,t}^r(\iota)^{\frac{\xi+1}{\xi}} \right]^{\frac{\xi}{\xi+1}} \tag{10}$$

A labour packer aggregates the labour supply of individual households in each industry according to:

$$n_{j,t} = \left( \int_0^1 n_{j,t}(\iota)^{\frac{\epsilon_w - 1}{\epsilon_w}} du \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{11}$$

Consequently, demand for different types of labour is given by:

$$n_{j,t}(\iota) = \left( \frac{w_{j,t}(\iota)}{w_{j,t}} \right)^{-\epsilon_w} n_{j,t} \tag{12}$$

where $w_{j,t}$ is the aggregate wage index in industry $j$. The household takes this labour demand function into account when making its wage decisions.
Price inflation is given by:

$$\Pi_{C,t} = \frac{P_{C,t}}{P_{C,t-1}} \tag{13}$$

$$\tag{14}$$

Market clearing for investment goods requires the aggregate volume of investment goods demanded by households to equal the sum of investment in all of the industries, that is:

$$I_t^r = \sum_{j=1}^{\mathcal{F}} z_{j,t}^r \tag{15}$$

Letting the Lagrange multipliers for the constraints on the budget constraint and the capital accumulation condition be $\Lambda_t / P_{C,t}$ and $\Lambda_t q_{j,t}$, the first order conditions for the household's problem are:

$$\frac{e^{\xi_{c,t}}}{C_t^r - hC_{t-1}^r} = \Lambda_t^r + \beta E_t \left\{ \frac{he^{\xi_{c,t+1}}}{C_{t+1}^r - hC_t^r} \right\} \tag{16}$$

$$\Lambda_t^r = \beta R_t E_t \left\{ \frac{\Lambda_{t+1}^r}{\Pi_{C,t+1}} \right\} \tag{17}$$

$$\Lambda_t^r q_{j,t} = \beta E_t \left\{ (1-\delta_j)\Lambda_{t+1}^r q_{j,t+1} + \Lambda_{t+1}^r \frac{r_{j,t+1}^K u_{j,t+1}}{\mathcal{M}} \right\} \tag{18}$$

$$\begin{aligned} \Lambda_t^r = {} & \Lambda_t^r q_{j,t} \left[ 1 - \mathcal{S}\left( \frac{z_{j,t}}{z_{j,t-1}} \right) - \mathcal{S}'\left( \frac{z_{j,t}}{z_{j,t-1}} \right) \frac{z_{j,t}}{z_{j,t-1}} \right] \\ & + \beta E_t \left\{ \Lambda_{t+1}^r q_{j,t+1} \mathcal{S}'\left( \frac{z_{j,t+1}}{z_{j,t}} \right) \frac{z_{j,t+1}^2}{z_{j,t}^2} \right\} \end{aligned} \tag{19}$$

$$r_{j,t}^k = a(u_{j,t}) \tag{20}$$

**Non-Ricardian households**
Non-Ricardian households maximise the utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[ e^{\xi_{c,t}} \log(C_t^{nr} - hC_{t-1}^{nr}) - \frac{A_N}{1+\nu} N_t^{nr}(\iota)^{1+\nu} \right] \tag{21}$$

subject to the budget constraint:

$$P_{C,t} C_t^{nr} \leq \sum_{j=1}^{\mathcal{F}} w_{j,t} n_{j,t}^{nr} + T_t^{nr} \tag{22}$$

The first order conditions for their problem are:

$$\frac{e^{\xi_{c,t}}}{C_t^{nr} - hC_t^{nr}} = P_{C,t}\Lambda_t^{nr} + \beta E_t \left\{ \frac{e^{\xi_{c,t+1}} h}{C_{t+1}^{nr} - hC_t^{nr}} \right\} \tag{23}$$

which defines the marginal utility of consumption for non-Ricardian households

### 1.1.2 Aggregate consumption and marginal utility of consumption

The 'aggregate' marginal utility of consumption, $\Lambda_t$ is a weighted average of the marginal utilities of the Ricardian and non-Ricardian households:

$$\Lambda_t = \omega_r \Lambda_t^r + (1-\omega_r)\Lambda_t^{nr} \tag{24}$$

Similarly, aggregate consumption is a weighted average of Ricardian and non-Ricardian consumption:

$$C_t = \omega_r C_t^r + (1-\omega_r) C_t^{nr} \tag{25}$$

### 1.1.3 Labour market:

In each industry, a continuum of perfectly competitive labour hiring firms combine the specialised labour types according to:

$$n_{j,t} = \left( \int_0^1 n_{j,t}(s)^{\frac{\epsilon_w - 1}{\epsilon_w}} ds \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{26}$$

The hiring firm's demand for each labour type $j$ is given by:

$$n_{j,t}(s) = \left( \frac{w_{j,t}(s)}{w_{j,t}} \right)^{-\epsilon_w} n_{j,t} \tag{27}$$

where $w_{j,t}$ is the industry wage index given by:

$$w_{j,t} = \left( \int_0^1 w_{j,t}(s)^{1-\epsilon_w} ds \right)^{\frac{1}{1-\epsilon_w}} \tag{28}$$

Workers of type $s$ unionise in order to take advantage of their monopoly power. These unions set nominal wages subject to the labour demand constraint and a Calvo friction that means that a random proportion, $\theta_{w,j}$ of households cannot re-optimise their wage each period.
Unions that do not re-optimise their wages re-scale them according to the indexation rule that depends on industry-specific lagged wage inflation $(\pi^{w,j}_{t-1})$ :

$$w_{j,t}(s) = (\pi^w_{j,t-1})^{\chi_{w,j}} w_{j,t-1}(s)$$

Define:

$$\Omega_{j,t,t+s} = \prod_{m=t}^{t+s-1} (\pi^w_{j,m})^{\chi_{w,j}}$$

to be the total indexation in period $s$ of a union that last updated its wage in period $t$.
Unions choose $w_{j,t}(s)$ to maximise:

$$\begin{aligned} \mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_{w,j})^s \Bigg[ & \Upsilon^w_j \frac{\Lambda_{t+s}}{P_{C,t+s}} w_{j,t}(s)\Omega_{j,t,t+s} \left( \frac{w_{j,t}(s)\Omega_{j,t,t+s}}{w_{j,t+s}} \right)^{-\epsilon_w} n_{j,t+s} \\ & - \frac{A_N}{1+\nu} \left[ \sum_{j=1}^{\mathcal{F}} \left( \int_0^1 \left( \frac{w_{j,t}(k)\Omega_{j,t,t+s}}{w_{j,t+s}} \right)^{-\epsilon_w} n_{j,t+s} dk \right)^{\frac{1+\xi}{\xi}} \right]^{\frac{\xi(1+\nu)}{1+\xi}} \Bigg] \end{aligned} \tag{29}$$

where $\Upsilon^w$ is a wage subsidy calibrated to offset the effect of imperfect labour market competition on employment.
The first order condition for this problem is:

$$\begin{aligned} 0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_{w,j})^s \Bigg[ & -(1-\epsilon_w)\Upsilon^w_j \frac{\Lambda_{t+s}}{P_{C,t+s}} \Omega_{j,t,t+s} \left( \frac{w_{j,t}(s)\Omega_{j,t,t+s}}{w_{j,t+s}} \right)^{-\epsilon_w} n_{j,t+s} \\ & + \epsilon_w A_N N_{t+s}^{\nu-\frac{1}{\xi}} n_{j,t+s}^{\frac{1+\xi}{\xi}} \left( \frac{\Omega_{j,t+s}}{w_{j,t+s}} \right)^{-\epsilon_w \frac{1+\xi}{\xi}} w_{j,t}(k)^{-\epsilon_w \frac{1+\xi}{\xi} - 1} \Bigg] \end{aligned} \tag{30}$$

which we can re-arrange to:

$$\left[ \frac{w_{j,t}(k)}{w_{j,t}} \right]^{\frac{\xi+\epsilon_w}{\xi}} = \frac{H_{w1,t}}{H_{w2,t}} \tag{31}$$

where:

$$H_{w1,t} = \sum_{s=0}^{\infty} (\beta\theta_{w,j})^s A_N N_{t+s}^{\nu-\frac{1}{\xi}} n_{j,t+s}^{\frac{1+\xi}{\xi}} \left( \frac{\Omega_{j,t+s}}{\pi_{w,j,t,t+s}} \right)^{-\epsilon_w \frac{1+\xi}{\xi}} \tag{32}$$

$$H_{w2,t} = \sum_{s=0}^{\infty} (\beta\theta_{w,j})^s \Lambda_{t+s} \frac{w_{j,t+s}}{P_{C,t+s}} n_{j,t+s} \left( \frac{\Omega_{j,t+s}}{\pi_{w,j,t,t+s}} \right)^{1-\epsilon_w} \tag{33}$$

We can-re-write $H_{w1,t}$ and $H_{w2,t}$ as:

$$H_{w1,t} = A_N N_t^{\nu-\frac{1}{\xi}} n_{j,t}^{\frac{1}{\xi}+1} + \beta\theta_{w,j} \mathbb{E}_t \left\{ \left( \frac{\pi_{j,t}^{w\chi_w}}{\pi_{j,t+1}^{w}} \right)^{-\epsilon_w \frac{1+\xi}{\xi}} H_{w1,t+1} \right\} \tag{34}$$

$$H_{w2,t} = \Lambda_t \frac{w_{j,t}}{P_{C,t}} n_{j,t} + \beta\theta_{w,j} \mathbb{E}_t \left\{ \left( \frac{\pi_{j,t}^{w\chi_w}}{\pi_{j,t+1}^{w}} \right)^{1-\epsilon_w} H_{w2,t+1} \right\} \tag{35}$$

From the definition of the wage index, we also know that:

$$1 = (1-\theta_{w,j}) \left( \frac{w_{j,t}(k)}{w_{j,t}} \right)^{1-\epsilon_w} + \theta_{w,j} \left( \frac{\pi_{j,t-1}^{w\chi_w}}{\pi_{j,t}^{w}} \right)^{1-\epsilon_w} \tag{36}$$

### 1.1.4 Firms:

Firms in industry $j$ produce output using capital, labour and intermediate goods according to the multi-layered production function:

$$y_{j,t}^{va}(\iota) = \left[ \omega_{n,j}^{\frac{1}{\zeta}} n_{j,t}(\iota)^{\frac{\zeta-1}{\zeta}} + (1-\omega_{n,j})^{\frac{1}{\zeta}} k_{j,t}^{s}(\iota)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} \tag{37}$$

$$x_{j,t}(\iota) = \left[ \sum_{k=1}^{\mathcal{F}} \omega_{k,j}^{\frac{1}{\psi}} x_{k,j,t}(\iota)^{\frac{\psi-1}{\psi}} \right]^{\frac{\psi}{\psi-1}} \tag{38}$$

$$y_{j,t}(\iota) = a_j \left[ \omega_{y,j}^{\frac{1}{\varphi}} y_{j,t}^{va}(\iota)^{\frac{\varphi-1}{\varphi}} + (1-\omega_{y,j})^{\frac{1}{\varphi}} x_{j,t}(\iota)^{\frac{\varphi-1}{\varphi}} \right]^{\frac{\varphi}{\varphi-1}} \tag{39}$$

where $y_{j,t}^{va}(\iota)$ is the value added of firm $\iota$ in industry $j$, $k_{j,t}^{s}(\iota)$ is the amount of capital hired by the firm, $x_{j,t}(\iota)$ is the amount of intermediate goods used by the firm and $y_{j,t}(\iota)$ is gross output of the firm. The total capital hired by industry $j$ and total capital available to be hired is related by:

$$k_{j,t}^{s} = u_{j,t} k_{j,t} \tag{40}$$

Marginal costs (deflated by industry-specific final prices) and the resulting demand functions are:

$$mc_{j,t}(\iota) = \frac{1}{a_j} \left[ \omega_{y,j} (p_{j,t}^{yva}(\iota)/p_{j,t})^{1-\varphi} + (1-\omega_{y,j})(p_{j,t}^{x}(\iota)/p_{j,t})^{1-\varphi} \right]^{\frac{1}{1-\varphi}} \tag{41}$$

$$(p_{j,t}^{yva}(\iota))^{\varphi} y_{j,t}^{va}(\iota) = \frac{\omega_{y,j}}{1-\omega_{y,j}} (p_{j,t}^{x}(\iota))^{\varphi} x_{j,t}(\iota) \tag{42}$$

where the price indices for value-added and intermediate goods in each industry are given by:

$$p_t^{yva} = \left[\omega_n w_{j,t}^{1-\zeta} + (1-\omega_{n,j}) r_{j,t}^{k1-\zeta}\right]^{\frac{1}{1-\zeta}} \tag{43}$$

$$p_{j,t}^{x} = \left[\sum_{k=1}^{\mathcal{F}} \omega_{k,j} p_{k,t}^{1-\psi}\right]^{\frac{1}{1-\psi}} \tag{44}$$

These imply the demand functions:

$$n_{j,t} = \omega_{n,j} \left(\frac{w_{j,t}}{p_{j,t}^{yva}}\right)^{-\zeta} y_{j,t}^{va} \tag{45}$$

$$k_{j,t} = (1-\omega_{n,j}) \left(\frac{r_{j,t}^{k}}{p_{j,t}^{yva}}\right)^{-\zeta} y_{j,t}^{va} \tag{46}$$

$$x_{kj,t} = \omega_{k,j} \left(\frac{p_{k,t}}{p_{j,t}^{x}}\right)^{-\psi} x_{j,t} \tag{47}$$

In each industry, individual firms face price stickiness a la Calvo. Each period a fraction of firms, $1-\theta_{pj}$ are able to change their prices. The remainder follow an indexing rule:

$$p_{j,t}(\iota) = (\pi_{j,t-1})^{\varphi_{j,p}} p_{j,t-1}(\iota)$$

Define:

$$\Omega_{j,t,t+s} = \prod_{m=t}^{t+s-1} (\pi_{j,m})^{\varphi_{j,p}}$$

as the cumulative change in prices between $t$ and $m$, conditional on not re-optimising.
The problem for a firm that is able to reset its prices at time $t$ is:

$$\max_{p_{j,t}^{*}(\iota)} E_t \sum_{s=0}^{\infty} (\beta\theta_{pj})^s \left\{\Lambda_{t+s}\left[\frac{p_{j,t}^{*}(\iota)\gamma_{j,t+s}\Omega_{j,t,t+s}}{p_{j,t+s}} y_{j,t+s}(\iota) - \frac{1}{1+\phi_{pj}} mc_{j,t+s}\gamma_{j,t+s} y_{j,t+s}(\iota)\right]\right\}$$

where $\gamma_{j,t+s} = p_{j,t+s}/P_{C,t+s}$ is the relative price of goods in industry $j$, subject to the demand condition given above. The parameter $\phi_j$ is a production subsidy to offset the steady-state distortion from imperfect competition. The first order condition for this problem is:

$$E_t \sum_{s=0}^{\infty} (\beta\theta_{pj})^s \left\{\Lambda_{t+s}\left[\frac{1-\epsilon_{jp}\gamma_{j,t+s}}{p_{j,t}(\iota)} \left(\frac{p_{jt}(\iota)\Omega_{j,t,t+s}}{p_{j,t+s}}\right)^{1-\epsilon_{jp}} y_{j,t+s} \right.\right.$$
$$\left.\left. + \frac{\epsilon_{jp}}{1+\phi_{jp}} \frac{mc_{j,t+s}}{p_{j,t}(\iota)} \left(\frac{p_{j,t}(\iota)\Omega_{j,t,t+s}}{p_{j,t+s}}\right)^{-\epsilon_{jp}} \gamma_{j,t+s} y_{j,t+s}\right]\right\} = 0 \tag{48}$$

Re-arranging gives:

$$\frac{p_{j,t}(\iota)}{p_{j,t}} = \frac{\epsilon_{j,p}}{(1+\phi_{jp})(\epsilon_{jp}-1)} \frac{h_{j,p1,t}}{h_{j,p2,t}}$$

where

$$h_{j,p1,t} = E_t \sum_{s=0}^{\infty} (\beta\theta_{pj})^s \Lambda_{t+s} \left(\frac{\Omega_{j,t,t+s}}{\pi_{j,t,t+s}}\right)^{-\epsilon_{jp}} mc_{j,t+s}\gamma_{j,t+s}y_{j,t+s} \tag{49}$$

$$h_{j,p2,t} = E_t \sum_{s=0}^{\infty} (\beta\theta_{pj})^s \Lambda_{t+s} \left(\frac{\Omega_{j,t,t+s}}{\pi_{j,t,t+s}}\right)^{1-\epsilon_{jp}} \gamma_{j,t+s}y_{j,t+s} \tag{50}$$

Note that:

$$h_{j,p1,t} = \Lambda_t mc_{j,t}\gamma_{j,t}y_{j,t} + \beta\theta_{pj} E_t \left(\frac{\Omega_{j,t,t+1}}{\pi_{j,t,t+1}}\right)^{-\epsilon_{jp}} h_{j,p1,t+1} \tag{51}$$

$$h_{j,p2,t} = \Lambda_t \gamma_{j,t}y_{j,t} + \beta\theta_{pj} \left(\frac{\Omega_{j,t,t+1}}{\pi_{j,t+1}}\right)^{1-\epsilon_{jp}} h_{j,p2,t+1} \tag{52}$$

Note also that the domestic price index can be expressed as:

$$1 = (1-\theta_{jp}) \left(\frac{p_{j,t}(\iota)}{p_{j,t}}\right)^{1-\epsilon_p} + \theta_{jp} \left(\frac{(\pi_{j,t-1})^{\varphi_{jp}}}{\pi_{j,t}}\right)^{1-\epsilon_{jp}} \tag{53}$$

### 1.1.5 Market clearing and aggregate price indices:

Goods market clearing requires that:

$$y_{j,t} = c_{j,t} + i_{j,t} + \sum_{k=1}^{\mathcal{F}} x_{j,k} \tag{54}$$

### 1.1.6 Monetary policy:

The monetary policy authority follows the policy rule:

$$\frac{R_t}{\bar{R}} = \left[\frac{R_{t-1}}{\bar{R}}\right]^{\rho^R} \left[(\Pi_t)^{\phi^\pi} (GAP_t)^{\phi^{gap}}\right]^{1-\rho^R} e^{\varepsilon^R_{j,t}} \tag{55}$$

where $GAP_t = Y_t/Y_t^{flex}$ is the output gap, defined as the deviation of real GDP from its flexible price level (defined below).

### 1.1.7 Fiscal policy:

The government budget constraint is:

$$\frac{B_{t+1}}{R_t} = B_t + P_{G,t}G_t - T_t \tag{56}$$

I assume that in steady state, government bonds are in zero net supply, so that $B_t = 0 \forall t$.
Fiscal policy purchases goods and services, $G_t$, according to the aggregate:

$$G_t = \left[\sum_{j=1}^{\mathcal{F}} \omega_{g,j}^{\frac{1}{\eta}} g_{j,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{57}$$

Implying the price index:

$$P_t^g = \left[\sum_{j=1}^{\mathcal{F}} \omega_{g,j} p_{j,t}^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{58}$$

and final output demands:

$$g_{j,t} = \omega_{g,j}\left(\frac{p_{j,t}}{P_t^g}\right)^{-\eta} G_t \tag{59}$$

Aggregate government spending evolves according to:

$$\frac{G_t}{G} = \left[\frac{G_{t-1}}{G}\right]^{\rho_g} \exp(\varepsilon_t^g) \tag{60}$$

Transfers consist of transfers to Ricardian and non-Ricardian households:

$$T_t = T_t^r + T_t^{nr} \tag{61}$$

### 1.1.8 Flexible price allocation

Flexible price variables are defined by the superscript $flex$. All equations are the same, except that in the flexible price system marginal costs are always at their steady state level and the marginal efficiency of labour equals its marginal product in all industries. This implies that:

$$MC = 0 \tag{62}$$

and

$$A_N N_{t+1}^{flex,\nu-\frac{1}{\zeta}} n_{j,t+s}^{flex,\frac{1+\zeta}{\zeta}} = \Lambda_{t+1}^{flex} \frac{w_{j,t+s}^{flex}}{P_{C,t+s}^{flex}} n_{j,t+s}^{flex} \tag{63}$$

## 1.2 Steady state

The steady state of the system is given by:
From the first order condition for bond holdings:

$$R = 1/\beta \tag{64}$$

From the first order condition for investment

$$q_j = 1 \tag{65}$$

From the first order condition for capital:

$$r_j^k = \mathcal{M}\left(\frac{1}{\beta} - 1 + \delta\right) \tag{66}$$

From the consumption choice for Ricardian households:

$$\frac{1-\beta h}{C^r(1-h)} = \frac{\Lambda^r}{e^{\xi_c}} \tag{67}$$

From the budget constraint for for non-Ricardian households:

$$C_t^{nr} = W_t N_t^{nr} + T_t^{nr} \tag{68}$$

From the consumption choice of non-Ricardian households

$$\frac{1-\beta h}{C^{nr}(1-h)} = \frac{\Lambda^{nr}}{e^{\xi_c}} \tag{69}$$

I set the level of transfers so that the marginal utility of consumption for Ricardian and non-Ricardian households are equal, i.e. $\Lambda^r = \Lambda^{nr} = \Lambda$.
From the definition of aggregate marginal utility:

$$\Lambda = \omega_r \Lambda^r + (1-\omega_r)\Lambda^{nr} \tag{70}$$

From the definition of aggregate consumption:

$$C = \omega_r C^r + (1-\omega_r)C^{nr} \tag{71}$$

From the wage choice:

$$\omega_{wj}^{\frac{1}{\xi}} A_N N^{\nu+\frac{1}{\xi}} n_j^{-\frac{1}{\xi}} = \Lambda w_j \tag{72}$$

From the definition of aggregate labour supply:

$$N = \left[\sum_{j=1}^{\mathcal{F}} \omega_{nj}^{-\frac{1}{\xi}} n_j^{\frac{\xi+1}{\xi}}\right]^{\frac{\xi}{\xi+1}} \tag{73}$$

From the capital accumulation condition:

$$z_j = \delta k_j \tag{74}$$

From the market clearing condition for investment:

$$I = \sum_{j=1}^{\mathcal{F}} z_j \tag{75}$$

From the demand function for consumption:

$$c_j = \omega_{cj} \gamma_j^{-\eta} C \tag{76}$$

From the demand function for investment:

$$i_j = \omega_{ij} \left(\frac{\gamma_j}{\gamma_I}\right)^{-\eta} I \tag{77}$$

From the demand function for government expenditure:

$$g_j = \omega_{g,j} \left(\frac{\gamma_j}{\gamma_G}\right)^{-\eta} G \tag{78}$$

From the price index for consumption:

$$1 = \left[ \sum^{\mathcal{N}} \omega_{c,j} \gamma_j^{1-\eta} \right] \tag{79}$$

From the price index for investment:

$$\gamma_I = \left[ \sum^{\mathcal{N}} \omega_{i,j} \gamma_j^{1-\eta} \right] \tag{80}$$

From the price index for government expenditure:

$$\gamma_G = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{g,j} \gamma_j^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{81}$$

From the definition of aggregate wages:

$$W = \left[ \sum^{\mathcal{N}} \omega_{l,j} w_j^{1+\xi} \right]^{\frac{1}{1+\xi}} \tag{82}$$

From the production function:

$$y_j = a_j \left[ \omega_{y,j}^{\frac{1}{\varphi}} y_j^{va\,\frac{\varphi-1}{\varphi}} + (1-\omega_{y,j})^{\frac{1}{\varphi}} x_j^{\frac{\varphi-1}{\varphi}} \right]^{\frac{\varphi}{\varphi-1}} \tag{83}$$

From the demand functions for intermediate goods:

$$x_{kj} = \omega_{kj} \left( \frac{\gamma_k}{\gamma_j} \right)^{-\psi} x_j \tag{84}$$

From the demand function for capital:

$$k_j = (1-\omega_n) \left( \frac{r_j^k}{\gamma_j^{yva}} \right)^{-\zeta} y_j^{va} \tag{85}$$

From the demand function for labour:

$$n_j = \omega_{nj} \left( \frac{w_j}{\gamma_j^{yva}} \right)^{-\zeta} y_j^{va} \tag{86}$$

From the definition of the price index of intermediate goods:

$$\gamma_j^x = \left[ \sum_{j=1}^{\mathcal{F}} \omega_{kj} \gamma_k^{1-\psi} \right]^{\frac{1}{1-\psi}} \tag{87}$$

From the definition of the price of value added:

$$\gamma_j^{yva} = \left[ \omega_{nj} w_j^{1-\zeta} + (1-\omega_{nj}) r_j^{k\,1-\zeta} \right]^{\frac{1}{1-\zeta}} \tag{88}$$

From the relative demand for inputs:

$$(\gamma_j^{yva})^{\varphi} y_j^{va} = \frac{\omega_{y,j}}{1-\omega_{y,j}} (\gamma_j^{x})^{\varphi} x_j \tag{89}$$

From goods market clearing:

$$y_j = c_j + i_j + g_j + \sum_{k=1}^{\mathcal{F}} x_{j,k} \tag{90}$$

From the definition of $h_{j,p2}$:

$$h_{j,p2} = \frac{\Lambda \gamma_j y_j}{1-\beta\theta_{pj}} \tag{91}$$

From the price index for good $j$:

$$h_{j,p2} = h_{j,p1} \tag{92}$$

From the definition of $h_{j,p1}$:

$$mc_j = 1 \tag{93}$$

From the definition of marginal costs:

$$a_j = \left[\omega_{y,j}(\gamma_j^{yva})^{1-\varphi} + (1-\omega_{y,j}(\gamma_j^{x})^{1-\varphi}\right]^{\frac{1}{1-\varphi}} \tag{94}$$

## 1.3 Linearised equations

Capital accumulation:

$$\hat{k}_{j,t+1} - \delta\hat{z}_{j,t} = (1-\delta)\hat{k}_{j,t} \tag{95}$$

Consumption price index:

$$0 = \sum_{j=1}^{\mathcal{F}} \omega_{c,j}\gamma_j^{1-\eta}\hat{\gamma}_{j,t} \tag{96}$$

Investment price index:

$$\gamma_I^{1-\eta} - \sum_{j=1}^{\mathcal{F}} \omega_{i,j}\gamma_j^{1-\eta}\hat{\gamma}_{j,t} = 0 \tag{97}$$

Government price index:

$$\gamma_G^{1-\eta} - \sum_{j=1}^{\mathcal{F}} \omega_{g,j}\gamma_j^{1-\eta}\hat{\gamma}_{j,t} = 0 \tag{98}$$

Consumption variety choice:

$$\hat{c}_{j,t} = \hat{C}_t - \eta\hat{\gamma}_{j,t} \tag{99}$$

Investment variety choice:

$$\hat{i}_{j,t} = \hat{I}_t - \eta(\hat{\gamma}_{j,t} - \hat{\gamma}_{I,t}) \tag{100}$$

Government expenditure variety choice:

$$\hat{g}_{j,t} = \hat{G}_t - \eta(\hat{\gamma}_{j,t} - \hat{\gamma}_{G,t}) \tag{101}$$

Aggregate labour supply:

$$N^{\frac{\xi+1}{\xi}}\hat{n}_t - \sum_{j=1}^{\mathcal{F}} \omega_{nj}^{-\frac{1}{\xi}} n_j^{\frac{\xi+1}{\xi}}\hat{n}_{j,t} = 0 \tag{102}$$

Investment price inflation:

$$\hat{\pi}_{I,t} - \hat{\pi}_t - \hat{\gamma}_{I,t} = -\hat{\gamma}_{I,t-1} \tag{103}$$

Wage inflation:

$$\hat{\pi}_{W,t} - \hat{\pi}_t - \hat{w}_t = -\hat{w}_{t-1} \tag{104}$$

Aggregate wage index:

$$W^{1+\xi}\hat{w}_t = \sum_{j=1}^{\mathcal{F}} \omega_{nj} w_j^{1+\xi}\hat{w}_{j,t} \tag{105}$$

Investment market clearing:

$$I\hat{i}_t - \sum_{j=1}^{\mathcal{F}} z_j \hat{z}_{j,t} = 0 \tag{106}$$

Consumption choice for Ricardian consumers:

$$h\hat{c}^r_{t-1} + \beta h E_t\{\hat{c}^r_{t+1}\} = (1+\beta h^2)\hat{c}^r_t + (1-h)(1-\beta h)\hat{\lambda}^r_t - (1-h)(\hat{\xi}_{c,t} - \beta\hat{\xi}_{c,t+1}) \tag{107}$$

Euler equation for Ricardian consumers:

$$\hat{\lambda}^r_t = \hat{r}_t + E_t\{\hat{\lambda}^r_{t+1}\} - E_t\{\hat{\pi}_{t+1}\} \tag{108}$$

Capital stock choice for Ricardian consumers:

$$\hat{\lambda}^r_t + \hat{q}_{j,t} = E_t\{\hat{\lambda}^r_{t+1}\} + \beta(1-\delta)E_t\{\hat{q}_{j,t+1}\} + \frac{\beta r^K_j}{\mathcal{M}} E_t\{\hat{r}^k_{j,t+1}\} \tag{109}$$

Relationship between capital supplied to firms and total capital stock:

$$\hat{k}^s_{j,t} = u_{j,t} + \hat{k}_{j,t} \tag{110}$$

Capital utilisation in industry $j$:

$$\mathcal{A}\hat{r}^k_{j,t} = \hat{u}_{j,t} \tag{111}$$

where $\mathcal{A}$ controls the degree of capital utilisation costs.
Investment choice:

$$(1+\beta)\hat{z}_{j,t} = \frac{\hat{q}_{j,t}}{S''} + \beta E_t\{\hat{z}_{j,t+1}\} + \hat{z}_{j,t-1} \tag{112}$$

Consumption choice for non-Ricardian consumers

$$C^{nr}\hat{c}^{nr}_t - WN(\hat{w}_t + \hat{n}_t) - TRANS\hat{trans}_t = 0 \tag{113}$$

Marginal utility of consumption for non-Ricardian consumers

$$h\hat{c}^{nr}_{t-1} + \beta h E_t\{\hat{c}^{nr}_{t+1}\} = (1+\beta h^2)\hat{c}^{nr}_t + (1-h)(1-\beta h)\hat{\lambda}^{nr}_t - (1-h)(\hat{\xi}_{c,t} - \beta\hat{\xi}_{c,t+1}) \tag{114}$$

Aggregate consumption:

$$\hat{c}_t - \omega_r \frac{C^r}{C}\hat{c}^r_t - (1-\omega_r)\frac{C^{nr}}{C}\hat{c}^{nr}_t = 0 \tag{115}$$

Aggregate marginal utility:

$$\hat{\lambda}_t - \omega_r \frac{\lambda^r}{\Lambda}\hat{\Lambda}^r_t - (1-\omega_r)\frac{\Lambda^{nr}}{\Lambda}\hat{\lambda}^{nr}_t = 0 \tag{116}$$

Wages choice:

$$\hat{\pi}^w_{j,t} - \frac{\beta}{1+\beta\chi_w}E_t\{\hat{\pi}^w_{j,t+1}\} - \frac{\kappa_{w,j}}{(1+\beta\chi_w)}\left[-\hat{\lambda}_t + (\nu - \frac{1}{\xi})\hat{n}_t + \frac{1}{\xi}\hat{n}_{j,t} - \hat{w}_{j,t}\right] = \frac{\chi_w}{1+\beta\chi_w}\hat{\pi}^w_{j,t-1} \tag{117}$$

where $\kappa_{w,j} = \frac{\xi}{\xi+\epsilon_w}(1-\beta\theta_{w,j})(1-\theta_{w,j})/\theta_{w,j}$
Gross output in sector j:

$$y_j^{\frac{\varphi-1}{\varphi}}\hat{y}_{j,t} - a_j^{\frac{\varphi-1}{\varphi}}\hat{a}_{j,t} - \omega_{y,j}^{\frac{1}{\varphi}}(y_j^{va})^{\frac{\varphi-1}{\varphi}}\hat{y}^{va}_{j,t} - (1-\omega_{y,j})^{\frac{1}{\varphi}}x_j^{\frac{\varphi-1}{\varphi}}\hat{x}_{j,t} = 0 \tag{118}$$

Marginal costs in sector j:

$$\begin{aligned}\hat{mc}_{j,t} + \hat{a}_{j,t} - \omega_{y,j}(\gamma_j^{va}/a_j\gamma_j)^{1-\varphi}\hat{\gamma}^{va}_{j,t} \\ - (1-\omega_{y,j})(\gamma_j^x/a_j\gamma_j)^{1-\varphi}\hat{\gamma}^x_{j,t} + \hat{\gamma}_{j,t} = 0\end{aligned} \tag{119}$$

Factor demand in sector j:

$$\varphi\hat{\gamma}^{va}_{j,t} + \hat{y}^{va}_{j,t} - \varphi\hat{\gamma}^x_{j,t} - \hat{x}_{j,t} = 0 \tag{120}$$

Value added price index in sector j:

$$\left(\gamma_j^{va}\right)^{1-\zeta}\hat{\gamma}^{va}_{j,t} - \omega_{nj}w_j^{1-\zeta}\hat{w}_{j,t} - (1-\omega_{nj})(r_j^k)^{1-\zeta}\hat{r}^k_{j,t} = 0 \tag{121}$$

Intermediate good price index in sector j:

$$(\gamma_j^x)^{1-\psi}\hat{\gamma}^x_{j,t} - \sum_{k=1}^{\mathcal{F}}\omega_{k,j}(\gamma_k)^{1-\psi}\hat{\gamma}_{k,t} = 0 \tag{122}$$

Labour demand in sector j:

$$\hat{y}^{va}_{j,t} - \zeta\hat{w}_{j,t} + \zeta\hat{\gamma}^{va}_{j,t} - \hat{n}_{j,t} = 0 \tag{123}$$

Capital demand in sector j:

$$\hat{y}^{va}_{j,t} - \zeta\hat{r}^k_{j,t} + \zeta\hat{\gamma}^{va}_{j,t} = \hat{k}_{j,t} \tag{124}$$

Intermediate good k demand in sector j:

$$\hat{x}_{j,t} - \hat{x}_{kj,t} - \psi\hat{\gamma}_{k,t} + \psi\hat{\gamma}^x_{j,t} = 0 \tag{125}$$

Defintion of relative price in sector j:

$$\hat{\gamma}_{j,t} - \hat{\pi}_{j,t} + \hat{\pi}_t = \hat{\gamma}_{j,t-1} \tag{126}$$

Phillips curve in sector j:

$$\hat{\pi}_{j,t} - \frac{\beta}{1+\beta\chi_p}E_t\{\hat{\pi}_{j,t+1}\} - \frac{(1-\theta)(1-\beta\theta)}{\theta(1+\beta\chi_p)}\hat{mc}_{j,t} = \frac{\chi_p}{1+\beta\chi_p}\hat{\pi}_{j,t-1} \tag{127}$$

Link between wage inflation and real wages in sector j:

$$\pi^w_{j,t} - w_{j,t} - \pi_t = -w_{j,t-1} \tag{128}$$

Market clearing in sector j:

$$y_j\hat{y}_{j,t} - c_j\hat{c}_{j,t} - i_j - \hat{i}_{j,t} - g_j\hat{g}_{j,t} - \sum_{k=1}^{\mathcal{F}} x_{jk}\hat{x}_{jk,t} = 0 \tag{129}$$

Taylor rule:

$$\hat{r}_t - (1-\rho_r)(\phi_\pi\hat{\pi}_t + \phi_{gap}g\hat{a}p_t) + \varepsilon_{r,t} = \rho_r\hat{r}_{t-1} \tag{130}$$

Output gap:

$$g\hat{a}p_t = \hat{y}_t - \hat{y}_t^{flex} \tag{131}$$

**Additional aggregate variables:**
Year-ended inflation:

$$\hat{\pi}_t^{ye} = \hat{\pi}_t + \hat{\pi}_{t-1} + \hat{pi}_{t-2} + \hat{\pi}_{t-3} \tag{132}$$

Aggregate value added:

$$y_t^{va} - \sum_{j=1}^{\mathcal{F}} nva_j\hat{y}^v a_{j,t} = 0 \tag{133}$$

where $nva_j$ is the steady-state share of sector j in nominal GDP.
**Shock processes:**
Productivity in sector j:

$$\hat{a}_{j,t} = \rho_{aj}\hat{a}_{j,t-1} + \varepsilon_{aj,t} \tag{134}$$

Aggregate government expenditure:

$$\hat{g}_t = \rho_g\hat{g}_{t_1} + \varepsilon_{gt} \tag{135}$$

Transfers:

$$trans_t = \rho_{trans}trans_{t-1} + \varepsilon_{trans,t} \tag{136}$$

# 2 Alternative technology specifications

This section derives the model for an alternative specification for technology, namely:

$$y_{j,t}^{va}(\iota) = \left[\omega_{n,j}^{\frac{1}{\zeta}}\left(a_{j,t}^{lap}n_{j,t}(\iota)\right)^{\frac{\zeta-1}{\zeta}} + (1-\omega_{n,j})^{\frac{1}{\zeta}}\left(a_{j,t}^{kap}k_{j,t}^{s}(\iota)\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}} \tag{137}$$

$$x_{j,t}(\iota) = \left[\sum_{k=1}^{\mathcal{F}}\omega_{k,j}^{\frac{1}{\psi}}x_{k,j,t}(\iota)^{\frac{\psi-1}{\psi}}\right]^{\frac{\psi}{\psi-1}} \tag{138}$$

$$y_{j,t}(\iota) = a_j\left[\omega_{y,j}^{\frac{1}{\varphi}}y_{j,t}^{va}(\iota)^{\frac{\varphi-1}{\varphi}} + (1-\omega_{y,j})^{\frac{1}{\varphi}}x_{j,t}(\iota)^{\frac{\varphi-1}{\varphi}}\right]^{\frac{\varphi}{\varphi-1}} \tag{139}$$

where $a_{j,t}^{lap}$ and $a_{j,t}^{kap}$ are labour and capital augmenting productivity in industry $j$ at time $t$.
With this specification of productivity, the price indices for value added become:

$$p_t^{yva} = \left[\omega_{n,j}\left(a_{j,t}^{lap}\right)^{\zeta-1}w_{j,t}^{1-\zeta} + (1-\omega_{n,j})\left(a_{j,t}^{kap}\right)^{\zeta-1}r_{j,t}^{k1-\zeta}\right]^{\frac{1}{1-\zeta}} \tag{140}$$

and factor demands are:

$$n_{j,t} = \omega_{n,j} \left(a_{j,t}^{lap}\right)^{\zeta-1} \left[\frac{w_{j,t}}{p_{j,t}^{yva}}\right]^{-\zeta} y_{j,t}^{va} \tag{141}$$

$$k_{j,t} = (1-\omega_{n,j}) \left(a_{j,t}^{kap}\right)^{\zeta-1} \left[\frac{r_{j,t}^{k}}{p_{j,t}^{yva}}\right]^{-\zeta} y_{j,t}^{va} \tag{142}$$

The steady state of these equations follows naturally.
The log-linearised price index for the value-added price index is:

$$\begin{aligned} 0 = \left[\gamma_j^{va}\right]^{(1-\zeta)} \hat{\gamma}_{j,t}^{va} - \omega_{n,j} \left[\frac{w_j}{a_j^{lap}}\right]^{1-\zeta} \left(\hat{w}_{j,t} - \hat{a}_{j,t}^{lap}\right) \\ - (1-\omega_{n,j}) \left[\frac{r_j^k}{a_j^{kap}}\right]^{1-\zeta} \left(\hat{r}_{j,t}^{k} - \hat{a}_{j,t}^{kap}\right) \end{aligned} \tag{143}$$

and for labour and capital demand:

$$\hat{n}_{j,t} + (1-\zeta)\hat{a}_{j,t}^{lap} + \zeta\hat{w}_{j,t} - \zeta\hat{\gamma}_{j,t}^{va} - \hat{y}_{j,t}^{va} = 0 \tag{144}$$

$$\hat{k}_{j,t}^{s} + (1-\zeta)\hat{a}_{j,t}^{kap} + \zeta\hat{r}_{j,t}^{k} - \zeta\hat{\gamma}_{j,t}^{va} - \hat{y}_{j,t}^{va} = 0 \tag{145}$$