Residual risk factors, portfolio composition and risk measurement

by

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Abstract

When risk managers develop firm-wide measures of risk, the efforts can impose substantial costs both of developing information systems and, on an ongoing basis, of computation and aggregation. These costs can lead risk managers to base risk measures on a set of risk factors (including asset prices) of lower dimension than the dimension of underlying sources of risk. Such truncation of the set of risk factors, however, could cause risk measures to systematically underestimate a portfolio's risk. This paper presents examples where risk aversion leads a firm to hedge risk factors that have high explanatory power for many asset returns; however, the firm may remain exposed to other, less-important risk factors if their market price of risk is sufficiently high. Statistical techniques for identifying sources of risk that choose risk factors based on the variability of asset prices without taking account of the market price of risk could systematically underestimate portfolio risks.

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1. Introduction

Two questions regarding measures of portfolio risk that are not equivalent are:

- What price shocks would lead to large losses?
- What losses would be caused by the class of price shocks that have occurred with sufficient regularity in historical data to be identified by statistical techniques?

The price shocks identified by the first question may be different from the price shocks used in the second question. At least two reasons for the difference can be mentioned.

- Selection bias: dealers' trading and hedging strategies may be conditioned on the same statistical regularities identified by the designers of stress tests. The dependence of observed statistical regularities on the sample period could cause stress tests to fail to identify the shock in the first question. See, Mahoney, 1996.[1]
- Portfolio composition and aggregation of shocks to risk factors.

This note addresses the last point, how portfolio composition determines the aggregation of shocks to the risk factors that drive asset price volatility. Computation burden and the information-system costs of firm-wide aggregation of risk can cause risk managers to construct risk measures that parsimoniously reduce the number of risk factors to a smaller dimension than the dimension of asset prices. Such truncation of the set of risk factors, however, could cause risk measures to systematically underestimate a portfolio's risk. This underestimate can occur even when the exercises uses the full dimensionality of the portfolio's sensitivity to asset prices (i.e. the sensitivity to every asset price is accounted for).

If a firm's risk aversion causes it to hedge risk factors that have high explanatory power in the variability of asset returns, smaller risk factors that appear to be less consequential may remain unhedged if those factors have a market price of risk. Statistical techniques that summarise the variability of asset prices without taking into account the sizes of market prices of risk could then systematically underestimate portfolio risks if the techniques ignore some factors with a market price of risk. While this claim should not be surprising, its implications in the measurement of portfolio risk should not be overlooked by designers of risk measures who may need to reduce the dimensionality of the measurement exercise due to constraints of data availability or computation burden. The following two examples illustrate this claim.

[1] "Empirical-based versus model-based approaches to value-at-risk: An examination of foreign exchange and global equity portfolios", James Mahoney; in *Risk measurement and systemic risk: Proceedings of a joint central bank research conference*. Board of Governors of the Federal Reserve System (1996).

2. An example with financing constraints

Our first example considers portfolio risks in a setting where the portfolio manager faces financing constraints.

Asset prices

Consider a portfolio that contains the four assets, whose returns are described by four independent random risk factors, f_j , j=1,..,4,

$$c_1 = r^* + (\lambda_1 + f_1)$$
$$c_2 = r^* + a_1(\lambda_1 + f_1) + (\lambda_2 + f_2)$$
$$y_1 = r^* + (\lambda_1 + f_1) + a_2(\lambda_2 + f_2) + (\lambda_3 + f_3)$$
$$y_2 = r^* + b_1 (\lambda_1 + f_1) + (\lambda_2 + f_2) + a_3(\lambda_3 + f_3) + (\lambda_4 + f_4)$$

where the remaining terms are constants. The constant terms λ_j are the market price of risk of the risk factors, f_j, and r* is the riskless interest rate. Each asset return is influenced by its "own" risk factor, and some are also influenced by other assets' risk factors as well. The market price of risk of each risk factor is assumed to be determined by its volatility and correlation with the return of a market portfolio consisting of equal amounts of all four assets.

Assume that the risk factors f_1 and f_2 explain more than 90% of the variability of each of the four asset prices, as is the case with the following parameter values: $a_1 = 0.5$, $a_2 = 0.25$, $a_3 = 0.25$, $b_1 = 0.75$, $\sigma_1 = 0.01$, $\sigma_2 = 0.01$, $\sigma_3 = 0.0033$, and $\sigma_4 = 0.004$, where σ_i is the standard deviation of the risk factor f_i , $E(f_i) = 0$, and $Cov(f_i,f_j) = 0$.

With the assumed parameter values, the two risk factors f_1 and f_2 would explain more than 90% of the variability of y_1 and more than 90% of the variability of y_2, while explaining 100% of the variability of c_1 and c_2 (see Table 1). In other words, the variability of all asset returns can be very well described by only two risk factors.[2] With this excellent explanatory power of only two risk factors, an analyst might be tempted to dismiss the remaining risk factors as unimportant residual terms. Can such residual risks be ignored in considering specific risk?

Table 1

Explanatory power of risk factors in *asset* returns

	c_1	c_2	y_1	y_2
Variance of asset returns	*0.01*	*0.0112*	*0.0108*	*0.0131*
Proportion of variance explained by: f_1 and f_2	1	1	0.91	0.90
Proportion of variance explained by: f_1 and f_3	1	0.2	0.95	0.33
Proportion of variance explained by: f_1, f_2, and f_3	1	1	1	0.91

2 The correlations between the asset prices are, $corr(c_1, y_1) = 0.93$, $corr(c_1, y_2) = 0.76$, $corr(c_1, c_2) = 0.45$ and $corr(c_2, y_2) = 0.94$.

Portfolio allocation

The risk factors f_1 and f_2 can be interpreted as market risk factors, since they have a large influence on all asset prices. The other risk factors represent spread risk. Consider the portfolio return described by,

$$y = [y_1 A + y_2(1-A) - c_1 B - c_2(1-B)] N, \tag{1}$$

where A and B are choice variables that determine portfolio composition, $0 \leq A \leq 1$ and $0 \leq B \leq 1$, and N is the portfolio size. This profit function has the following interpretations.

> *Interpretation 1*: A bank can invest and fund its positions in different markets (countries). $\mathbf{y}_j$ is the investment return in market j, and $\boldsymbol{c}_j$ is the funding cost in market j, for j = 1,2. In contrast to the model in section 3, here the bank does not have access to funding at the risk-free rate. Hence, its portfolio must earn a return above the risk-free rate with the corresponding risk.
>
> *Interpretation 2*: A bank takes long positions in two assets with returns y_j, j=1,2, and short positions in two assets with returns c_j, j=1,2.

Variance of portfolio return

Depending on portfolio composition, the risk factors that appear to have high explanatory power in asset returns can have much smaller explanatory power in portfolio returns. Table 2 compares portfolio risk and estimates of that risk based on an incomplete model (with the portfolio size variable N=100). Panel A shows results in terms of portfolio variance for contrast with Table 1, while Panel B shows results in terms of portfolio standard deviation. The portfolio weights in Table 2, are optimal portfolios corresponding to different values of the risk aversion parameter (ρ) in the utility function,

$$U = E(y(A, B, f_1, f_2, f_3, f_4)) - \frac{\rho}{2} V(y(A, B, f_1, f_2, f_3, f_4)) \tag{2}$$

where *E* and *V* denote expected value and variance, and *y* is portfolio returns as defined in (1), where portfolio size is held constant.

While the first two risk factors explain more than 90% of the variability of all asset prices, their ability to describe the risks in portfolio returns can be much smaller. In the case of the portfolio chosen by a moderately risk averse firm, the first two risk factors explain only 31% of the variance of portfolio returns (56% in terms of portfolio standard deviation). Moreover, adding the third risk factor would increase the explanatory power to only 47% of the variance of portfolio returns (69% in terms of portfolio standard deviation).

Table 2

Explanatory power of risk factors in *portfolio* returns

Panel A: ***Variance*** **of portfolio returns**				
	Risk neutrality: A=0 B=1	Slightly risk averse: A=0 B=0.4	Moderate risk aversion: A=0.3 B=0.4	Extreme risk aversion: A=0.7 B=0.6
Portfolio variance	*1.23*	*0.33*	*0.15*	*0.10*
Proportion of variance explained by: f_1 and f_2	0.86	0.49	0.31	0.21
Proportion of variance explained by: f_1, f_2, and f_3	0.87	0.51	0.47	0.85
Panel B: ***Standard deviation*** **of portfolio returns**				
	Risk neutrality: A=0 B=1	Slightly risk averse: A=0 B=0.4	Moderate risk aversion: A=0.3 B=0.4	Extreme risk aversion: A=0.7 B=0.6
Portfolio Standard Deviation, $\sigma(y)$	*1.11*	*0.57*	*0.39*	*0.31*
Proportion of $\sigma(y)$ explained by: f_1 and f_2	0.93	0.70	0.56	0.46
Proportion of $\sigma(y)$ explained by: f_1, f_2, and f_3	0.93	0.72	0.69	0.92

Another feature of these results is that the amount of risk left unaccounted for does not always decline as portfolio risk decreases. Table 3 shows the amount of portfolio risk (as measured by portfolio standard deviation), as well as modelled risk (using f_1 and f_2 only) and the amount of risk left unmeasured. The last row of Table 3 shows that the amount of risk left unaccounted for need not decrease as portfolio becomes risk becomes smaller.

Table 3

The effect of neglected risk factors: portfolio risk left unmeasured

	Risk neutrality: A=0 B=1	Slightly risk averse: A=0 B=0.4	Moderate risk aversion: A=0.3 B=0.4	Extreme risk aversion: A=0.7 B=0.6
Portfolio Standard Deviation, $\sigma(y)$	*1.11*	*0.57*	*0.39*	*0.31*
Modelled Standard Deviation using f_1 and f_2	*1.03*	*0.40*	*0.22*	*0.14*
Risk left unmeasured	*0.08*	*0.17*	*0.17*	*0.17*

Stress test results

As with the variance of portfolio returns, risk factors that have high explanatory power in asset prices can have much smaller explanatory power in stress tests of portfolio returns. The results in Table 4 apply to the moderately risk averse portfolio (A=0.3, B=0.4).

Table 4

The effect of neglected risk factors: stress tests of portfolio returns

Actual shock in risk factors	Stress test specification	Proportion of actual change predicted by the stress test
$f_i = \sigma_i$, i = 1 to 4	$f_i = \sigma_i$, i = 1,2 $f_j = 0$, j = 3,4	0.41
Same as above	$f_i = \sigma_i$, i = 1,2,3 $f_4 = 0$	0.62

In contrast to their higher explanatory power in the space of asset prices, the truncated set of risk factors has much weaker explanatory power in stress tests of portfolio returns. For example, while the first two risk factors account for 79% of the change in y_1 and 78% of the change in y_2 due to a one standard deviation shock to all risk factors, they account for only 41% of the actual change in portfolio value.[3] Adding the third risk factor to the stress test, would account for all the change in y_1 and 82% of the change in y_2, but yet account for only 62% of the true change in portfolio value.

3. Portfolio risk

Like the first example, our second example investigates sources of portfolio risk when market returns follow the restrictions of linear arbitrage pricing theory and the portfolio is selected by a risk averse portfolio manager. The difference between the two examples is the unconstrained choice in the second example, where the portfolio manager can choose a portfolio with the risk-free return. We demonstrate that portfolio risk will likely be mismeasured by risk-management methodologies that do not include all sources of non-diversifiable risk in asset returns. Specifically, the example suggests that an analysis of portfolio risk that uses only factors accounting for a large fraction of return variance or that leaves out factors with high expected returns will often understate portfolio risk.

[3] The first two risk factors account for 100% if the variability of c_1 and c_2.

Asset returns

We assume that returns on the n risky assets follow a k factor model:

$$r = r^e + \beta f + \varepsilon$$

where returns above the risk-free rate (r), expected excess returns (r^e) above the risk-free rate, and idiosyncratic errors (ε) are n x 1 vectors; the factors (f) are a k x 1 vector; and the matrix of factor loadings (β) is a n x k matrix. The expected value of the factors and idiosyncratic errors is zero. For convenience, we assume that the factors are uncorrelated both with each other and with the idiosyncratic errors, that the factors are normalised so $\beta'\beta = I_k$ (the k dimension identity matrix), and that the variances of the idiosyncratic errors are equal so $Var(\varepsilon) = \sigma_\varepsilon^2 I_n$ (i.e. proportional to the n x n identity matrix).

Thus, the variance of excess returns is given by:

$$Var(r) = \beta\, Var(f)\, \beta' + Var(\varepsilon) = \beta\, Var(f)\, \beta' + \sigma_\varepsilon^2 I_n$$

where $Var(f)$ is a diagonal matrix. The normalisation assumption on β and the correlation assumptions for f imply that β is the matrix of eigenvectors of $\beta\, Var(f)\, \beta'$ corresponding to the (positive) eigenvalues on the diagonal of $Var(f)$. The full decomposition is given by:

$$\beta Var(f)\beta\beta' = \begin{bmatrix}\beta & \tilde{\beta}\end{bmatrix}\begin{bmatrix}Var(f) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\beta & \tilde{\beta}\end{bmatrix} \text{ and } \begin{bmatrix}\beta & \tilde{\beta}\end{bmatrix}\begin{bmatrix}\beta'\\ \tilde{\beta}'\end{bmatrix} = \begin{bmatrix}\beta'\\ \tilde{\beta}'\end{bmatrix}\begin{bmatrix}\beta & \tilde{\beta}\end{bmatrix} = I_n$$

where $\tilde{\beta}$ is the n x (n-k) matrix of eigenvectors corresponding to the zero eigenvalues of $\beta Var(f)\beta'$. With these assumptions, the variance matrix of returns can be expressed as:

$$Var(r) = \sum_1^n \beta_j \beta_j' \; (\sigma_j^2 + \sigma_\varepsilon^2)$$

where $\sigma_j^2 = \begin{cases} Var(f_j), & j \le k \\ 0, & j > k \end{cases}$, either the variance of the j-th factor or 0, and β_j is the j-th eigenvector of $Var(r)$, a column of either β or $\tilde{\beta}$.[4] The inverse of the variance matrix of returns is:

$$Var^{-1}(r) = \sum_1^n \beta_j \beta_j' \; (\sigma_j^2 + \sigma_\varepsilon^2)^{-1}$$

A portfolio is defined by the shares (ω) held in the n risky assets. Returns on a portfolio are:

$$\omega'(r + r^0 i) + r^0(1-\omega' i) = \omega' r + r^0$$

where r^0 is the risk-free rate and i is a n x 1 vector of ones. Expected portfolio returns and the variance of returns are:

$$E(\omega' r + r^0) = \omega' r^e + r^0 \text{ and}$$

$$Var(\omega' r + r^0) = \omega'\beta(Var(f)\beta'\omega + \sigma_\varepsilon^2\, \omega'\omega.$$

Arbitrage profits could be obtained with these assets unless riskless portfolios earn the risk-free rate of interest (i.e., have a zero expected excess return). When the number of risky assets is large, idiosyncratic risk can be largely eliminated by diversification. For example, a portfolio whose

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4 The decompositions of A and A + bI are closely related, where A is a n x n matrix, b is a number, and I is the n x n identity matrix. It can be shown that the eigenvalues of A + bI equal b plus the eigenvalues of A while the eigenvectors of the two matrices are identical.

weights are $\tilde{\omega} = \tilde{\beta} i / (n - k)$ has variance $\sigma_\varepsilon^2 / (n - k)$ and will have little risk when the number of assets, n, is large. Thus, portfolios whose risk arises only from idiosyncratic risk will have a return equal to the risk-free rate.[5] A portfolio whose risk arises from idiosyncratic risk has weights on risky assets satisfying $\omega'\beta = 0$, with not all weights equal to zero. If these portfolios have zero expected excess return then $\omega' r^e = 0$. These conditions imply that the vector of expected excess returns, r^e, must be a linear combination of the columns of β. Thus, absence of arbitrage opportunities implies that expected returns satisfy:

$$r^e = \beta\lambda$$

where the elements of λ represent the marginal expected excess return from additional investments in portfolios reproducing particular factors.

Portfolio choice

To investigate the possible implications of choosing too few factors to describe returns and the risk of a portfolio, we consider portfolios formed by mean-variance optimisation. Mean-variance optimises generally choose portfolios to reduce risk unless the returns to bearing risk are sufficiently attractive. We use mean-variance optimisation to illustrate possible tradeoffs that might occur in real portfolio choices.

We assume that the portfolio manager selects the portfolio weights on risky assets, ω, to maximise an objective function that rewards expected return and penalises variance of returns:

$$E(\omega' r + r^0) - 0.5\rho Var(\omega' r + r^0).$$

In this objective function, the coefficient ρ describes risk aversion; managers with higher values of ρ are more risk averse. The optimising portfolio weights are:

$$\omega^{OPT} = (\rho Var(r))^{-1} r^e.$$

This well-known result shows that the portfolio manager tends to give higher weights to assets with high expected excess returns and lower weights to assets with high variances. This portfolio has realised return equal to:

$$r^e + \rho^{-1} r^{e\prime} Var(r)^{-1} r^e + \rho^{-1} r^{e\prime} Var(r)^{-1} (\beta f + \varepsilon)$$

with variance:

$$\rho^{-2} r^{e\prime}\, Var(r)^{-1} r^e.$$

If all market participants are mean-variance optimisers and if there are a large number of assets (none of which is a large part of the market portfolio), then in market equilibrium:

$$r^e = (\rho^{avg}\, Var(r))\, \omega^{market} \approx \beta\left[Var(f)\, \beta' \omega^{market} \rho^{avg}\right] = \beta\, \lambda$$

5 As the discussion suggests, this property holds exactly only in a limiting case as the number of assets grows. See J. Ingersoll, *Theory of financial decision making*, (Rowman & Littlefield, 1987), Chapter 7 for a more detailed discussion. We assume that the result holds exactly here to simplify the algebra that follows.

where ω^{market} represents the portfolio weights in the market portfolio and ρ^{avg} is a wealth-weighted average of market participants' risk aversion.[6] It follows that the elements of λ are proportional to the factor variances, or $\lambda_l = \sigma_l^2\ \kappa_l$, where the constant of proportionality, κ_l, will be large either if the factor has large β for many assets or if the large elements of β correspond to assets with large shares in the market portfolio. This second condition could occur if the relative factor loadings corresponded closely to the shares in the market portfolio.

Factor contributions to return and portfolio variance

We can combine these results to compare factor contributions to the variance of returns or to the variance of the portfolio. These contributions will suggest when a factor is more important to return or portfolio variances.

One measure of the contribution of a factor to a set of returns is the share of summed return variances that can be attributed to the factor. With the structure assumed above, this calculation is very easy because each factor is uncorrelated with the other sources of risk. Specifically, the fraction of total return variances contributed by factor *l* is given by:

$$\frac{trVar(\beta_l f_l)}{trVar(r)} = \frac{tr\beta_l \beta_l'\ \sigma_l^2}{tr\sum_{j=1}^{n}\beta_j\ \beta_j'\ (\sigma_j^2 + \sigma_\varepsilon^2)} = \frac{\sigma_l^2}{\sum_{j=1}^{n}(\sigma_j^2 + \sigma_\varepsilon^2)}$$

where tr represents the trace of a matrix - the sum of the diagonal elements.

Turning to portfolio returns, the variance of realised returns contributed by factor *l* is:

$$Var\,(\rho^{-1}\cdot(r^e)'\cdot Var\,(r)^{-1}\cdot\beta_l f_l) = \rho^{-2}\cdot(r^e)'\cdot Var\,(r)^{-1}\cdot Var\,(\beta_l f_l)\cdot Var\,(r)^{-1}\cdot r^e =$$
$$\rho^{-2}\cdot\sigma_l^2\cdot(r^e)'\cdot Var\,(r)^{-1}\cdot\beta_l\cdot\beta_l'\cdot Var\,(r)^{-1}\cdot r^e$$

This expression can be simplified considerably by using the expression for the inverse of $Var(r)$ derived above, the property that $\beta_j\ \beta_l = 0$, $j \neq l$, and the arbitrage-free value of r^e to obtain:

$$\rho^{-2}\cdot\sigma_l^2\cdot\frac{\lambda_l}{\left(\sigma_l^2 + \sigma_\varepsilon^2\right)^2}$$

where λ_l is marginal expected excess return for factor *l*.

The contribution of factor *l* to portfolio variance can be compared to the overall variance of the portfolio. Recall that the variance of the optimising portfolio is:

$$\rho^{-2}\ r^{e\,\prime}\ Var\,(r)^{-1}\ r^e$$

Substituting for the inverse of $Var(r)$ and for the arbitrage-free value of r^e gives the following expression for the portfolio variance:

$$\rho^{-2}\cdot(\beta\cdot\lambda)'\cdot\sum_{j=1}^{n}\beta_j\beta_j'\left(\sigma_j^2 + \sigma_\varepsilon^2\right)^{-1}\cdot\beta\cdot\lambda = \rho^{-2}\cdot\sum_{j=1}^{n}\lambda_j^2\left(\sigma_j^2 + \sigma_\varepsilon^2\right)^{-1}$$

where $\lambda_l = 0, l > k$

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6 Specifically, $\rho^{avg} = (\Sigma\ \rho_i^{-1}\ W_i/W)^{-1}$ where ρ_i and W_i are risk aversion and wealth, respectively, of participant *i* and *W* is total wealth of all market participants.

Thus, the fraction of portfolio variance that is attributable to a factor is:

$$\frac{\dfrac{\sigma_l^2 \cdot \lambda_l^2}{\left(\sigma_1^2+\sigma_\varepsilon^2\right)^2}}{\displaystyle\sum_{j=1}^{n}\frac{\lambda_j^2}{\left(\sigma_j^2+\sigma_\varepsilon^2\right)}} = \frac{\dfrac{\lambda_l^2}{\left(\sigma_1^2+\sigma_\varepsilon^2\right)} \cdot \dfrac{\sigma_l^2}{\left(\sigma_1^2+\sigma_\varepsilon^2\right)}}{\displaystyle\sum_{j=1}^{n}\frac{\lambda_j^2}{\left(\sigma_j^2+\sigma_\varepsilon^2\right)}} = \frac{\dfrac{\sigma_l^4\kappa_l^2}{\left(\sigma_1^2+\sigma_\varepsilon^2\right)} \cdot \dfrac{\sigma_l^2}{\left(\sigma_1^2+\sigma_\varepsilon^2\right)}}{\displaystyle\sum_{j=1}^{n}\frac{\sigma_j^4\kappa_j^2}{\left(\sigma_j^2+\sigma_\varepsilon^2\right)}}$$

Note that the fractions do not add up to one because the idiosyncratic, asset specific, sources of risk also contribute some risk to a portfolio.

Contributions to portfolio and return variances

A factor will contribute substantially to portfolio variance either if κ_l is large or if σ_l^2 is large. Note that the first case, when κ_l is large, is one where the factor may account for a larger share of portfolio variance than of overall return variance. (This is most likely when κ_l is large and σ_l^2 is small.) The second case highlights that factors with large variance will contribute usually substantially to both portfolio variance and to the variance of returns, $\sigma_l^2 \,/\, \sum_{j=1}^{n}\left(\sigma_j^2+\sigma_\varepsilon^2\right)$.

4. Conclusion

This paper presents two simple but somewhat realistic examples of portfolio exposure to the sources of risk in the underlying assets. Both examples suggest that all factors with priced risk should be included in risk measurement systems.

The first example considers a risk-averse portfolio manager who optimises subject to some financing constraints. The example shows that the resulting portfolio will include investments with high market prices of risk; these investments may generate exposure to factors that do not account for a substantial part of asset return variance.

The second example also models a risk-averse portfolio manager selecting a portfolio, but without financing constraints. The model shows that the market risk of a portfolio may be determined by factors that contribute fairly little to asset return variance. Thus risk measurement methodologies that do not include all sources of priced risk could substantially understate the risk faced by some participants.