Debt management and optimal fiscal policy with long bonds[1]

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Abstract

We study Ramsey optimal fiscal policy under incomplete markets in the case where the government issues only long bonds of maturity $N > 1$. We find that many features of optimal policy are sensitive to the introduction of long bonds, in particular tax variability and the long-run behaviour of debt. When government is indebted, it is optimal to respond to an adverse shock by promising to reduce taxes in the distant future as this achieves a cut in the cost of debt. Hence, debt management concerns override typical fiscal policy concerns such as tax-smoothing. In the case where the government leaves bonds in the market until maturity, we find two additional reasons why taxes are volatile due to debt management concerns: debt has to be brought to zero in the long run and there are N-period cycles. We formulate our equilibrium recursively applying the Lagrangean approach for recursive contracts. However even with this approach the dimension of the state vector is very large. To overcome this issue we propose a flexible numerical method, the "condensed PEA", which substantially reduces the required state space. This technique has a wide range of applications. To explore issues of policy coordination and commitment we propose an alternative model where monetary and fiscal authorities are independent.

Keywords: Computational methods, debt management, fiscal policy, government debt, maturity structure, tax-smoothing, yield curve

JEL classification: C63, E43, E62, H63

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1. Introduction

Table 1 shows the average maturity of outstanding government debt for a variety of countries and displays clear differences across nations. Any theory of debt management needs to explain the costs and benefits for fiscal policy of varying the average maturity in this manner. As the current European sovereign debt crisis emphasises, the maturity structure of government debt is a key variable. Deciding fiscal policy independently of funding conditions in the market is a doomed concept: taxes, public spending and fiscal deficits should all take into account the funding conditions in the market for bonds. Therefore debt management should not be subservient to fiscal policy and should not focus simply on "minimising costs". Rather, fiscal policy and debt management should be studied jointly.

Table 1

Average maturity government debt 2010

Country	Average maturity (years)
United Kingdom	13.7
Denmark	7.9
Greece	7.7
Italy	7.2
Austria	7
France	6.9
Ireland	6.8
Spain	6.7
Switzerland	6.7
Portugal	6.5
Czech Republic	6.4
Sweden	6.4
Germany	5.8
Belgium	5.6
Japan	5.4
Netherlands	5.4
Canada	5.2
Poland	5.2
Australia	5
Norway	4.9
United States	4.8
Finland	4.3
Hungary	3.3

Sources: OECD; The Economist.

A number of recent contributions have studied this interaction between debt management and taxation policy in a Ramsey equilibrium setting. Angeletos (2002), Barro (2003), Buera and Nicolini (2004) use models of complete markets. Nosbusch (2008) explores a simplified model of incomplete markets and Lustig, Sleet and Yeltekin (2009) examine an incomplete market model with multiple maturities and nominal bonds. In this paper we build on our recommendations in Faraglia, Marcet and Scott (2010) and extend the setup of Aiyagari, Marcet, Sargent and Seppälä (2002), who studied optimal fiscal policy with incomplete markets and short bonds, to the case when bonds mature N periods after having been issued. We describe the behaviour of optimal policy with long bonds and we show how to navigate computational problems. The equilibrium in our model shows some well known features of optimal fiscal policy under incomplete markets: the government tries to smooth taxes, taxes follow a near-martingale behaviour and debt is used as a buffer stock to spread tax increases over all periods after an unexpected adverse shock. We also find that if the government is indebted and an adverse shock occurs the government should promise to cut taxes in future periods, when the newly issued long bonds generate a payoff. These future tax cuts "twist" current long interest rates so as to reduce the burden of past debt. This means that a typical debt management concern, ie reducing the costs of debt, overrides a typical concern of fiscal policy, namely tax-smoothing. This promise to cut taxes is the reason that optimal policy is time-inconsistent: if the government could, it would renege on the promise to cut taxes.

A further problem that arises only when dealing with long bonds is what decision to make about outstanding debt at the end of each period. Most of the literature assumes that the government buys back each period all previously issued debt and then reissues new bonds. This assumption is innocuous in models of complete markets, but matters under incomplete markets. Furthermore, as shown in Marchesi (2004), governments rarely buy back outstanding debt before redemption. To quote the UK Debt Management Office (2003), "the UK's debt management approach is that debt once issued will not be redeemed before maturity". For this reason we also study optimal policy when the government leaves long bonds in circulation until the time of maturity. We call this the "hold to redemption" case. In this case, at any moment in time the government has a full spectrum of outstanding debt with maturity until redemption of N, $N-1$ through to one year even though the government only ever issues N period debt. The maturity profile of government debt is therefore much more complex with long bonds and hold to redemption and this will potentially impact debt management and fiscal policy. We find that optimal tax policy is even more volatile in this case: the government promises to cut taxes permanently and there are N-period cycles in tax policy.

Obtaining numerical simulations is not straightforward. A first difficulty lies in obtaining a recursive formulation of the model. To do so we extend the recursive contracts treatment of Aiyagari et al (2002). A second difficulty arises because the vector of state variables is typically of dimension $2N+1$. Hence it grows rapidly with maturity: many OECD countries issue 30-year bonds, and both France and the UK issue 50-year bonds. Solving a non-linear dynamic model with these many state variables is not feasible.[5]

To reduce this computational complexity, we propose a new method, the "condensed PEA", that reduces the dimensionality of the state vector while allowing, in principle, for arbitrary precision. We show how in the case of a 20-year bond the state space is effectively only four variables. We believe this computational method has wide applicability to other cases.

The fact that the fiscal authority finds it optimal to twist interest rates to minimise funding costs raises issues of commitment and policy coordination. To assess this, we introduce a model where the fiscal authority is separate from the monetary authority setting interest rates. In this

[5] Linearisation of the policy function is undesirable. First, because it turns out that non-linear terms in the policy function play a crucial role even near the steady state mean. Second, because of the presence of debt limits.

way the "twisting" of interest rates is not possible, since the fiscal authority takes interest rates as given. This setup provides a framework to understand the role of commitment in the Ramsey policy, and in the case with buyback it reduces the dimensionality of the state vector, as the usual co-state variables of optimal Ramsey policy are no longer present. We find that the second moments of the model are not highly dependent on maturity. In a calibrated example, allocations, interest rates and persistence of debt are similar across maturities and across the three models of policy considered. The main difference is the long-run level of debt, as longer maturities are associated with more debt.

The structure of the paper is as follows. Section 2 outlines our main model, a Ramsey model with incomplete markets and long bonds when the government buys back all outstanding debt each period. Section 2 shows some properties of the model using analytic results. Section 3 studies numerical issues, introducing the condensed PEA and describing the behaviour of the model numerically. Section 4 studies the model of independent powers, whilst Section 5 considers the case of hold to redemption. A final section concludes.

2. The model: analytic results

Our benchmark model is of a Ramsey policy equilibrium, with perfect commitment and coordination of policy authorities, in which the government buys back all existing debt each period. In Sections 4 and 5 we relax these assumptions.

The economy produces a single non-storable good with a technology

$$c_t + g_t \leq 1 - x_t, \tag{1}$$

for all t, where x_t, c_t and g_t represent leisure, private consumption and government expenditure respectively. The exogenous stochastic process g_t is the only source of uncertainty. The representative consumer has utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u\left(c_t\right) + v\left(x_t\right) \right\} \tag{2}$$

and is endowed with one unit of time that it allocates between leisure and labour and faces a proportional tax rate τ_t on labour income. The representative firm maximises profits, both consumers and firms act competitively by taking prices and taxes as given. Consumers, firms and government have full information, ie they observe all shocks up to the current period, and all variables dated t are chosen contingent on histories $g^t = \left(g_t, ..., g_0\right)$. All agents have rational expectations.

Agents can only borrow and lend in the form of a zero-coupon, risk-free, N-period bond so that the government budget constraint is:

$$g_t + p_{N-1,t} b_{N,t-1} = \tau_t \left(1 - x_t\right) + p_{N,t} b_{N,t} \tag{3}$$

where $b_{N,t}$ denotes the number of bonds the government issues at time t, each bond pays one unit of consumption good in N-periods of time with complete certainty. The price of an i-period bond at time t is p_{it}. In this section, we assume that at the end of each period the government buys back the existing stock of debt and then reissues new debt of maturity N; these repurchases are reflected in the left side of the budget constraint (3). In addition, government debt has to remain within upper and lower limits $\underline{M}$ and $\overline{M}$ so ruling out Ponzi schemes eg

$$\underline{M} \leq \beta^N b_{N,t-1} \leq \overline{M} \tag{4}$$

The term β^N in this constraint reflects the value of the long bond at steady state, so that the limits $\underline{M}, \overline{M}$ appropriately refer to the value of debt and they are comparable across maturities.[6]

We assume that, after purchasing a long bond, the household has only two options: one is to resell the government bond in the secondary market in the period immediately after having purchased it, the other possibility is to hold the bond until maturity.[7] Letting $s_{N,t}$ be the sales in the secondary market, the household's problem is to choose stochastic processes $\{c_t, x_t, s_{N,t}, b_{N,t}\}_{t=0}^{\infty}$ to maximise (2) subject to the sequence of budget constraints:

$$c_t + p_{N,t}b_{N,t} = (1-\tau_t)(1-x_t) + p_{N-1,t}s_{N,t} + b_{N,t-N} - s_{N,t-N+1}$$

with prices and taxes $\{p_{N,t}, p_{N-1,t}, \tau_t\}$ taken as given. The household also faces debt limits analogous to (4). We assume for simplicity that these limits are less stringent than those faced by the government so that, in equilibrium, the household's problem always has an interior solution.

The consumer's first-order conditions of optimality are given by

$$\frac{v_{x,t}}{u_{c,t}} = 1-\tau_t \tag{5}$$

$$p_{N,t} = \frac{\beta^N E_t\left(u_{c,t+N}\right)}{u_{c,t}} \tag{6}$$

$$p_{N-1,t} = \frac{\beta^N E_t\left(u_{c,t+N-1}\right)}{u_{c,t}} \tag{7}$$

2.1 The Ramsey problem

We assume the government has full commitment to implementing the best sequence of (possibly time-inconsistent) taxes and government debt knowing equilibrium relationships between prices and allocations. Using (5), (6) and (7) to substitute for taxes and consumption, the Ramsey equilibrium can be found by solving

$$\begin{aligned}&\max_{\{c_t, b_{N,t}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{u(c_t) + v(x_t)\right\}\\ &\text{s.t.}\quad \beta^{N-1} E_t\left(u_{c,t+N-1}\right) b_{N,t-1} = S_t + \beta^N E_t\left(u_{c,t+N}\right) b_{N,t}\end{aligned} \tag{8}$$

and (4), and x_t implicitly defined by (1).

To simplify the algebra we define $S_t = (u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t}g_t$ as the "discounted" surplus of the government and set up the Lagrangian

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[6] Obviously the actual value of debt is $p_{N,t}b_{N,t}$, we substitute $p_{N,t}$ by its steady state value β^N for simplicity, nothing much changes if the limits are in terms of $p_{N,t}b_{N,t}$.

[7] We need to introduce secondary market sales $s_{N,t}$ in order to price the repurchase price of the bond.

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + v(x_t) + \lambda_t \left[S_t + \beta^N u_{c,t+N} b_{N,t} - \beta^{N-1} u_{c,t+N} b_{N,t-1} \right] \right.$$
$$\left. + \nu_{1,t} \left(\bar{M} - \beta^N b_{n,t} \right) + \nu_{2,t} \left(\beta^N b_{N,t} - \underline{M} \right) \right\}$$

where λ_t is the Lagrange multiplier associated with the government budget constraint and $\nu_{1,t}$ and $\nu_{2,t}$ are the multipliers associated with the debt limits.

The first-order conditions for the planner's problem with respect to c_t and $b_{N,t}$ are

$$\begin{aligned} &u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + g_t \right) - v_{x,t} \right) \\ &\quad + u_{cc,t} \left(\lambda_{t-N} - \lambda_{t-N+1} \right) b_{N,t-N} = 0 \end{aligned} \tag{9}$$

$$E_t \left(u_{c,t+N} \lambda_{t+1} \right) = \lambda_t E_t \left(u_{c,t+N} \right) + \nu_{2,t} - \nu_{1,t} \tag{10}$$

with $\lambda_{-1} = ... = \lambda_{-N} = 0$.

These FOC help characterise some features of optimal fiscal policy with long bonds. Following the discussion in Aiyagari et al (2002), we see that in the case where debt limits are non-binding (10) says that λ_t is a risk-adjusted martingale with risk-adjustment measure $\frac{u_{c,t+N}}{E_t \left(u_{c,t+N} \right)}$, indicating that in this model the presence of the state variable λ in the policy function imparts persistence in the variables of the model. The term $D_t = \left(\lambda_{t-N} - \lambda_{t-N+1} \right) b_{N,t-N}$ in (9) indicates that a feature of optimal fiscal policy will be that what happened in period $t-N$ has a special impact on today's taxes. Since we have $u_{c,t} - v_{x,t} = 0$ and zero taxes in the first best, a high D_t pulls the model away from the first best and zero taxes. If $D_t > 0$ it can be thought of as introducing a higher distortion in a given period. In periods when g_{t-N+1} is very high, we have that the cost of the budget constraint is high, so λ_{t-N+1} is high, and if the government is in debt $D_t < 0$ so that taxes should go down at t. Of course this is not a tight argument, as λ_t also responds to the shocks that have happened between t and $t-N$ and λ_t also plays a role in (9), but this argument is at the core of the interest rate twisting policy we identify below. In order to build up intuition for the role of commitment and to provide a tighter argument, we now show two examples that can be solved analytically.

2.2 A model without uncertainty

Assume now that government spending is constant, $g_t = \bar{g}$. The only budget constraint of the government is then

$$\begin{aligned} &\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = b_{N,-1} p_0^{N-1}, \text{or} \\ &\sum_{t=0}^{\infty} \beta^t S_t = b_{N,-1} \beta^{N-1} u_{c,N-1} \end{aligned} \tag{11}$$

where $\tilde{S}_t = \frac{S_t}{u_{c,t}}$ is the "non-discounted" surplus of the government. This shows that, for a given set of surpluses, the funding costs of initial debt $b_{-1}^N > 0$ can be reduced by manipulating consumption such that $c_t < c_{N-1}$ for all $t \neq N$. As long as the elasticity of consumption with respect to wages is positive, as occurs with most utility functions, this can be achieved by setting

$$\begin{aligned} &\tau_t = \overline{\tau} \text{ for all } t \neq N-1 \\ &\overline{\tau} > \tau_{N-1} \end{aligned} \tag{12}$$

This achieves a reduction of $u_{c,N-1}$, reducing the cost of outstanding debt. In other words, the long end of the yield curve needs to be twisted up.[8] Interestingly, even though there are no fluctuations in the economy, (12) shows that optimal policy implies that the government desires to introduce variability in taxes. In other words, optimal policy violates tax-smoothing. This policy is clearly time-inconsistent: if the government is able to reoptimise by surprise at some period $t' > 0$, $t' < N-1$ it will then promise instead a cut in taxes in period $t' + N - 1$.

2.3 A model with uncertainty at $t = 1$

The previous subsection abstracted from uncertainty. We now introduce uncertainty into our model. In the interest of obtaining analytic results we assume uncertainty occurs only in the first period, ie g is given by:[9]

$$\begin{cases} g_t = \overline{g} \text{ for } t = 0 \text{ and } t \geq 2 \\ g_1 \sim F_g \end{cases}$$

for some non-degenerate distribution F_g. Since future consumption and λ's are known, the martingale condition implies $u_{c,t+N}\lambda_{t+1} = \lambda_t u_{c,t+N}$ and

$$\lambda_t = \lambda_1 \quad t > 1$$

It is clear that in the case of short bonds $(N = 1)$ equilibrium implies c_t and τ_t constant for $t \geq 2$, reflecting the fact that, even though markets are incomplete, the government smooths taxes after the shock is realised.

For the case of long bonds when $N > 1$, the FOC with respect to consumption (9) is satisfied for $D_t = (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N}$

$$D_t = 0 \text{ for } t \geq 0 \text{ and } t \neq N-1, N \tag{13}$$

$$D_{N-1} = \lambda_0 b_{N,-1},\ D_N = (\lambda_0 - \lambda_1) b_{N,0} \tag{14}$$

Hence equilibrium satisfies

$$c_t = c^*(g_1) \text{ for } t \geq 2 \text{ and } t \neq N, N-1 \tag{15}$$

for a certain function c^*. ie consumption is the same in all periods $t \geq 2$ and $t \neq N, N-1$, although this level of constant consumption depends on the realisation of the shock g_1. Clearly, c_{N-1}, c_N also depend on the realisation of g_1.

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8 This is, of course, a manifestation of the standard interest rate manipulation already noted by Lucas and Stokey (1983), except that in our case the twisting occurs in N periods.

9 Formally, this economy is very similar to that of Nosbusch (2008).

Therefore, there is more tax volatility than in the case of short bonds: taxes vary in periods $N-1$ and N, even though by the time the economy arrives at these periods no more shocks have occurred for a long time.

2.3.1 An analytic example

To make this argument precise consider the utility function

$$\frac{c_t^{1-\gamma_c}}{1-\gamma_c} - B\frac{\left(1-x_t\right)^{1+\gamma_\iota}}{1+\gamma_\iota} \tag{16}$$

for $\gamma_c, \gamma_\iota, B > 0$.

Result 1 *Assume utility (16) and* $b_{N,-1} > 0$.

For a sufficiently high realisation of g_1 *we have*

$\tau_1 = \tau_t$ for all $t \geq 1, t \neq N-1, N$

$\tau_1 > \tau_{N-1}, \tau_N$

The inequalities are reversed if $b_{N,-1} < 0$ *or if the realisation of* g_1 *is sufficiently low.*

Proof

Since $\lambda_t = \lambda_1$ $t > 1$ the FOC of optimality yield

$$\frac{u_{c,t}}{v_{x,t}} - \frac{B + \left(\gamma_\iota + 1\right)\lambda_1}{\left(1 + \left(-\gamma_c + 1\right)\lambda_1\right)B} + \left(\lambda_{t-N} - \lambda_{t-N+1}\right)F_t = 0 \quad \text{for } t \geq 1$$

where $F_t \equiv \dfrac{u_{cc,t} b_{N,t-N}}{\left(1 + \left(-\gamma_c + 1\right)\lambda_1\right)B}$.

Consider $t = 1$. For any long maturity $N > 1$ we have that $\lambda_{t-N} = \lambda_{t-N+1} = 0$ when $t = 1$ so that

$$\frac{u_{c,1}}{v_{x,1}} = \frac{B + \left(\gamma_\iota + 1\right)\lambda_1}{\left(1 + \left(-\gamma_c + 1\right)\lambda_1\right)B} \tag{17}$$

Therefore we can write

$$\frac{u_{c,t}}{v_{x,t}} - \frac{u_{c,1}}{v_{x,1}} = \left(\lambda_{t-N} - \lambda_{t-N+1}\right)F_t = 0 \quad \text{for } t \geq 1 \tag{18}$$

That $\tau_t = \tau_1$ for all $t > 1$ and $t \neq N-1, N$ follows from (15).

Now we show that $F_t < 0$ for $t = N-1, N$. Since $\lambda_1, B, \gamma_\iota > 0$ we have that $B + \left(\gamma_\iota + 1\right)\lambda_1 > 0$.

Since $u_{c,1}, v_{x,1} > 0$ clearly (17) implies that $\left(1 + \left(\gamma_c + 1\right)\lambda_1\right)B > 0$. Since we consider the case of initial government debt $b_{N,-1} > 0$ this leads to $b_{N,0} > 0$ and since $u_{cc,1} < 0$ we have $F_t < 0$ for $t = N-1, N$.

Since for $t = N-1$ we have $\lambda_{t-N} - \lambda_{t-N+1} = -\lambda_0 < 0$ it follows

$$\frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \Rightarrow \tau_{N-1} < \tau_t \text{ for all } t > 1, t \neq N-1, N.$$

Also, it is clear from (17) that high g_1 implies a high λ_1. Since the martingale condition implies $E_t\left(u_{c,N}\lambda_1\right)=\lambda_0 E_0\left(u_{c,N}\right)$ for slightly high g_1 we have $\lambda_1>\lambda_0$. Therefore, for $t=N$ and if g_1 was high enough we have $\lambda_{t-N}-\lambda_{t-N+1}=\lambda_0-\lambda_1<0$ so that (18) implies

$$\frac{u_{c,N}}{v_{x,N}},\frac{u_{c,N-1}}{v_{x,N-1}}<\frac{u_{c,1}}{v_{x,1}}\Rightarrow\tau_N,\tau_{N-1}<\tau_1 \qquad \blacksquare$$

Intuitively, in period $t=N-1$ there is a tax cut for the same reasons as in Section 2.2. New in this section is the tax cut (for high g_1) at $t=N$. The intuition for this is clear: when an adverse shock to spending occurs at $t=1$ the government uses debt as a buffer stock so that $b_{N,1}>b_{N,0}$, as this allows tax-smoothing by financing part of the adverse shock with higher future taxes. But since future surpluses are higher than expected, as the higher interest has to be serviced, the government can lower the cost of existing debt by announcing a tax cut in period N, since this will reduce the price $p_{N-1,0}$ of period $t=1$ outstanding bonds $b_{N,0}$. The tax cut at $t=N$ is a stochastic analog of the tax cut described in Section 2.2.

2.3.2 Contradicting tax-smoothing

The above result shows that in this model tax policy is subordinate to debt management. In models of optimal policy, the government usually desires to smooth taxes. Taxes would be constant in the above model if the government had access to complete markets. But we find that the government increases tax volatility in period N, long after the economy has received any shock. Therefore, government forfeits tax-smoothing in order to enhance a typical debt management concern, namely reducing the average cost of debt.

Obviously this policy is time-inconsistent: if the government could unexpectedly reoptimise in period $t=2$ given its debt $b_{N,1}$ it would renege on the promise to cut taxes in period N. Instead it would promise to lower taxes in period $N+1$.

It is clear from this discussion that what will matter for the policy function is the term $D_N=\left(\lambda_0-\lambda_1\right)b_{N,0}$. Therefore it is the interaction between past λs and past bs that determines the size and the sign of today's tax cut. A linear approximation to the policy function would fail to capture this feature of the model and it would be quite inaccurate.

To summarise, we have proved that in the presence of an adverse shock to spending the government has to take three actions: (i) increase taxes permanently, (ii) increase debt, and (iii) announce a tax cut when the outstanding debt matures. Effects (i) and (ii) are well known in the literature of optimal taxation under incomplete markets, effect (iii) is clearly seen in this model with long bonds since the promise is made N periods ahead. Obviously in the case of short maturity $N=1$ of Aiyagari et al, the effect of D_1 would be felt in deciding optimally τ_1, but this effect would be confounded with the fact that g_1 is stochastic, so effect (iii) is harder to see in a model with short bonds.

3. Optimal policy: simulation results

We now turn to the case where g_t is stochastic in all periods. As is well known, analytic solutions for this type of model are infeasible, so we utilise numerical results. The objective is to compute a stochastic process $\left\{c_t,\lambda_t,b_{N,t}\right\}$ that solves the FOC of the Ramsey planner, namely (8), (9) and (10). First we obtain a recursive formulation that makes computation possible, then we describe a method for reducing the dimensionality of the state space and finally we discuss the behaviour of the economy.

3.1 Recursive formulation

Using the recursive contract approach of Marcet and Marimon (2011), the Lagrangean can be rewritten as:

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \{ & u(c_t) + v(x_t) + \lambda_t S_t + u_{c,t}(\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N} \\ & + \nu_{1,t}(\bar{M} - \beta^N b_{N,t}) + \nu_{2,t}(\beta^N b_{N,t} - \underline{M}) \} \end{aligned} \tag{19}$$

for $\lambda_{-1} = ... = \lambda_{-N} = 0$.

Assuming g_t is a Markov process, as suggested by the form of this Lagrangean, Corollary 3.1 in Marcet and Marimon (2011) implies the solution satisfies the recursive structure[10]

$$\begin{bmatrix} b_{N,t} \\ \lambda_t \\ c_t \end{bmatrix} = F(g_t, \lambda_{t-1}, ..., \lambda_{t-N}, b_{N,t-1}, ..., b_{N,t-N})$$

$$\lambda_{-1} = ... = \lambda_{-N} = 0, \quad \text{given } b_{N,-1}$$

for a time-invariant policy function F. This allows for a simpler recursive formulation than the promised utility approach, as the co-state variables λ do not have to be restricted to belong to the set of feasible continuation variables.

The state vector in this recursive formulation has dimension $2N+1$. It is unlikely that further reductions in this dimension can be achieved purely by theoretical results. In order to overcome the problem of dimensionality, some authors model long bonds as perpetuities with decaying coupon payments where the rates of decay mimic differences in maturity (Woodford (2001), Broner, Lorenzoni and Schmulker (2007), Arellano and Ramanarayanan (2008)). One justification for assuming a decaying payoff is that it mimics a bond portfolio with fixed shares that decay with maturity. However, since our goal is to build a model of debt management where the object is precisely to study the appropriate portfolio weights, the assumption of fixed portfolio weights would be inappropriate. Further, although modelling bond payoffs in this way would yield smaller state space vectors, it is contrary to the structure of most government portfolios, where most of the payoff occurs at the time of maturity, as in this model.

3.2 The condensed PEA

We wish to find non-linear solutions, first, because the debt limits are likely to be occasionally binding if we want to keep debt at levels similar to those observed in the real world, second, because per our discussion at the end of Section 2.3, a linear approximation of the policy function F will miss key aspects of optimal policy. Since bonds of maturity $N = 10, 30$ or 50 years are not uncommon a non-linear approach rapidly becomes intractable for a state

[10] In this model it is possible to reduce the state space even further by recognising that the only relevant state variables are N lags of $s_t = b_{N,t}(\lambda_t - \lambda_{t-1})$. We do not exploit this feature of the model as it is very specific to this version of the model. For example, the no buyback case of Section 5 needs all state variables.

vector of dimension $2N+1$. To overcome this difficulty, we introduce a solution method based on the parameterised expectation algorithm of den Haan and Marcet (1990). This allows us to reduce the dimensionality of the policy function actually solved for while keeping an accurate solution. Using PEA is useful because it does capture the relevant non-linearities described in Section 2.3 even if the expectations are parameterised as linear functions and because it allows for a natural space reduction method that we call “condensed PEA”.

This method goes as follows. Denote the state vector as $X_t = \left(g_t, \lambda_{t-1}, ..., \lambda_{t-N}, b_{N,t-1}, ..., b_{N,t-N}\right)$. The idea is that, even though theoretically all elements of X_t are necessary in determining decision variables at t, it is unlikely that in the steady state distribution each and every one of these variables plays a substantial role in determining the solution. For us, most likely some function of these lags will be sufficient to summarise the features from the past that need to be remembered by the government in order to take an optimal decision. In the context of PEA this can be expressed in the following way.

One of the expectations requiring approximation is

$$E_t \left\{u_{c,t+N}\right\} \tag{20}$$

appearing in (10). This expectation is a function, in principle, of all elements in X_t, but it is likely that in practice a few linear combinations of X_t may be sufficient to predict $u_{c,t+N}$. There are two reasons for this. First, the very structure of the model suggests that elements of X_t are very highly correlated with each other, suggesting that a few linear combinations of X_t have as much predictive power as the whole vector. Another way of saying this is that it is enough to project any variable on the principal components of X_t. Other methods available for reducing the dimensionality of state vectors have emphasised this aspect. The second reason is that some principal components of X_t may be irrelevant in predicting $u_{c,t+N}$ in equilibrium and, therefore, they can be left out of the approximated conditional expectation. So the goal is to include only linear combinations of X_t that have some predictive power for $u_{c,t+N}$, the remaining linear combinations can be understood as appearing in the conditional expectation with a coefficient of zero.

More precisely, we partition the state vector into two parts: a subset of n state variables $\left\{X_t^{core}\right\} \subset \left\{X_t\right\}$, where $n < 2N - n$ is small and an omitted subset of state variables $\left\{X_t^{out}\right\} = \left\{X_t\right\} - \left\{X_t^{core}\right\}$ of dimension $1 + 2N - n$. We first solve the model including only X_t^{core} in the parameterised expectations. If the error $\phi_{t+N} \equiv u_{c,t+N} - E_t\left\{u_{c,t+N}\right\}$ found using just these core variables is unpredictable with X_t^{out} we would claim the solution with core variables is the correct one. If X_t^{out} can predict this error, we then find the linear combination of X_t^{out} that has the highest predictive power for ϕ_{t+N}. We add this linear combination to the set of state variables, solve the model again with this sole additional state variable, check if X^{out} can predict ϕ_{t+N} and so on.

Formally, given the set of core variables we define the condensed PEA as follows.[11,12]

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[11] This definition assumes we are interested in the steady state distribution. Of course, it could be modified in the usual way to take transitions into account.

Step 1 Parameterise the expectation as

$$E_t\left\{u_{c,t+N}\right\}=\left(1,X_t^{core}\right)\cdot\beta^1 \tag{21}$$

Find values for $\beta^1\in R^{n+1}$, denoted $\beta^{1,f}$, that satisfy the usual PEA fixed point ie where the series generated by $\left(1,X_t^{core}\right)\cdot\beta^{1,f}$ causes this to be the best parameterised expectation. This solution is of course based on a restricted set of state variables. It is therefore necessary to check if the omission of X^{out} affects the approximate solution. The next step orthogonalises the information in X_t^{out}, which will be helpful in arriving at a well conditioned fixed point problem in Step 4.

Step 2 Using a long-run simulation, run a regression of each element of X_t^{out} on the core variables.

Letting $X_{i,t}^{out}$ be the i −th element, we run the regression

$$X_{i,t}^{out}=\left(1,X_t^{core}\right)\cdot b_i^1+u_{i,t}^1$$

$b_i^1\in R^{2N+2-n}$ and calculate the residuals

$$X_{i,t}^{res,1}=X_{i,t}^{out}-\left(1,X_t^{core}\right)\cdot b_i^1. \tag{22}$$

It is clear that $X^{res,1}$ adds the same information to X^{core} as X^{out}, but $X^{res,1}$ has the advantage that it is orthogonal to X^{core}.

Step 3 Using a long-run simulation find $\alpha^1\in R^{n+1}$ such that

$$\alpha^1=\arg\min_{\alpha}\sum_{t=1}^{T}\left(u_{c,t+N}-X_t^{core}\cdot\beta^1-X_t^{res,1}\cdot\alpha\right)^2 \tag{23}$$

If α^1 is close to zero the solution with only X^{core} is sufficiently accurate and we can stop here. Otherwise go to

Step 4 Apply PEA adding $X_t^{res,1}\cdot\alpha^1$ as a state variable, ie parameterising the conditional expectation as

$$E_t\left\{u_{c,t+N}\right\}=\left(X_t^{core},X_t^{res,1}\alpha^1\right)\cdot\beta^2$$

where $\beta^2\in R^{n+2}$. Find a fixed point $\beta^{2,f}$ for this parameterised expectation. Since $\beta^{1,f}$ is a fixed point, since X_t^{core} and $X_t^{res,1}$ are orthogonal and since the linear combination α^1 has high predictive power, it makes sense to use as initial condition for the iterations of the fixed point

$$\underset{(n+2)\times 1}{\beta^{2,f}}=\begin{pmatrix}\beta^{1,f}\\ 1\end{pmatrix}$$

[12] For convenience we describe these steps with reference only to the expectation $E_t\left\{u_{c,t+N}\right\}$. In practice the expectations $E_t\left\{u_{c,t+N}\lambda_{t+1}\right\}$ and $E_t\left\{u_{c,t+N-1}\right\}$ appearing in the FOC also need to be parameterised concurrently and the steps need to be applied jointly to all conditional expectations.

Go to Step 2 with $\left(X_t^{core}, \alpha^1 X_t^{res,1}\right)$ in the role of X_t^{core}, find a new linear combination etc.

Two remarks end this subsection. In the presence of many state variables, it has been customary in dynamic economic models to try each state variable in order. The idea is to add state variables one by one until the next variable does not much change the solution found. For example, if many lags are needed, we add the first lag, then the second lag, and so on. If, at some step, the solution changes very little, it is claimed that the solution is sufficiently accurate. But it is easy to find reasons why this argument may fail. For instance, perhaps the variables further down the list are more relevant.[13] This is the case, by the way, in our model, where state variable λ_{t-N} and $b_{N,t-N}$ play a key role in determining the solution at t. Or it can be that all the remaining variables together make a difference but they do not make a difference one by one. Our method gives a chance to all these variables to make a difference in the solution. It is therefore more efficient in finding relevant state variables, as Step 3 indicates automatically if they are needed and which of them are to be introduced.

The whole argument in this section is made for linear conditional expectations, as in (21). Of course the same idea works for higher-order terms. In order to check the accuracy for higher-order terms, one can use the condensed PEA with the higher-order polynomial terms, ie one can check if linear combinations of, say, quadratic and cubic terms of X_t have predictive power in Step 2, include these in X_t^{out} and go through Steps 2 to 4 above.

The variables included in X_t^{core} are not the only ones influencing the solution. Due to the nature of PEA, past variables can have an effect even if they are excluded from the parameterised expectation. For example, even if we find a solution $X_t^{core} = \left(\lambda_{t-1}, b_{N,t-1}, g_t\right)$ that excludes λ_{t-N} and $b_{N,t-N}$ from the parameterised expectation these state variables will influence the solution at t through their presence in (9).

3.3 Solving the model with condensed PEA

The utility function (16) was convenient for obtaining the analytic results of Section 2.3. In this section we use a utility function more commonly used in DSGE models:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2}$$

We choose $\beta = 0.98$, $\gamma_1 = 1$ and $\gamma_2 = 2$. The choice of discount factor implies that we think of a period as one year. We set η such that if the government's deficit equals zero in the non-stochastic steady state, agents work a fraction of leisure of 30% of the time endowment. For the stochastic shock g, we assume the following truncated AR(1) process:

$$g_t = \begin{cases} \overline{g} & \text{if } (1-\rho)g^* + \rho g_{t-1} + \varepsilon_t > \overline{g} \\ \underline{g} & \text{if } (1-\rho)g^* + \rho g_{t-1} + \varepsilon_t < \underline{g} \\ (1-\rho)g^* + \rho g_{t-1} + \varepsilon_t & \text{otherwise} \end{cases}$$

[13] For another example, incomplete market models with a large number of agents need as state variable all the moments of the distribution of agents, which is an infinite number of state variables. Usually these models are solved first by using the first moment as a state variable, and checking that, if the second moment is added, nothing much changes. But it could be, of course, that the third or fourth moment are the relevant ones, especially since the actual distribution of wealth is so skewed.

We assume $\varepsilon_t \sim N(0{,}1.44)^2, g^* = 25$, with an upper bound $\bar{g}$ equal to 35% and a lower bound $\underline{g} = 15\%$ of average GDP and $\rho = 0.95$. $\bar{M}$ is set equal to 90% of average GDP and $\underline{M} = -\bar{M}$.

We choose $X_t^{core} = \left(\lambda_{t-1}, b_{N,t-1}, g_t\right)$ hence $X_t^{out} = \left(b_{N,t-2},\ldots,b_{N,t-N},\lambda_{t-2},\ldots,\lambda_{t-N}\right)$. To test if sufficient variables are included for an accurate solution in Step 3 we use as our tolerance statistic:

$$dist = \frac{R_{aug}^2 - R^2}{R^2}$$

where R^2 and R_{aug}^2 denote the goodness of fit of the original regression based on the condensed PEA and augmented with the linear combination of residuals respectively. We use for tolerance criterion $dist \leq 0.0001$. Table 2 summarises the number of linear combinations needed for each maturity whilst Table 3 gives details and shows the number of linear combinations needed for each approximations and the R^2 and $dist$.

Table 2

Holding-to-redemption model:
linear combinations introduced with condensed PEA

N	$2N+1$	Number of linear combinations		
		Φ_λ	Φ_{uc_N}	$\Phi_{uc_{N-1}}$
1	3	–	–	–
2	5	0	1	0
5	11	0	1	0
10	21	0	1	0
15	31	1	1	0
20	41	1	1	1

Note: recall that N denotes maturity and $2N+1$ is the dimension of the state vector. In all cases X^{core} has three variables. "# of linear comb" refers to how many linear combinations of X^{out} had to be added to satisfy the accuracy criterion. We denote each expectation to be approximated by $\Phi_\lambda = E_t\left(u_{c,t+N}\lambda_{t+1}\right)$, $\Phi_{uc_N} = E_t\left(u_{c,t+N}\right)$ and $\Phi_{uc_{N-1}} = E_t\left(u_{c,t+N-1}\right)$.

The advantages of the condensed PEA are readily apparent. In nearly half the cases the core variables are sufficient to solve the model and, at most, only one linear combination of omitted variables is required to improve accuracy. Clearly the condensed PEA can be used to solve models with large state spaces with relatively small computational cost, since the state vector is in principle of dimension 41 but utilising a dimension of 4 is sufficient. Whilst we have focused on a case of optimal fiscal policy and debt management, this methodology clearly has much broader applicability.

Table 3

Benchmark model: accuracy measures in condensed PEA

N		Adding 1 linear comb			Adding 1 linear comb		
		Φ_{λ}	Φ_{uc_N}	$\Phi_{uc_{N-1}}$	Φ_{λ}	Φ_{uc_N}	$\Phi_{uc_{N-1}}$
2	# lin comb in	0	0	0	0	1	0
	R^2_{aug}	0.9208	0.7533	0.8669	0.9209	0.7535	0.8669
	dist	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
5	# lin comb in	0	0	0	0	1	0
	R^2_{aug}	0.9069	0.5022	0.5751	0.9070	0.5026	0.5754
	dist	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
10	# lin comb in	0	0	0	0	1	0
	R^2_{aug}	0.8911	0.2630	0.2991	0.8909	0.2632	0.2993
	dist	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000
15	# lin comb in	0	0	0	1	1	0
	R^2_{aug}	0.8814	0.1422	0.1609	0.8831	0.1446	0.1635
	dist	0.0001	0.0002	0.0000	0.0000	0.0000	0.0000
20	# lin comb in	0	0	0	1	1	1
	R^2_{aug}	0.8751	0.0788	0.0886	0.8771	0.0807	0.0907
	dist	0.0002	0.0003	0.0002	0.0000	0.0000	0.0000

Note: see Note of previous Table. R^2_{aug} and *dist* are defined in Section 3.3.

3.4 Optimal policy: the impact of maturity

3.4.1 Interest rate twisting

We compute the policy functions[14] and display the implied response functions of key variables to an unexpected shock in g_t in Figures 1 and 2. The vertical axis is in units of each of the variables and expresses deviations from the value that would occur for the given initial condition if $g_t = g^{ss}$.

Figure 1 is for the case when the government has zero debt on impact. It shows minor differences between long and short bonds. As usual in models of incomplete markets, it is optimal to use debt as a buffer stock so that debt displays considerable persistence.

[14] Since debt is very persistent, to ensure we visit all possible realisations in the long-run simulations of PEA we initialise the model at nine different initial conditions, simulate it for 5,000 periods for each initial condition, doing this 1,000 times per initial condition, and compute conditional expectations discarding the first 500 observations for each simulation.

Figure1

Responses to a shock in g_t ,
benchmark model maturities 1 and 10: $b_{N,-1} = 0$

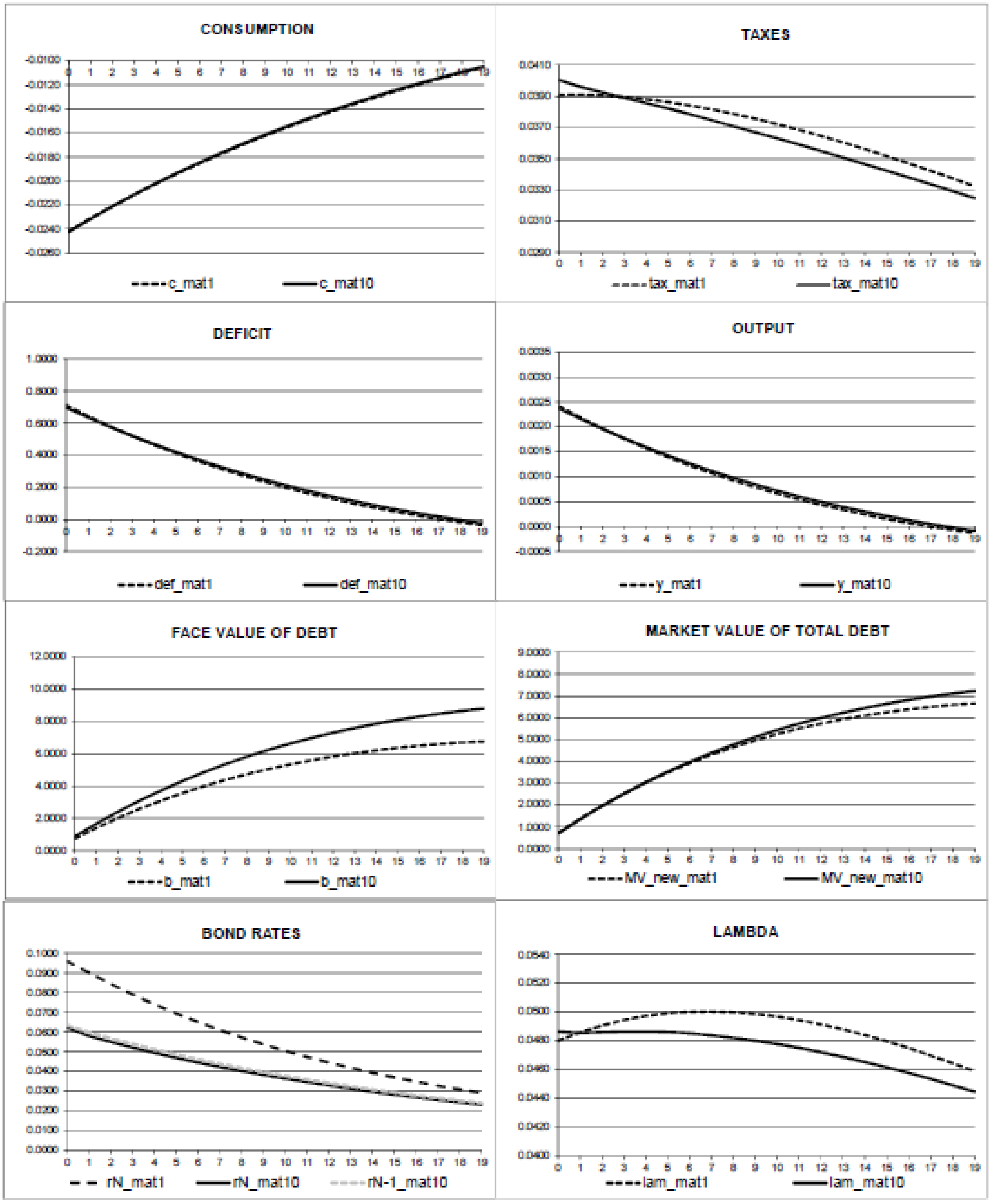
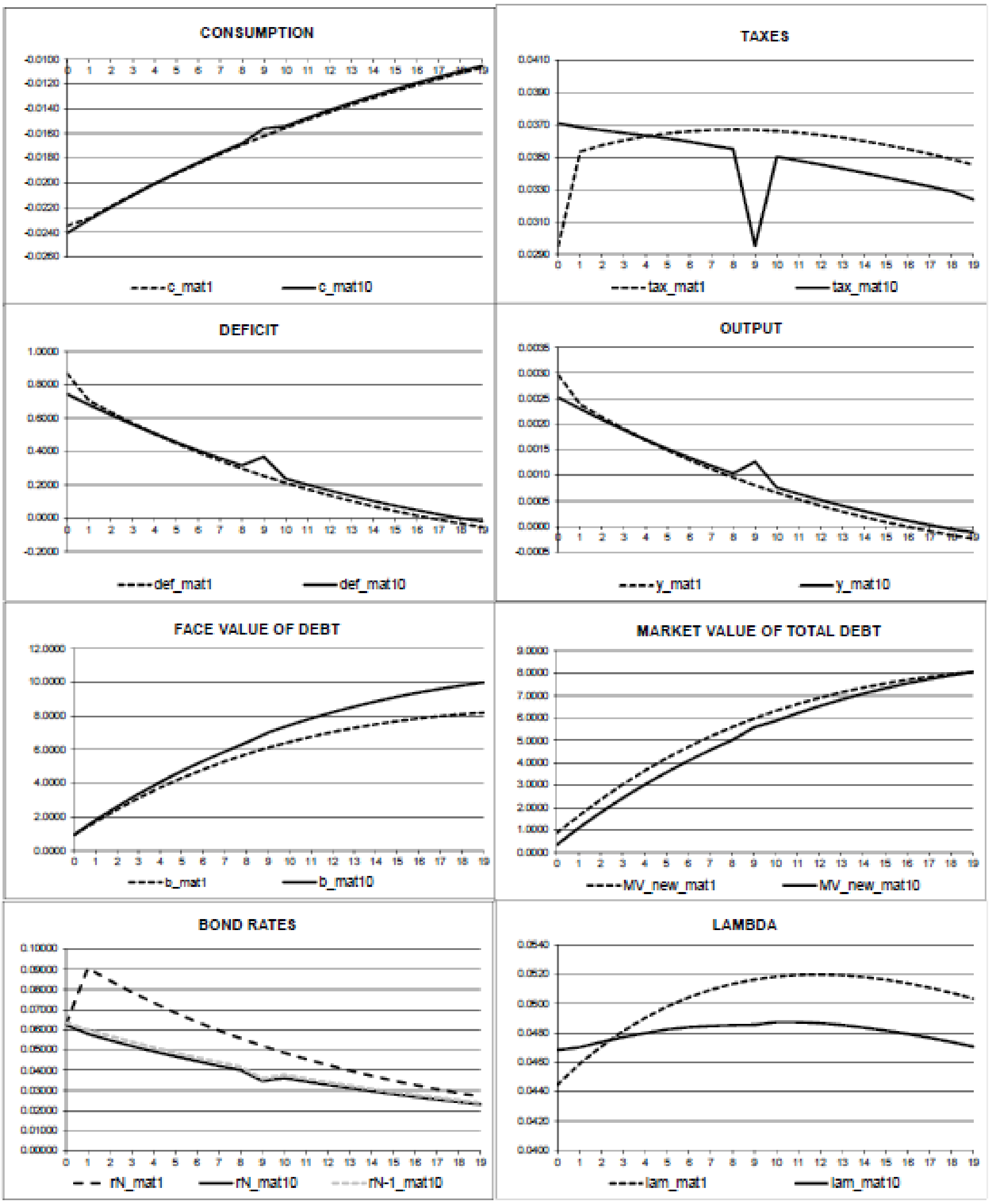


Figure 2 shows the same impulse response functions when we assume the government is indebted on impact, more precisely $b_{N,t-1} = 0.5\, y^* / \beta^N$ where y^* is steady state output.

Figure 2

Responses to a positive shock in g_t,

benchmark model maturities 1 and 10: $b_{N,-1} = \dfrac{0.5y^*}{\beta^N}$



We see that with long bonds of maturity $N = 10$ there is a blip in taxes at the time of maturity of the outstanding bonds. This is a reflection of the promise to cut taxes with the aim to twist interest rates as discussed in Section 2.3, only now the interest rate twisting occurs each period there is an adverse shock.

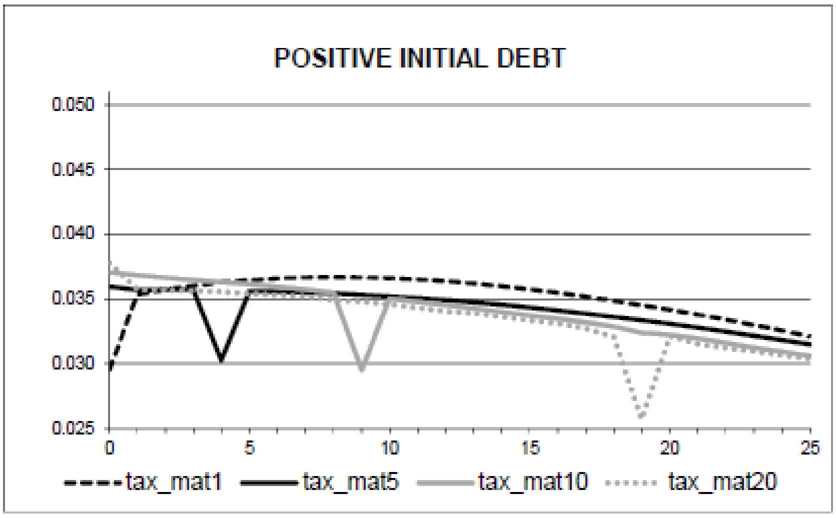
3.4.2 Optimal policy with short bonds

This discussion helps to elucidate the role of commitment in the model of short-term bonds as in Aiyagari et al (2002). Consider the case where the government is indebted when an adverse shock occurs, as in Figure 2. As we explained in Section 2.3, optimal policy is to increase current taxes but promise a tax cut in $N-1$ periods. In the case of long bonds, the promised tax cut is clearly distinct from the current increase in taxes. But in the case of short bonds $N=1$ the two effects are confounded as they happen in the same period.

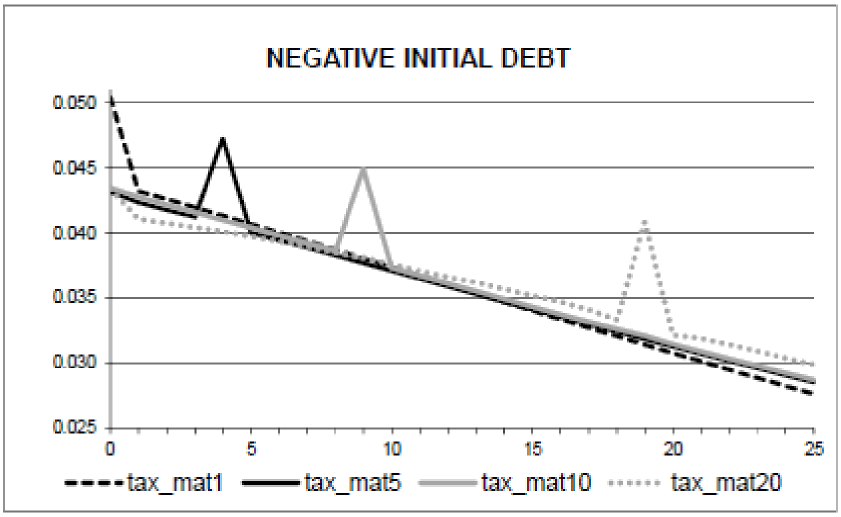
Figure 3

Responses to a positive shock in g_t,
benchmark model maturities 1, 5, 10 and 20:

Taxes: $b_{N,-1} = \dfrac{0.5y^*}{\beta^N}$



Taxes: $b_{N,-1} = -\dfrac{0.5y^*}{\beta^N}$



This is clearly seen in the response of taxes depicted in Figure 3 for maturities $N=1,5,10,20$. Given our previous discussion, it is clear why the blip in taxes keeps moving to the left as we decrease the maturity until the blip simply reduces the reaction of taxes on impact at $N=1$.

Therefore optimal policy for short bonds is to increase taxes on impact but less than would be done if considerations of interest rate twisting were absent.

In the case where the government has assets, the blip in taxes goes upwards, as the government desires to increase the value of assets. This is shown in the response of taxes for the case of assets shown in Figure 3. So, comparing the dashed lines in the response of taxes in Figures 3, it is clear that, for short bonds, the increase in taxes on impact if the government initially has assets is much larger than if the government is indebted.

3.4.3 The level of debt, persistence

Table 4 shows second moments for the economy at steady state distribution for different maturities. Most of the moments differ only to the second or third decimal place across maturities. The main exceptions are the levels of debt and deficit: the government on average holds assets but less under longer maturity. The value of assets when bonds are of 20 years halves the average debt for short bonds.

Table 4

Second moments, steady state

Model: Benchmark model

	N	c	y	τ	Deficit	R_N	$MV = p_N b_N$	λ
mean	**1**	52.60	70.11	0.243	0.42	2.02	–24.68	0.057
	5	52.58	70.08	0.245	0.32	2.02	–19.21	0.058
	10	52.56	70.06	0.246	0.25	2.03	–16.28	0.058
	20	52.54	70.05	0.247	0.17	2.03	–12.46	0.059
std	**1**	3.49	0.35	0.044	1.46	0.5	27.26	0.013
	5	3.48	0.37	0.043	1.57	0.4	30.96	0.013
	10	3.48	0.38	0.044	1.59	0.3	31.97	0.013
	20	3.48	0.39	0.044	1.66	0.2	32.84	0.014

Note: to provide a more interpretable quantity we report annualised interest rates instead of bond prices, namely $R_N = \left(\left(p_N \right)^{-\frac{1}{N}} - 1 \right) 100$.

The intuition for the lower level of assets as maturity grows is as follows. It is well known that in models of optimal policy with incomplete markets, if the government has the same discount factor as agents, the government accumulates assets in the long run. More precisely, it is easy to extend the results in Aiyagari et (2002) Section III for the case of a linear utility of consumption $u(c) = c$ to prove that government assets go to a very high level.

Therefore it is not surprising that all steady states for debt have a negative mean. On the other hand it is also well known that, with long bonds, fiscal insurance recommends that the government issues long bonds. As argued in Angeletos (2002) and Buera and Nicolini (2004), governments should issue long bonds in a model without capital accumulation because long interest rates are higher when the government runs deficits, so that issuing long bonds provides fiscal insurance. Nosbusch (2008) argues that the same tendency for issuing long debt is present in an incomplete markets model. For the same reason, if a government accumulates debt in long bonds, the implied volatility of taxes will be higher. It is therefore not surprising that long-run debt is lower for longer maturities, as holding long bonds causes taxes to be more volatile. In other words, accumulating assets of long maturity is detrimental to fiscal insurance. This is not the case with short bonds, as they provide fiscal insurance when issued. Therefore the level of assets is lower for longer maturities.

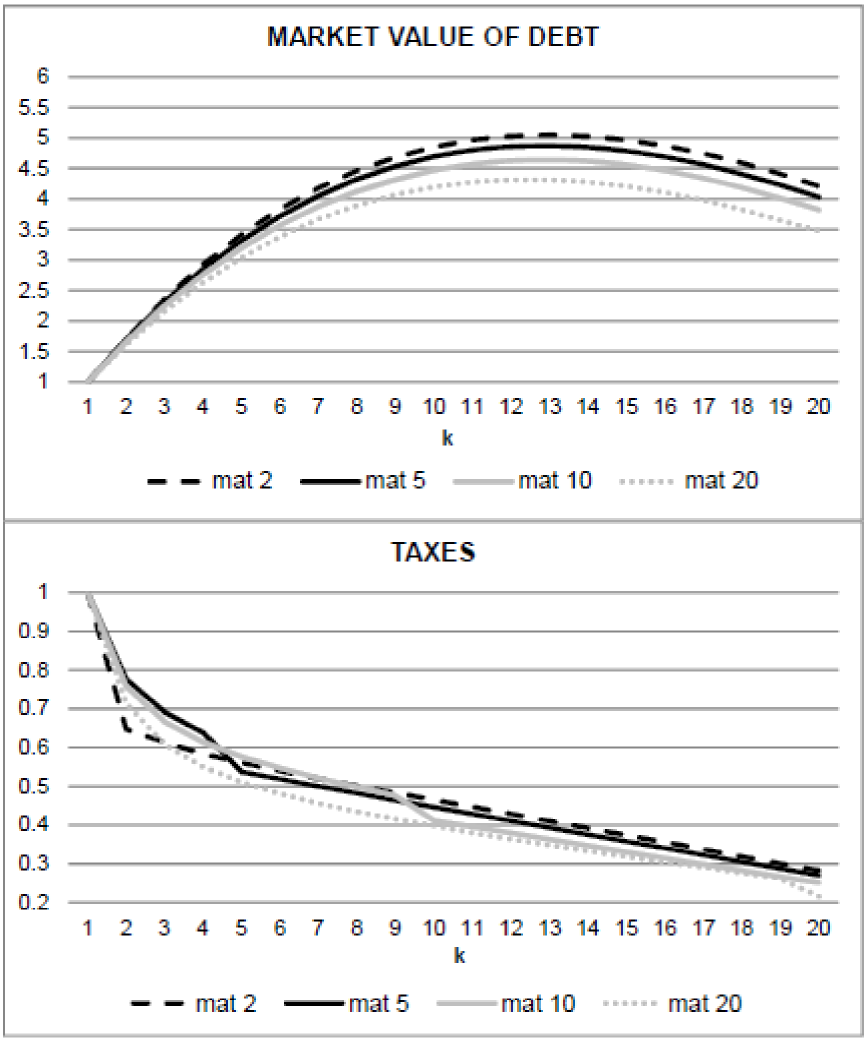
Given that average asset holdings are lower, it is natural that average primary deficits are lower for higher N, since the value of assets is equal to the expected present value of primary deficits also under incomplete markets. For this reason, also, taxes are higher in steady state for higher N.

Another way of examining the impact of varying the average maturity of debt is to see whether this influences how close to the complete market outcome these incomplete market models can get. Marcet and Scott (2009) show that measures of relative persistence are a good way of assessing the extent of market incompleteness and so Figure 4 shows for various variables the measure:

$$P_y^k = \frac{Var\left(y_t - y_{t-k}\right)}{kVar\left(y_t - y_{t-1}\right)}.$$

Figure 4

k-variances, benchmark model maturities 2, 5, 10 and 20:



The closer to 0 this measure, the less persistence the variable shows, whereas the closer to 1 the measure, the more the variable shows unit root persistence. Although the long bond model shows less persistence, suggesting that, in the case of persistent government expenditure shocks, the issuance of longer bonds helps provide more fiscal insurance, the difference between the two cases are minor. Given that taxes are distortionary, we are not in a Modigliani-Miller world and how the government finances its expenditure can affect the real economy. However, the fact that the differences across maturities are so small is perhaps not surprising. With the government only issuing one type of bond in each case and the yield curve showing broadly similar behaviour at different maturities, the tax-smoothing properties of debt issuance are achieved mainly through the role of debt as a buffer rather than through fiscal insurance. Further, we are at this point following the rest of the literature in assuming that every period the government buys back all existing debt and then reissues. So, although the government is issuing 10-period bonds, it always buys them back after a year. Thus, it is effectively always borrowing through one-period debt, reducing the distinction between one-period and 10-period bonds. We shall return to this issue in a later section.

4. Independent powers

In Sections 2 and 3, we found that full commitment implied a tight connection between interest rate policy, debt management and tax policy: when government is in debt and spending is high the government promises a tax cut in $N-1$ periods, knowing that this will increase future consumption and thus increase long interest rates in the current period. The reader may think that this optimal policy is not relevant for the "real world" for at least two reasons. First, as different authorities influence interest rates and fiscal policy, it is unlikely that they will coordinate in the way described above and, second, it is unlikely that governments can commit to a tax cut in the distant future and actually carry through with the promise. Some papers in the literature react to this type of criticism by writing down models where government policy is discretionary. But the assumption that the government has no possibility of committing is also problematic, as governments frequently do things for the very reason they have previously committed to do so.

For these reasons, we change the way policy is decided in this model. We relax the assumption of perfect coordination and assume the presence of a third agent, a monetary authority that fixes interest rates in every period. The fiscal authority now takes interest rates as given and implements optimal policy given these interest rates. We examine an equilibrium where the two policy powers play a dynamic Markov Nash equilibrium with respect to the strategy of the other policy power and they both play Stackelberg leaders with respect to the consumer. More precisely, the fiscal authority chooses taxes and debt given a sequence for interest rates, while the monetary authority simply chooses interest rates that clear the market and the fiscal authority maximises the utility of agents. This assumption sidesteps the issues of commitment, because there is now no room for interest rate twisting on the part of the fiscal authority. It is easy to think of models where, even if the monetary authority is independent, it can not deviate too much from the equilibrium interest rates of the flexible price model. Therefore we take a limit case and assume that the monetary authority simply sets in equilibrium interest rates as:

$$p_{N,t} = \frac{\beta^N E_t\left(u_{c,t+N}\right)}{u_{c,t}} \tag{24}$$

$$p_{N-1,t} = \frac{\beta^{N-1} E_t\left(u_{c,t+N-1}\right)}{u_{c,t}}.$$

given agents' consumption. Now the fiscal authority will not be able to manipulate interest rates, so it will lose any interest in making promises to cut future taxes. To solve this model we are looking for an interest rate policy function $\Re : R^2 \to R^2$ such that if long interest rates at t are given by

$$\left(p_{N,t}^{-1}, p_{N,t-1}^{-1}\right) = \Re\left(g_t, b_{N,t-1}\right) \tag{25}$$

then (24) holds and with the fiscal authority maximising consumer utility in the knowledge of all market equilibrium conditions but, taking the stochastic process for interest rates as given, it chooses a bond policy such that (25) holds. For the fiscal authority, the problem now is a standard dynamic programming one and as a result the state space now only consists of the variables $b_{N,t-1}$ and g_t. An advantage of this model is that there is no reason now for longer lags to enter this state vector, as past Lagrange multipliers do not play a role. Therefore, this separation of powers approach is an alternative way to reducing the state space and simplifying the model's solution.

In this case of independent powers, the Lagrangian of the Ramsey planner becomes

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & u(c_t) + v(x_t) + \lambda_t \left[S_t + p_{N,t} b_{N,t} - p_{N,t-1} b_{N,t-1} \right] \\ & + \nu_{1,t} \left(\bar{M} - \beta^N b_{N,t} \right) + \nu_{2,t} \left(\beta^N b_{N,t} - \underline{M} \right) \Big\} \end{aligned} \tag{26}$$

The first-order condition with respect to consumption is

$$\mathrm{u}_{c,t} - \mathrm{v}_{x,t} + \lambda_t (\mathrm{u}_{cc,t} \mathrm{c}_t + \mathrm{u}_{c,t} + \mathrm{v}_{xx,t} (\mathrm{c}_t + \mathrm{g}_t) - \mathrm{v}_{x,t}) + \mathrm{u}_{cc,t} \lambda_t (\mathrm{p}_{N,t} \mathrm{b}_{N,t} - \mathrm{p}_{N-1,t} \mathrm{b}_{N,t-1}) = 0$$

and using the government's budget constraint gives

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + g_t \right) - v_{x,t} \right) + u_{cc,t} \lambda_t \left(g_t - \left(1 - \frac{v_{x,t}}{u_{c,t}} \right) (1 - x_t) \right) = 0 \tag{27}$$

To see the impact of independent powers, we calibrate the model as in Section 3 and consider the case $N = 10$. Figure 5 compares the impulse responses to a one standard deviation shock to the innovation in the level of government spending when the government has debt between independent powers and the benchmark model of Section 3. As can be seen, the model of independent powers does not show the blip in taxes at maturity. In this case, debt management is subservient to tax-smoothing and is aimed at lowering the variance of deficits.

To better understand the magnitude of the interest twisting channel, we can compare our independent powers model with our earlier benchmark model. We simulated the model at different time horizons $T = 40$, $T = 200$ and $T = 5000$ discarding the first 500 periods. We calculated the standard deviation of taxes for each realisation and we averaged it across simulations. We repeat the same exercise for $\mathrm{N} = 2, 5, 10, 15, 20$. Figure 6 shows the results.

In shorter sample periods, the effect of twisting interest rates in connection with initial period debt is significant and provides a higher level of tax volatility in the benchmark model. Naturally, as we increase the sample size the initial period effect diminishes.

The second moments of the model in this section are shown in Table 5. They are extremely similar to those of the benchmark model in Table 4. We have essentially a very similar amount of bond issuance, debt persistence, tax-smoothing etc, the only difference being that the interest rate twisting adds some tax volatility, but this volatility shows up only in second moments with short samples as shown in Figure 6. We conclude that the model of independent powers may be a good model to have in the toolkit as it retains many of the interesting features of the Ramsey models, it has the same steady state moments, it avoids the technicalities arising from the very large state vector and it avoids discussion on the role to commitment at very long horizons. There are, however, issues of tax volatility showing up in small samples where the two models differ.

Figure 5

Responses to a positive shock in g_t, benchmark and independent power model

Maturity 10: $b_{N,-1} = \dfrac{0.5y^*}{\beta^N}$

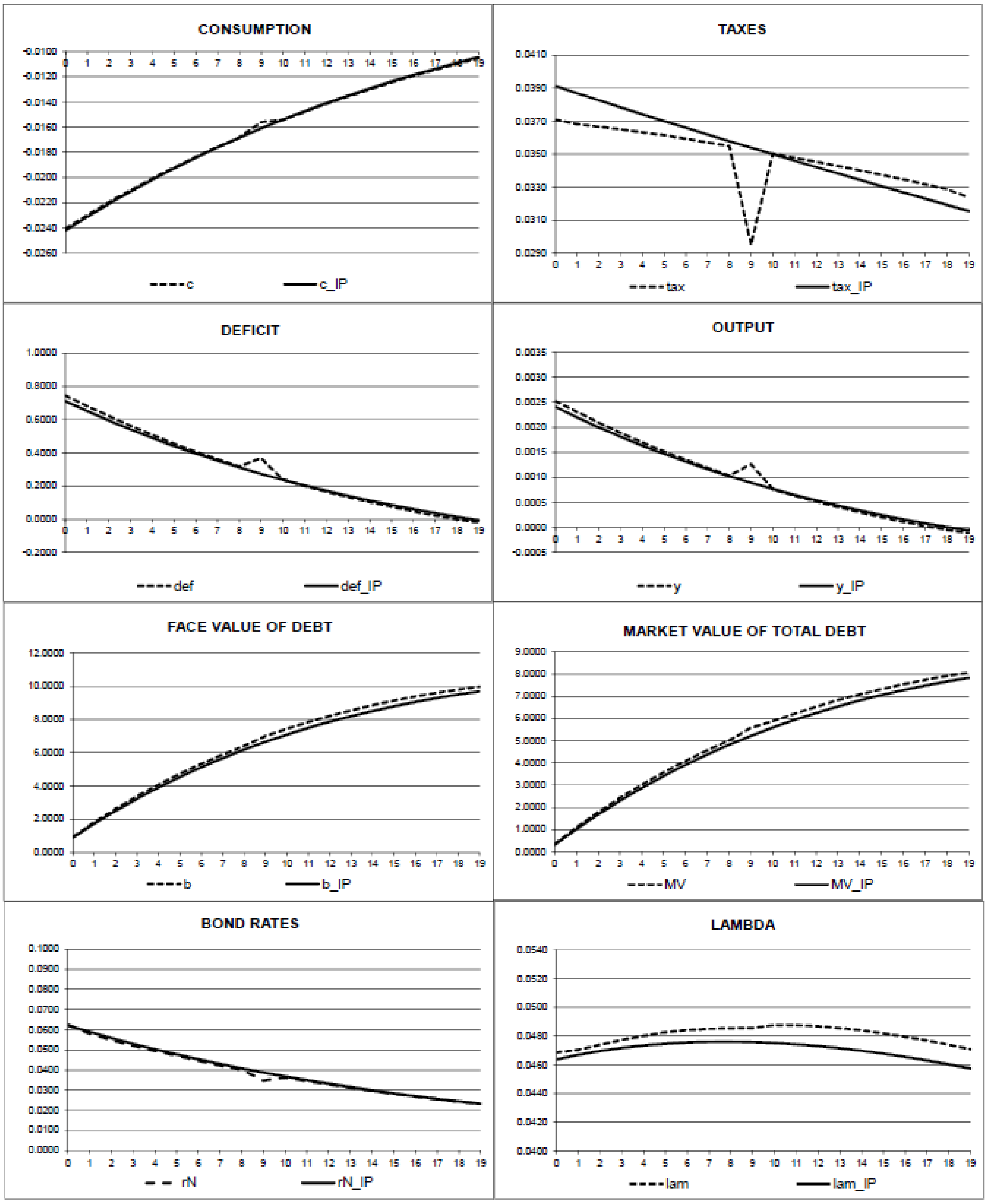


Figure 6

Tax volatility at different horizons: benchmark and independent powers model

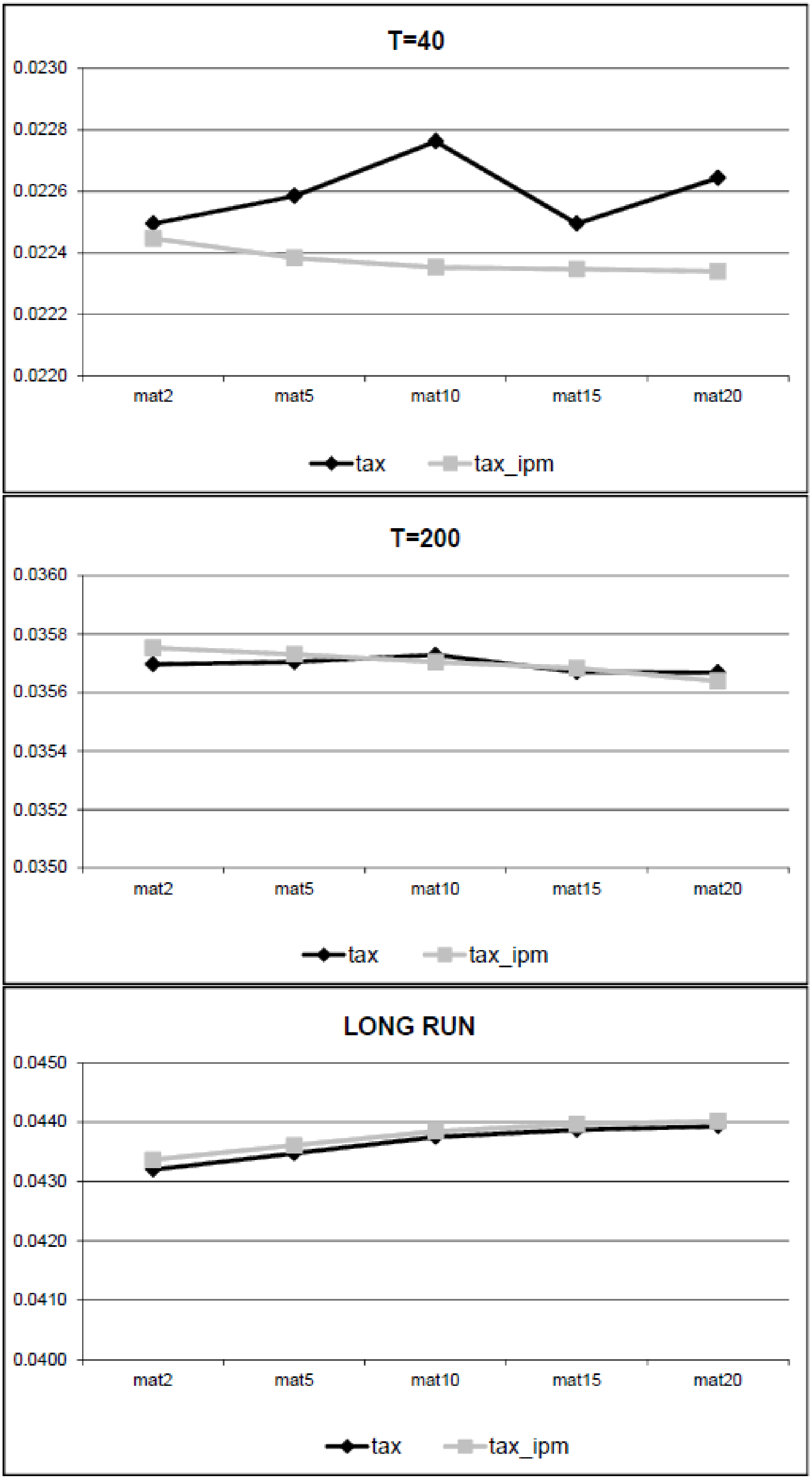


Table 5

Second moments, steady state

Model: Independent powers

	N	c	y	τ	deficit	R_N	$MV = p_N b_N$	λ
mean	**1**	52.60	70.10	0.244	0.41	2.02	–23.54	0.057
	5	52.58	70.08	0.245	0.32	2.02	–19.49	0.058
	10	52.56	70.07	0.246	0.26	2.03	–16.40	0.058
	20	52.54	70.05	0.247	0.17	2.03	–12.31	0.059
std	**1**	3.49	0.34	0.044	1.43	0.5	27.88	0.013
	5	3.48	0.36	0.044	1.51	0.4	31.11	0.013
	10	3.48	0.37	0.044	1.54	0.3	32.20	0.013
	20	3.49	0.37	0.044	1.56	0.2	33.20	0.014

5. Hold to redemption

With long bonds, the government has a choice to make at the end of every period. It can buy back the N period bonds issued last period, as assumed in Sections 2 and 3. Alternatively it can leave some or all of the outstanding bonds in circulation until they mature at their specified redemption date. In models of complete markets, whether or not there is buyback in each period is immaterial: all prices and allocations remain unchanged. But in this paper there are two reasons why the outcome is different. The first reason is that the stream of payoffs generated by each policy is quite different from the point of view of the government: with buyback the bond pays the random payoff $p_{N-1,t+1}$ next period; if the bond is left in circulation until maturity the bond pays 1 with certainty at $t+N$. As is well known, under incomplete markets not only the present value of payoffs of an asset are relevant; the timing of payoffs also matters. A second reason for the differences is that the possibilities for governments to twist interest rates are different.

In Section 2 we made the extreme assumption that the government each period buys back the whole stock of outstanding bonds issued last period. As shown in Marchesi (2004), it is normal practice for governments not to buy back debt – debt is issued and it is paid off at maturity. In this section, we assume that bonds are left to mature to their redemption date. In the case of buyback there are only N-period bonds outstanding. In the case of holding to redemption, there exist bonds at all maturities between 1 and N even though the government only issues N period bonds. Although we model the implications of holding to redemption, an explanation for why no buyback is standard practice[15] is considered beyond the scope of this paper.

In this section, we set up a model where debt managers do not buy back debt at the end of each period, show how full commitment gives rise to a different kind of interest rate twisting,

15 Conversations with debt managers suggest some combination of transaction costs, a desire to create liquid secondary markets at most maturities or worries over refinancing risk. For simplicity we rule out a third possibility – that governments choose to buy back only a certain proportion of outstanding debt.

outline how to use condensed PEA to solve for optimal fiscal policy and we show the behaviour of the model. Since we follow closely the analysis of Sections 2 and 3 we omit some details and focus on the differences.

The economy is as before except that the government budget constraint is now

$$b^{HTR}_{N,t-N} = \tau_t \left(1 - x_t\right) - g_t + p_{N,t} b^{HTR}_{N,t} \tag{28}$$

so that the payment obligations of the government at t are the amount of bonds issued at $t - N$.

We include the debt limits

$$\underline{M} \leq b^{HTR}_{N,t} \sum_{i=1}^{N} \beta^i \leq \bar{M} \tag{29}$$

Again, this limit mimics the value of the newly issued debt at steady state prices: if the government issued b_N bonds at all periods it would have b_N units of bonds of maturities $1, 2, ..., N$ outstanding so the total value of debt at steady state would be $\sum_{i=1}^{N} \beta^i b^{HTR}_N$. The budget constraint of the household's problem changes in a parallel way.

5.1 Optimal policy with maturing debt

Substituting in equilibrium bond prices and wages net of taxes (28) becomes

$$s.t. \qquad b^{HTR}_{N,t.N} u_{c,t} = s_t + \beta^N E_t \left(u_{c,t+N}\right) b^{HTR}_{N,t} \tag{30}$$

The Ramsey problem is now to maximise utility (2) over choices of $\left\{c_t, b^{HTR}_{N,t}\right\}$ subject to this constraint and the debt limits (29) for all t . The Lagrangian becomes

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & u\left(c_t\right) + v\left(x_t\right) + \lambda_t \left[S_t + \beta^N u_{c,t+N} b^{HTR}_{N,t} u_{c,t} \right] \\ & + \nu_{1,t} \left(\underline{M}^{HTR} - b^{HTR}_{N,t} \right) + \nu_{2,t} \left(b^{HTR}_{N,t} - \bar{M}^{HTR} \right) \Big\} \end{aligned}$$

where λ_t is the Lagrange multiplier associated with (30), $\nu_{1,t}$ and $\nu_{2,t}$ are the ones associated with the debt limits and $\underline{M}^{HTR} \equiv \underline{M} \left(\sum_{i=1}^{N} \beta^i \right)^{-1}$, $\bar{M}^{HTR} \equiv \bar{M} \left(\sum_{i=1}^{N} \beta^i \right)^{-1}$.

The first-order conditions with respect to c_t and $b^{HTR}_{N,t}$ are

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + g_t \right) - v_{x,t} \right) + u_{cc,t} \left(\lambda_{t-N} - \lambda_t \right) b^{HTR}_{N,t-N} = 0 \tag{31}$$

$$E_t \left(u_{c,t+N} \lambda_{t+N} \right) = \lambda_t E_t \left(u_{c,t+N} \right) + \nu_{2,t} - \nu_{1,t} \tag{32}$$

With $\lambda_{-1} = ... = \lambda_{-\mathrm{M}} = 0$.

In short, these FOC have two differences relative to the buyback case: in equation (31) we now have $(\lambda_{t-N}-\lambda_{t})$ instead of $(\lambda_{t-N}-\lambda_{t-N+1})$ and we now have λ_{t+N} instead of λ_{t+1} in the martingale condition (32)[16].

5.2 No uncertainty and hold to redemption

Let us now consider the no uncertainty case when $g_t = \bar{g}$. Proceeding in an analogous way to the case of Section 2.2 we could write the implementability constraint as

$$\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = \sum_{i=1}^{N} b_{N,-i}^{HTR} p_{N-i,0} \text{, or} \tag{33}$$

$$\sum_{t=0}^{\infty} \beta^t S_t = \sum_{i=1}^{N} b_{N,-i}^{HTR} \beta^{N-i} u_{c,N-i} \tag{34}$$

for $p_{0,t} \equiv 1$. Bonds issued in periods $i=-1,-2,...,.-N$ appropriately appear in the right side of the above constraint, as what matters now is the total value of debt initially.

Let us consider the problem of maximising utility when (34) is the sole implementability constraint. If the government is in debt with $b_{N,-i}^{HTR} > 0$ for all $i = 1,...,N$ it is clear that in this case interest rate twisting will involve changing interest rates in the first $N-1$ periods hence the government will promise to cut taxes in all periods between $t = 0,...,N-1$. The FOC for consumption indicates that the tax cut will be larger for periods $t = 0,...,N-1$ where the maturing debt $b_{N,t-N}^{HTR}$ is larger. Therefore tax cuts now last N periods. For $t \geq N$ consumption and taxes are constant.

But assuming that (34) is the sole implementability constraint as we did in the previous paragraph is not correct for our model. It would be correct in a slightly different model, where the debt limits would be in terms of the total value of debt, for example, if debt limits would be

$$\underline{M}^{MV} \leq \sum_{i=1}^{N} b_{N,t-i}^{HTR} p_{N-i,t} \leq \bar{M}^{MV} \tag{35}$$

Take for simplicity the case $N = 2$. It is clear that the optimal allocation described in the previous paragraph can be implemented for bond issuances satisfying $b_{N,t-2}^{HTR} + \beta \frac{u_{c,t+1}}{u_{c,t}} b_{N,t-1}^{HTR} = \sum_{j=0}^{\infty} \beta^j S_{t+j}$ for all $t = 0,1,...$. Given initial conditions this provides a difference equation on b_N that satisfies the period-t budget constraint (30) and the value of debt limits if $\underline{M}^{HTM}$ and $\bar{M}^{HTM}$ were sufficiently large in absolute value.

But, for our model, (34) is not sufficient for an equilibrium. This is perhaps surprising, as we think that without uncertainty and one asset one can always complete the markets for sufficiently high debt limits. To see this point, notice that for the optimal allocation described above the surplus is constant, equal to a level, say $\tilde{S}$, for all $t \geq N$. The bonds that would satisfy the period t budget constraint satisfy $b_{N,t-2}^{HTR} + \beta b_{N,t-1}^{HTR} = \frac{\tilde{S}}{1-\beta}$ for all $t = N, N+1,...$. This path for bonds would satisfy the difference equation

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16 In the case of hold to redemption, the assumption of independent powers would not simplify the analysis in terms of reducing the state space. One would still need N lags of b_N as state variables.

$$b_{N,t}^{HTR} = \frac{\tilde{S}}{(1-\beta)\beta} - \beta^{-1} b_{N,t-1}^{HTR} \; t = N, N+1, ... \tag{36}$$

which in general is an unstable difference equation in $b_{N,t}^{HTR}$. Normally the values of $b_{N,t}^{HTR}$ satisfying this equation will explode geometrically to plus and minus infinity, alternating sign. The sequence that is compatible with the non-explosive wealth of the government implies that the debt limits (29) are violated. Therefore, (34) is not sufficient for an equilibrium.

The intuition that one asset completes the markets for no uncertainty if the debt limits are sufficiently loose is only correct if the debt limits are in terms of the value of debt, but not in terms of the actual asset issued. Bond issuance each period in absolute value goes to infinity, constant wealth is only achieved because of the alternation in signs of b_t^{HTR} each period. Of course, one modelling solution would be to assume that debt limits are in terms of the value of debt as in (35), but we believe limits on bonds as in (29) are the more relevant constraint. After all the bond markets are extremely concerned with gross issuance of bonds each period.

This argument shows that, with long bonds, we can not use (34) as the only implementability condition; we need to keep the budget constraint (30) in all periods in the analysis. The following result shows the actual behaviour of optimal policy. Essentially, we show that optimal policy induces higher tax volatility for two reasons: (i) there are cycles of length N, (ii) interest rate twisting is permanent, and the reduction in taxes lasts N periods.

Result 2. *Assume $b_{N,-i}^{HTR} > 0$ for all $i = 1,...,N$. Optimal policy for the model in this section is that there are cycles of order N in taxes and in bonds. More precisely*

$$\tau_i = \tau_{tN+i} \qquad i = N,...,2N-1 \text{ for all } t = 1,2,...$$

and

$$b_{N,i}^{HTR} = b_{N,tN+i}^{HTR} \quad i = 0,...,N-1, \text{ for all } t = 1,2,...$$

Assume further the standard utility function where higher λ (in a complete markets case) would imply lower taxes, as for example happens with the utility (16), then

$$\tau_{i+N} > \tau_i \quad i = 0,...,N-1$$

Furthermore, if $b_{2,-2}^{HTR} > b_{2,-1}^{HTR}$ then $\tau_0 < \tau_1$

Proof

Consider the case $N = 2$. It is clear from the martingale condition (32) that

$$\lambda_t = \lambda_0 \text{ for all } t > 0, t \text{ even}$$

$$\lambda_t = \lambda_1 \text{ for all } t > 1, t \text{ odd}$$

Therefore

$$\begin{aligned} & u_{c,t} - v_{x,t} + \lambda_0 \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + \bar{g} \right) - v_{x,t} \right) = 0 \text{ for all } t \geq 2, t \text{ even} \\ & u_{c,t} - v_{x,t} + \lambda_1 \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + \bar{g} \right) - v_{x,t} \right) = 0 \text{ for all } t \geq 3, t \text{ odd} \end{aligned} \tag{37}$$

notice the only difference between even and odd is in the Lagrange multiplier λ. This proves

$$\begin{aligned} & c_t = c_2, \tau_t = \tau_2 \text{ for all } t > 2, t \text{ even} \\ & c_t = c_3, \tau_t = \tau_3 \text{ for all } t > 3, t \text{ odd} \end{aligned} \tag{38}$$

The budget constraint (30) can be rolled forward as follows

$$b_{2,t-1}^{HTR} = S_t + \beta^2 \frac{u_{c,t+2}}{u_{c,t}} b_{2,t}^{HTR} = S_t + \beta^2 \frac{u_{c,t+2}}{u_{c,t}} S_{t+2} + \beta^4 \frac{u_{c,t+4}}{u_{c,t}} b_{N,t}^{HTR} = \ldots$$

Using debt limits we conclude

$$b_{2,t-2}^{HTR} = \sum_{j=0}^{\infty} \beta^{2j} \frac{u_{c,t+2j}}{u_{c,t}} S_{t+2j} \quad \text{for all } t = 0,1,\ldots$$

This combined with (38) implies

$$b_t^{HTR} = b_0^{HTR} = \frac{S_2}{1-\beta^2} \text{ for all } t \geq 0, t \text{ even}$$

$$b_t^{HTR} = b_1^{HTR} = \frac{S_3}{1-\beta^2} \text{ for all } t \geq 1, t \text{ odd}$$

The only statement left to prove are the tax cuts in periods $t = 0,1$. For periods $t = 0,1$ we have

$$u_{c,0} + v_{x,0} + \lambda_0 \left(u_{cc,0} c_0 + u_{c,0} + v_{xx,0} \left(c_0 + \bar{g} \right) - v_{x,0} \right) - u_{cc,0} \lambda_0 b_{2,-2}^{HTR} = 0$$
$$u_{c,1} + v_{x,1} + \lambda_1 \left(u_{cc,1} c_1 + u_{c,1} + v_{xx,1} \left(c_1 + \bar{g} \right) - v_{x,1} \right) - u_{cc,1} \lambda_1 b_{2,-1}^{HTR} = 0$$

Notice that the difference with (37) for $t > 1$ is the presence of the terms $u_{cc,0} \lambda_0 b_{2,-2}^{HTR}$ and $u_{cc,1} \lambda_1 b_{2,-1}^{HTR}$. These are clearly negative, implying that for the considered utility functions we have

$$\tau_2 > \tau_0$$
$$\tau_3 > \tau_1$$

The statement in the last line follows immediately from the last FOC written. ■

These results could be easily extended to the case of uncertainty only in period $t = 1$ as in Section 2.3.1, to show that if an adverse shock to g occurs taxes are lowered for the next $N - 1$ periods and there is a cycle of order N.

5.3 Numerical solutions

To write the model recursively, we observe that the Lagrangean can be rewritten as

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + v(x_t) + \lambda_t S_t + u_{c,t} \left(\lambda_{t-N} - \lambda_t \right) b_{N,t-N}^{HTR} + \nu_{1,t} \left(\underline{M}^{HTR} - b_{N,t}^{HTR} \right) + \nu_{2,t} \left(b_{N,t}^{HTR} - \bar{M}^{HTR} \right) \Big\} \tag{39}$$

for $\lambda_{-1} = \ldots = \lambda_{-N} = 0$. In a recursive formulation we would have the $2N+1$ states $\left[\lambda_{t-1}, \ldots, \lambda_{t-N}, b_{N,t-1}^{HTR}, \ldots, b_{N,t-N}^{HTR}, g_t \right]$ just as before. We use condensed PEA again. The FOC show that this problem is easier to solve as there are only two expectations to approximate, $E_t \left(u_{c,t+N} \lambda_{t+N} \right)$, and $E_t \left(u_{c,t+N} \right)$. We choose the core $X_t^{core} = \left(\lambda_{t-N}, b_{N,t-N}^{HTR}, g_t \right)$. We keep the same tolerance level as in the model with buyback. Table 6 summarises the number of linear combinations we needed to approximate our expectations. Relative to Section 3.3, the required state space is larger – in some cases two linear combinations of residuals are needed. Effectively this just means a total of five state variables is enough. The condensed PEA still dramatically reduces the state space and it makes feasible the computation of a non-linear solution.

Table 6

Holding-to-redemption model

N	$2N+1$	# lin. comb. in	
		Φ_{λ}	Φ_{uc_N}
1	3	–	–
2	5	0	0
5	11	0	0
10	21	2	2
15	31	2	2
20	41	2	2

Note: same as in Table 2 except we denote expectations to be approximated by $\Phi_{\lambda} = E_t\left(u_{c,t+N}\lambda_{t+N}\right)$, $\Phi_{uc_N} = E_t\left(u_{c,t+N}\right)$.

Figure 7 shows the impulse response functions for a 10-period bond under hold to redemption with the same calibration as in the previous sections. We compare the policy with the case of a one- and 10-period bond and buyback. The figure is for the case when the government initially has no debt, so it is comparable to Figure 1. We see from the impulse response functions for tax rates that varying the maturity of the bond does affect optimal policy, even for initial zero debt.

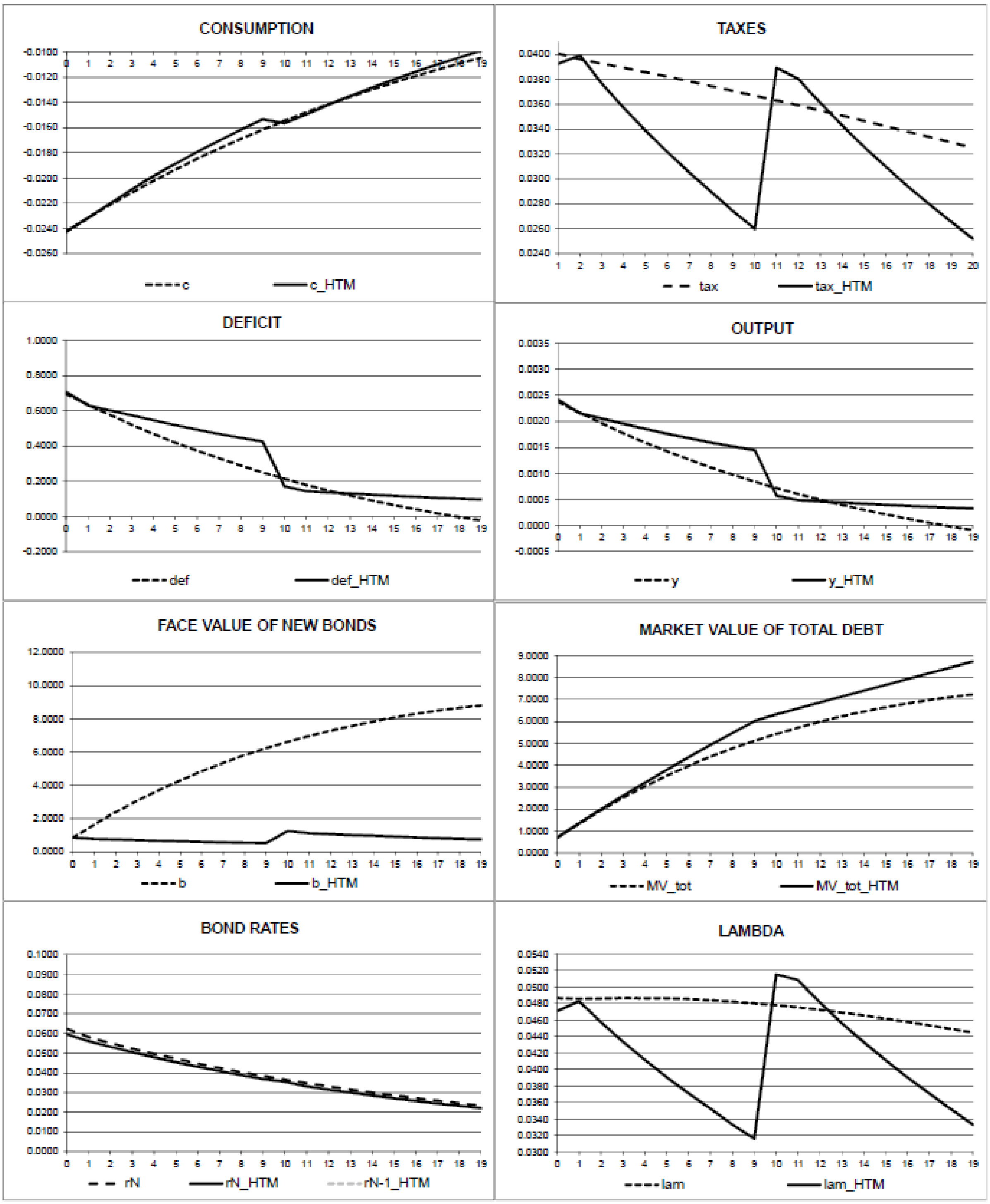
In the buyback case of Sections 2 and 3, when initial debt is zero, $b_{N,-1} = 0$, Figure 1 showed that the government does not promise a cut in taxes. Only when the government is in debt $b_{N,-1} > 0$ (or has assets), as in Figures 2 or 3, did we observe the promise to cut (increase) taxes in $N-1$ periods. Figure 7, however, shows that, even in the case of zero initial debt, taxes show fluctuations. Taxes increase on impact: the response is decreasing for $N-1$ periods, then it jumps at the time of maturity to start going back down after that and so on. The positive but decreasing response for the first $N-1$ periods is standard in optimal taxation models with serially correlated shocks. It would also occur under complete markets: the higher g_t on impact indicates that g_t will also be higher in the next periods, and this generates higher taxes for the next few periods for the utility function considered. The jump in the response function at lag N is a reflection of the fact that there are cycles of order N, as suggested by Result 2 and as can be seen directly from the martingale condition (32). Strictly speaking λ is not a risk-adjusted martingale but one can say that it is a risk-adjusted martingale of cycle N[17]. The initial high and decreasing response echoes N periods later. This is because a high g_t bumps up λ_t so it is optimal to set higher λ_{t+N} and so on. Even if g_{t+N} may be close to its mean, the effect of today's shock on λ_{t+N} drives taxes back up at N lags and the cycle starts again.

[17] Formally, we could say that letting $\xi_t^i = \lambda_{i+tN}$ for $i = 0,1,...,N-1$, each ξ_t^i is a risk-adjusted martingale.

Figure 7

Responses to a positive shock in g_t,
benchmark and holding-to-redemption model

Maturity 10: $b_{N,-1}^{HTR} = ... = b_{N,-1}^{HTR} = 0$



The intuitive reason that there are cycles of order N is the following. One could think of writing the budget constraints under incomplete markets in discounted form as

$$\sum_{j=0}^{\infty} \beta^{j} \frac{u_{c,t+j}}{u_{c,t}} \tilde{S}_{t+j} = \sum_{i=1}^{N} b_{N,t-i}^{HTR} p_{N-i,t} \text{ for all } t \tag{40}$$

These discounted constraints hold in all periods if and only if the period$-t$ budget constraints (30) hold. But, as should be clear from the proof of Result 2, this is not a very relevant condition: even if (40) holds we would easily violate the debt limits (29), since solutions of this equation for b_N given a sequence of surpluses usually generates an unstable solution for issued bonds.

We could instead write the budget constraints as follows:

$$\sum_{j=0}^{\infty} \beta^{jN} \frac{u_{c,t+Nj}}{u_{c,t}} \tilde{S}_{t+Nj} = b_{N,t-N}^{HTR}, \text{ for all } t$$

These are also necessary and sufficient for (30), with the advantage that they guarantee that if we use these conditions to solve for the b_N 's given surpluses, bonds do not go to infinity. These conditions show that what is relevant is the link between today's issued bonds and the surpluses in $N, 2N, 3N, \ldots$ periods from now. If today we have a bad shock and we issue N period bonds, when these bonds mature N periods from now there will be a need for higher taxes and a higher deficit, so $b_{N.t+N}$ will increase. Hence there will be a need for higher taxes and higher deficits in $2N$ periods and so on. Therefore it is reasonable that there is a cycle of period N and that optimal policy has the shape displayed in Figure 7. The optimal response to an unexpected shock is to promise future taxes that in part accommodate the additional debt servicing in the periods when today's debt will have to be repaid.

Result 2 suggests that taxes in the first $N-1$ periods should be lower if the government is in debt. This suggests that optimal policy will be to lower taxes during the first cycle of N periods relative to later cycles. An additional role of commitment is indeed to promise a cut in taxes during the first cycle relative to the cycles later down the line. This is why, in Figure 9, which looks at the case of initial debt, the main difference to Figure 7 is that the second peak in taxes is lower than the first peak, while the opposite is true in Figure 7.

Table 7

Holding-to-redemption model with different maturities

	Maturity	c	y	τ	deficit	R_N	MV	λ
average	**1**	52.60	70.11	0.243	0.43	2.02	–24.69	0.057
	5	52.57	70.07	0.246	0.28	2.02	–17.43	0.058
	10	52.55	70.05	0.247	0.22	2.03	–14.53	0.058
	20	52.54	70.05	0.247	0.19	2.03	–12.77	0.059
std	**1**	3.49	0.35	0.044	1.46	0.5	27.26	0.013
	5	3.47	0.40	0.044	1.67	0.4	32.26	0.014
	10	3.48	0.41	0.044	1.72	0.3	33.98	0.014
	20	3.50	0.41	0.046	1.71	0.2	33.81	0.015

Table 7 shows summary statistics for the model with no buyback and bonds of varying maturities. The results are exceptionally similar to the case of buyback. Because debt is held to maturity each period, the government now issues fewer bonds per period. As in the no buyback case the short sample second moments do show more volatility of tax rates, as shown in Figure 8.

Figure 8

Tax volatility at different horizons benchmark and holding-to-redemption model

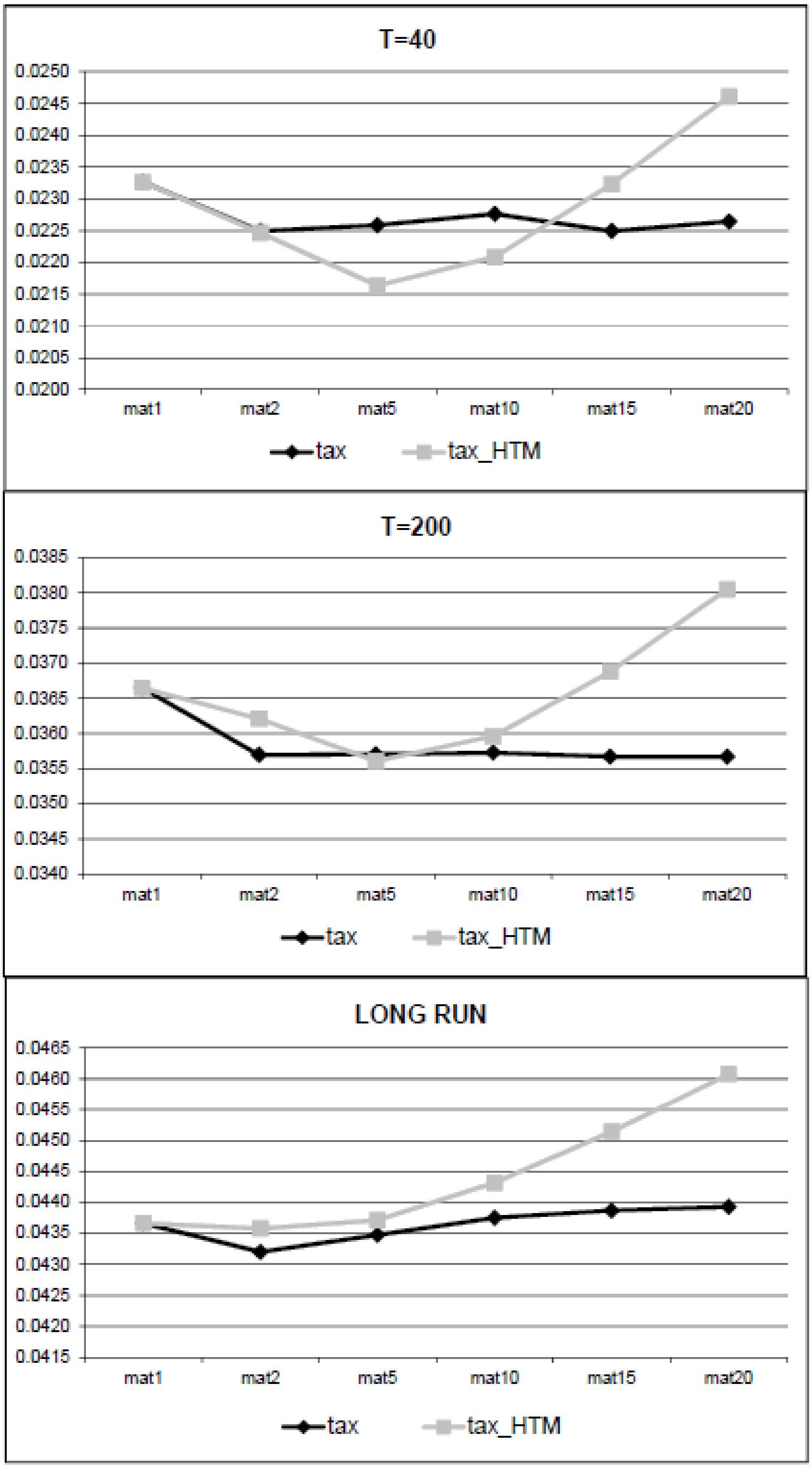
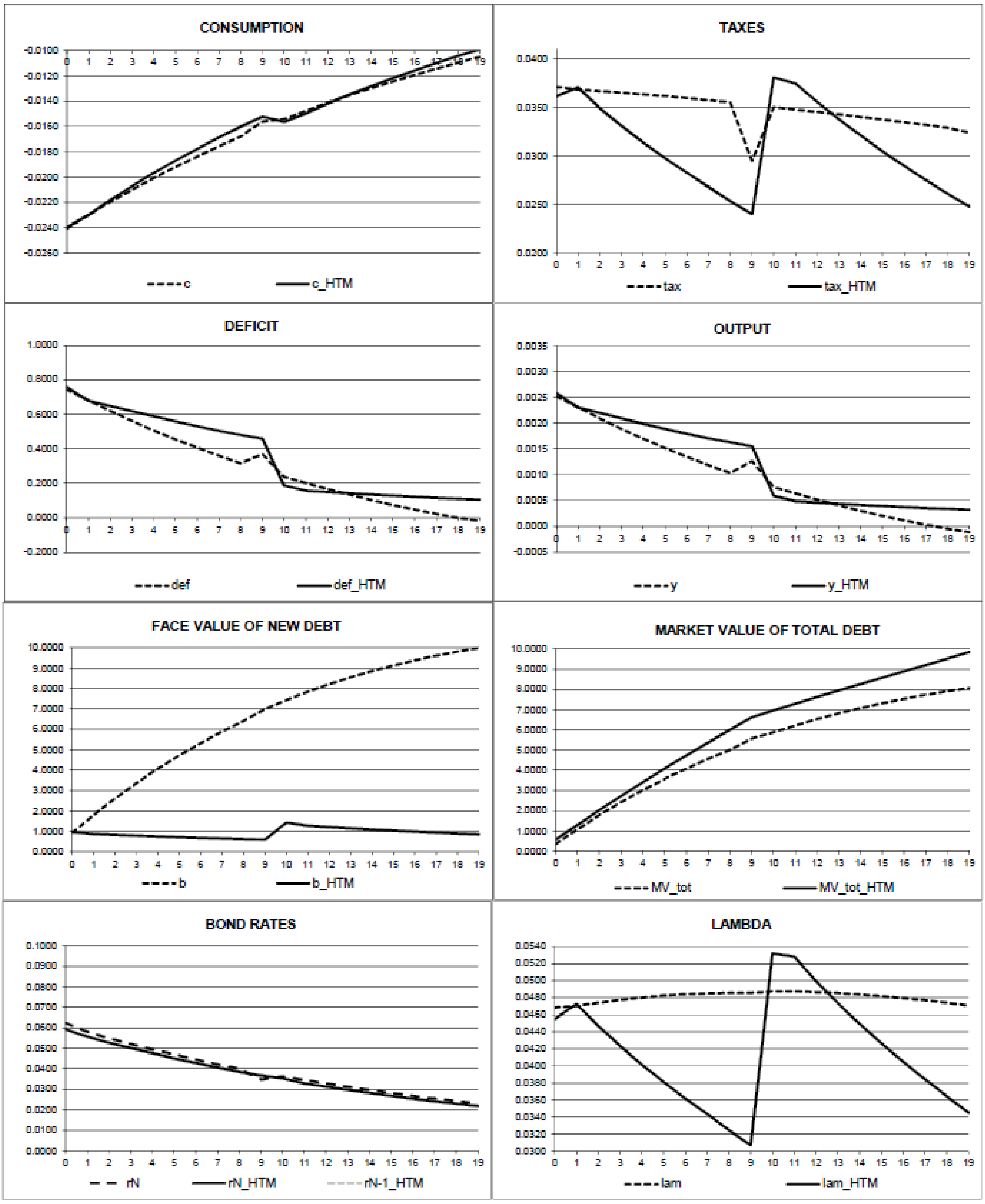


Figure 9

Responses to a positive shock in g_t,
benchmark and holding-to-redemption model

Maturity 10: $b_{N,-1}^{HTR} = \ldots = b_{N,-1}^{HTR} = \dfrac{0.5y_{ss}}{\sum_{i=0}^{N-1}\beta^i}$



6. Conclusions

This paper has had two interrelated aims. The first has been to study optimal fiscal policy when governments issue bonds of long maturity. The second has been to propose a general method for solving models with a large state space – the condensed PEA.

A number of additional considerations arise when governments issue long-term bonds. If the government inherits debt, it has an incentive to twist interest rates to minimise costs of funding debt. This is achieved by violating tax-smoothing and promising a tax cut in $N-1$ periods, when existing bonds mature. A typical debt management concern, namely lowering the cost of debt, therefore shapes the path of fiscal policy. This suggests that it is important to consider debt management and fiscal policy jointly.

The model with long bonds helps to clarify the role of commitment in models of fiscal policy and incomplete markets. In the case of short bonds, the change in taxes needed to adjust to a shock and the promise to cut taxes at time of maturity are conjoined; what is observed is that taxes increase on impact much less if the government is in debt.

In the case of long bonds these two effects are separated. The commitment to cut future taxes is time-inconsistent and also leads to a potentially very large state space of dimension $2N+1$. Using the condensed PEA enables us to solve this model accurately with a much reduced state space allowing for the computation of non-linear numerical solutions.

We also propose an alternative model of government policy, where a central bank determines interest rates and a fiscal authority separately decides on debt and taxes. This model of independent powers is of interest per se, as policy authorities may not be able to coordinate as much as is required to implement the full commitment solution. Also, it does not display policies where promises that will be implemented very far in the future matter for today's solution. As such it serves to highlight the role of commitment and to look at a solution in which the state space is not enormous.

We started with the case usually considered in the literature where government buys back the existing stock of debt each period. To get closer to actual practice we study the case where government bonds are left in circulation until maturity. This model gives rise to even more tax volatility due to debt management concerns: promises to cut taxes for interest twisting purposes are now permanent and policy creates $N-$period cycles, giving rise to even more tax volatility.

There is little quantitative difference in fiscal policy or economic allocations at steady state second moments as the maturity of debt is varied, justifying the observation in Table 1 that similar countries may have very different average maturity of debt. The main difference is in the steady state level of debt: longer maturities imply lower asset accumulation because long bonds provide a volatile deficit if the government holds assets. However, for second moments computed with short-run moments we do find more tax volatility with long bonds.

A number of further issues remain. We have throughout this paper assumed the government can issue only one bond and has varied its maturity. In order to fully understand debt management, we need to consider the case when the government can issue several bonds of different maturity and choose the optimal portfolio. Another important issue is to consider why governments do not buy back debt – presumably because of concerns over transaction costs. We have abstracted from crucial elements of actual debt management practice such as refinancing risk, rollover risk, transaction costs, default etc. We hope the methodologies of this paper will enable us to provide a detailed study of optimal debt management and to introduce some of these features in the analysis.

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