Including linkers in a sovereign bond portfolio: an HJM approach

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1. Introduction

An inflation-linked bond (ILB) is a debt security which generates cash-flows linked to the evolution of a given price index. The aim of the indexation is to protect the "real" value of the investment. Contrary to conventional sovereign[2] fixed or floating rate securities, which offer investors certain nominal rates of return, inflation-linked bonds tie part of their economic result to the evolution of a price index, assuring in this sense a real rate of return. By so doing, the risk/return characteristics of these instruments differ from those of conventional bonds, while still offering the same credit exposure. The question naturally arises whether there are any advantages, from a risk/return perspective, on including this kind of instruments in a bond portfolio made up of conventional fixed/floating rate bonds and money market instruments. In other words, do ILBs constitute a different asset class able to enhance the efficient frontier if included in an otherwise conventional bond portfolio?

Inflation-linked securities have a long history, with the State of Massachusetts having issued a first bond linked to a basket of commodities as long ago as 1780. The modern development of the market is widely regarded to have started in 1981, the year in which the index-linked gilts were first issued by the UK Treasury.

Today, the global (government) market is worth well above EUR 1,000 billion and the main global issuers are the US, UK, France and Italy. Euro-denominated inflation-indexed bonds are issued mainly by the French Treasury (AFT-Agence France Tresor), and by the Italian and German Treasuries as well. The market is dominated by sovereign issuers. However, corporate issuance has also seen growth in recent years. It has been facilitated by the rapid development of inflation-indexed derivatives (such as inflation swaps), which enable greater flexibility in terms of determining the desired cash flows. Mainly due to its relative size versus the other euro-denominated markets, the French market for the inflation-linked bonds seems the most appropriate for an analysis of the impact of including linkers in a bond portfolio. For this reason, all references are made primarily to the (French) HICP-linked bonds. We start by a quick review of the main elements characterizing ILBs, then we address the issue of their inclusion in a bond portfolio so that we can later develop a model for pricing linkers and derivatives.

2. Inflation-linked bonds (ILBs)

The fundamental feature of inflation-indexed securities is that they offer investors the promise of a certain "real" yield or rate of return (r) on their investments as compared to

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[1] Disclaimer: the views expressed here are those of the authors and do not necessarily correspond to those of the European Commission.

[2] Our references to sovereign bonds include agency and supranational securities.

conventional bonds (either zero-coupon or coupon-paying bonds, fixed or floating) which offer investors the promise of a certain "nominal" rate of return (i).[3]

The classic Fisher equation suggests that the expected annual rate of inflation over the life of the two bonds would on average amount to:

$$\pi^e = (i - r)$$

If the actual average annual inflation proves later to be higher, ie $\pi = \pi^e + \varepsilon$, with $\varepsilon > 0$, then the ex-post real annual yield (rate of return) on the nominal bond will turn out to be just:

$$(i - \pi^e - \varepsilon < r).$$

On the other hand, the ILB will still have yielded its promised annual real rate (r).

Naturally, the ex-post real yield on the nominal bond could end up being higher than that on the inflation-linker, should the average inflation rate prove lower than π^e per annum.

The key feature of linkers is that they provide a mean to guarantee *ex-ante* a certain real rate of return, whereas real return on conventional bonds is only known ex-post, depending on the actual inflation rate realized over the investment period.

The actual mechanism inflation linkers use to ensure protection against inflation varies in details across the different countries. In general, however, most issuers, including France, have chosen a relatively simple framework, first introduced by Canada. Specifically, bonds are quoted in real terms, and both principal and coupons are adjusted for changes in the relevant consumer price index between issue date and cash-flow payment date, subject to a certain indexation lag. Such a cash-flow structure is commonly referred to as *capital*-indexed.

We will concentrate on French government linkers as they are the most liquid in the Euro-denominated ILB market.[4]

The following table presents the situation of the euro-denominated sovereign inflation-linked debt in the largest European markets, as of the end of November 2009, and its relative importance in the corresponding total government debt market.

	France	**Germany**	**Italy**	**Greece**	**Total**
Nom.	133.90	22.70	78.50	13.40	248.50
% of LT debt	20.9	3.9	8.6	N/A	11.0
% of EUR debt	13.2	2.3	6.1	5.1	7.0

Source: Periodic bulletins of the respective debt agencies and Barclays Capital.

The prices of inflation-linked bonds are quoted in real terms. Settlement values and cash-flows then adjust for accrued inflation. This mechanism makes linkers entirely equivalent to a conventional bond denominated in a foreign currency: Everything is traded, computed and negotiated in the foreign currency (in real terms in our case) and then the resulting

3 This promise of a real return is just a promise, on the same ground as the promise of a certain nominal yield offered by conventional bonds is subject to a series of assumptions such as reinvestment conditions etc.

4 French linkers account for more than 50% of the euro-linker government market, followed by Italy, representing about 30%, and then Germany and Greece both accounting for less than 10% each.

magnitudes so calculated (accrued interests, principal, coupons) are just multiplied by the “exchange rate” (the index ratio).

This means that for settlement amounts, real accrued interests are calculated as for ordinary OATs. Clean prices and accrued interests are then each multiplied by the index ratio to arrive at a cash settlement amount. As for the coupons paid, the (real) annual coupon rate is multiplied by the index ratio, and likewise for the par redemption amount (with the cash value subject to a par floor).

3. Including government ILBs in a bond portfolio

Now we come to the question if are there any advantages from a risk/return perspective in including ILBs on a bond portfolio made up of conventional fixed/floating rate bonds and possibly some money market instruments?[5] In other words, do ILBs have the potential to enhance the efficient frontier of such a portfolio?

The answer from a theoretical perspective is clearly yes. From an efficient frontier point of view, linkers can significantly enhance the risk/return characteristics of an otherwise classical portfolio. This argument effectively relies on the beta relationship between real and nominal yields. Recalling the Fisher equation that was introduced earlier, which relates nominal rates to real rates, inflation and risk; the offered yield on a nominal bond (i) can be decomposed into the required real return (r), a necessary compensation for inflation (π^e) and a certain risk premium (ρ), as previously stated.

In its loose version the Fisher equation states that

$$i = r + \pi^e + \rho$$

Based on the definition of breakeven inflation, $bei = \pi^e + \rho$, we can write:

$$i = r + bei$$

The variance of the nominal yield can then be written as:

$$Var(i) \equiv Cov(i,i) = Cov(i, r + bei) = Cov(i,r) + Cov(i,bei)$$

Based on this expression, dividing both sides by $Var(i)$ we can get:

$$1 = \frac{Cov(i,r)}{Var(i)} + \frac{Cov(i,bei)}{Var(i)}$$

Again from a theoretical perspective, we should expect some positive covariation between nominal yields and expected inflation, or more precisely between nominal yields and breakeven inflation, ie $Cov(i,bei) > 0$, which means that:

$$\frac{Cov(i,r)}{Var(i)} \equiv \beta(i,r) < 1$$

In other words, this means that part of the variability in nominal yields is accounted for by the variability in breakevens, which leaves the real yield relatively more stable, which in turn translates into additional stability in real prices and real returns on linkers. This also means

[5] References to risk are made to market risk and leave aside credit risk, which is completely similar to that already existing on conventional bonds.

that the sensitivity of linkers to changes in nominal yields will usually (but not necessarily) be less than 1.[6]

The attractiveness of an asset to a portfolio is usually measured in terms of the risk and return trade-off; so if the theory holds in reality, linkers should stand a very good chance of being included in a fixed income portfolio.

Several empirical studies have shown that the efficient frontier of portfolios including linkers as an asset class moves upward, meaning that better rewards are achieved for the same levels of risk.

Barclays ([3]) has tested empirically this assertion for several markets, but we concentrate on the euro-linkers. By the end of 2007 the size of the euro-linker market had surpassed that of the UK, making it the second-largest in the world. In their empirical analysis (with data covering 1998–2007) Barclays found that adding linkers to a portfolio of MM, conventional bonds and equities significantly improved the efficient frontier. Barclays also ran the exercise restricting the weight in the portfolio to 20%, to reflect the fact that linkers share a portion in the market that is lower than 20%, and the improvement remained significant (see Barclays([3])).

Société Générale (SG) also discusses the case of a portfolio investing in European securities (MM, conventional bonds and equities[7]). The study shows that the portfolio becomes more efficient when including linkers from a historical perspective.

In the following section we develop a 3-factor HJM to characterize the economy, with time-dependent (non-stochastic) volatilities. If validated, the fixed income market can be characterized as Gaussian and so the inclusion of linkers in a bond portfolio can be analyzed in the context of the classical portfolio analysis, ie building an efficient frontier just based in the variance-covariance matrix of returns.

4. An HJM approach to pricing bonds

We start from the modeling of the market itself by applying an HJM model to consistently price both ILB and conventional euro-denominated (French) sovereign bonds. As explained in the description of linkers, a foreign currency analogy is naturally suited to implementing this methodology. In the vein of Jarrow and Turnbull [9] and Jarrow and Yildirim [10] we consider a hypothetical world under the no-arbitrage assumption where nominal euros correspond to the domestic currency, real euros correspond to the foreign currency, and the HICP corresponds to the spot exchange rate. In this setup, the fluctuations of the real and nominal interest rates and the inflation rate will be correlated.

Following the foreign bonds analogy, nominal bonds will play the role of “national” bonds (upper-scripted N), the role of “foreign bonds” will be played by the real bonds (upper-scripted R) and the HICP will play the role of the “exchange rate”. The following notation will be used:

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[6] If this beta were always a stable number, it would be easy to calculate the equivalent nominal duration for an inflation bond. Equally, though, if it were that easy, then there would be no additional value to inflation-linked bonds as a diversifying asset class (Barclays Capital).

[7] Total return for nominal bonds and linkers computed from total return Barclays Capital Euro Indices (France), money market returns based on one-month Euribor rates and equity returns derived from the total return MSCI Equity index for France.

1. $f_{t,T}^{h}$ stands for country's h forward rate (with $h \in \{N,R\}$), set at t ,for borrowing over $[T, T+dt], T > t$.[8]

2. $P_{t,T}^{h}$ stands for the price at t of country's h zero-coupon bond, maturing at $T > t$.

3. I_t stands for the HICP, ie the "exchange rate" for a unit of "foreign currency" expressed in terms of the local currency.[9]

4. r_t^h stands for country's h instantaneous risk-free interest rate.

5. $B_t^h = e^{\int_0^t r_u^h du}$ stands for country's h money market account.

In a general HJM-world, $f_{t,T}^{h}$ evolves according to:

$$df_{t,T}^{h} = \alpha_{t,T}^{h} dt + \underline{\sigma}_{t,T}^{h} \cdot \underline{dW}_t$$

with $\underline{\sigma}_{t,T}^{h} = \left(\sigma_{t,T}^{h1}, \cdots, \sigma_{t,T}^{hk}\right)$

and $\underline{W}_t = \left(W_t^1, \cdots, W_t^k\right)$
a k-dimensional Brownian Motion.[10]

In the spirit of Jarrow and Yildirim [10], we will assume a three-factor model, where nominal bonds depend on W_N , real bonds depend on W_R and the HICP depends on W_I , with:

$$dW_N \cdot dW_R = \rho_{N,R} dt$$

$$dW_N \cdot dW_I = \rho_{N,I} dt$$

$$dW_R \cdot dW_I = \rho_{R,I} dt$$

The price of a zero-coupon bond, $P_{t,T}^{h}, h \in \{N,R\}$, may be expressed as a function of these forward rates as:

$$P_{t,T}^{h} = e^{-\int_t^T f_{t,u}^h du} . \tag{1}$$

Letting $f_{0,T}^{h}$ be the forward rate curve at time 0, it is possible to express $f_{t,T}^{h}$ as:

$$f_{t,T}^{h} = f_{o,T}^{h} + \int_0^t \alpha_{s,T}^{h} ds + \int_0^t \sigma_{s,T}^{h} dW_s^h$$

As a particular case, the short rate $r_t^h = f_{t,t}^h$ results:

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8 The dynamic is with respect to calendar time, t , whereas the maturity, T , acts as a parameter.

9 Each ILB has associated a particular initial HICP value, I_0 , which depends on its issuance date, and which constitutes the basis to calculate the applicable "exchange rate" at any particular time, so $\left(I_t / I_o\right)$ and not I_t should be used. For easiness of exposition however, we will assume that both, the HICP's basis and the initial ILB's index coincide, unless otherwise required by the context, which will be made clear in the text.

10 $\alpha_{t,T}$ and $\underline{\sigma}_{t,T}$ are adapted with respect to the σ-algebra generated by $W_s^j, 1 \le j \le k, s \le t$ (the filtration $F_t^{\underline{W}}$) and satisfy the boundary conditions $\mathrm{E}\left(\int_0^T \left|\alpha_{t,T}\right| dt\right) < \infty$ and $\int_0^T \mathrm{E}\left(\left|\underline{\sigma}_{t,T}\right|^2\right) dt < \infty$.

$$r_t^h = f_{t,t}^h = f_{0,t}^h + \int_0^t \alpha_{s,t}^h ds + \int_0^t \sigma_{s,t}^h dW_s^h$$

with dynamics:

$$dr_t^h = \left(\frac{\partial f_{0,t}^h}{\partial t} + \alpha_{t,t}^h + \int_0^t \frac{\partial \alpha_{s,t}^h}{\partial t} ds \right) dt + \sigma_{t,t}^h dW_t^h + \int_0^t \frac{\partial \sigma_{s,t}^h}{\partial t} dW_s^h dt$$

(assuming that $\alpha_{t,T}^h$ and $\sigma_{t,T}^h$ are differentiable with respect to maturity).[11]

Bond prices as given by (1) satisfy a SDE:

$$d_t P_{t,T}^h = P_{t,T}^h \left(\left(r_t^h - \alpha_{t,T}^{h*} + \frac{1}{2} \left| \sigma_{t,T}^{h*} \right|^2 \right) dt - \sigma_{t,T}^{h*} \cdot dW_t^h \right) \tag{2}$$

where we have put:

$$\alpha_{t,T}^{h*} = \int_t^T \alpha_{t,U}^h dU \ ,$$

$$\sigma_{t,T}^{h*} = \int_t^T \alpha_{t,U}^h dU$$

for, respectively, the integrated drift and the integrated volatility with respect to maturity.

The HICP (or "exchange rate") I_t satisfies a SDE as well:

$$dI_t = I_t \left(\mu_t dt + \sigma_t^I dW_t^I \right) \tag{3}$$

Real ("foreign") bonds and the real current account are non-tradeable assets in the domestic economy, ie it is precise to express them (price them) in nominal terms (the domestic currency), in order for them to be tradeable:

- Let $P_{t,T}^T = I_t \times P_{t,T}^R$ be the price in "domestic currency" at t, of the real zero-coupon bond, maturing at $T > t$, ie $P_{t,T}^T$ is the price of a zero-coupon linker.
- Similarly, for the "foreign" money market account, let us define $P_t^T = I_t \times B_t^R$, the value at t, in the domestic currency, of the foreign money market holdings.

These two assets are governed by the following stochastic processes (as a simple application of Ito's rules):

$$\begin{pmatrix} d_t P_{t,T}^T \\ d_t P_t^T \end{pmatrix} = \begin{pmatrix} P_{t,T}^T \left[\left(r_t^R - \alpha_{t,T}^{R*} + \frac{1}{2} \left| \sigma_{t,T}^{R*} \right|^2 + \mu_t - \rho_{R,I} \sigma_t^I \sigma_{t,T}^{R*} \right) dt + \sigma_t^I dW_t^I - \sigma_{t,T}^{R*} dW_t^R \right] \\ P_t^T \left[\left(r_t^R + \mu_t \right) dt + \sigma_t^I dW_t^I \right] \end{pmatrix} \tag{4}$$

In order to price claims in this economy, we need:

1. A replicating *self-financing trading strategy* (SFTS).
2. An *equivalent martingale measure* (EMM) for discounted bond prices.

As there are three sources of uncertainty in our model, we need three securities and a savings account to build a SFTS. We choose a (any) nominal zero-coupon bond $\left(P_{t,T_1}^T \right)$, a

[11] This is in general not a SDE, due to the final integral. In fact, r_t^h is in general not a Markov process.

(tradeable) inflation-linked zero-coupon bond $\left(P_{t,T_2}^{T}\right)$, the (tradeable) real saving account $\left(P_t^{T}\right)$ and the nominal saving account $\left(B_t^{N}\right)$.

The SFTS will be a vector of adapted processes $\left(\underline{\varphi}_t, \psi_t\right) = \left(\varphi_t^1, \varphi_t^2, \varphi_t^3, \psi_t\right)$ on this set of securities, such that, if $V_t\left(\underline{\varphi}_t, \psi_t\right)$ is the portfolio's value at t ,

$$V_t\left(\underline{\varphi}_t, \psi_t\right) = \varphi_t^1 P_{t,T_1}^N + \varphi_t^2 P_{t,T_2}^T + \varphi_t^3 P_t^T + \psi_t B_t \,,$$

then, for $t < \min(T_1, T_2)$:

$$dV_t\left(\underline{\varphi}_t, \psi_t\right) = \varphi_t^1 P_{t,T_1}^N + \varphi_t^2 P_{t,T_2}^T + \varphi_t^3 dP_t^T + \psi_t dB_t \,,$$

(where we have put $B_t = B_t^N$).

If $\left(\varphi_t, \psi_t\right)$ is self-financing for $\left(P_{t,T_1}^N, P_{t,T_2}^T, P_t^T, B_t\right)$, then it is also self-financing for the discounted bond prices $\left(Z_{t,T_1}^N, Z_{t,T_2}^T, Z_{t,t}^T, 1\right)$ with:

$$Z^h := \frac{P^h}{B_t}, h \in \{N, T\}$$

From equations (2) and (4), the definition of Bt and the rules of Ito’s calculus, it results that:

$$\begin{pmatrix} \dfrac{dZ_{t,T_1}^N}{Z_{t,T_1}^N} \\ \dfrac{dZ_{t,T_2}^R}{Z_{t,T_2}^T} \\ \dfrac{dZ_t^T}{Z_t^T} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2}\left(\sigma_{t,T_1}^{N*}\right)^2 - \alpha_{t,T_1}^{N*}\right)dt - \sigma_{t,T_1}^{N*} dW_t^N \\ \left(\frac{1}{2}\left(\sigma_{t,T_2}^{R*}\right)^2 - \alpha_{t,T_2}^{R*} + r_t^R - r_t^N + \mu_t - \rho_{R,I}\sigma_t^I \sigma_{t,T_2}^R\right)dt + \sigma_t^I dW_t^I - \sigma_{t,T_2}^{R*} dW_t^R \\ \left(r_t^R - r_t^N + \mu_t\right)dt + \sigma_t^I dW_t^I \end{pmatrix}$$

We now turn into the issue of the EMM for Z^h . Let us define:

$$\underline{W}_t = \underline{\hat{W}}_t - \int_0^t \underline{\gamma}_s ds$$

$$\underline{dW}_t = \underline{d\hat{W}}_t - \underline{\gamma}_t dt \,,$$

where $\underline{\hat{W}}_t$ is a three-dimensional Brownian motion with respect to a new probability measure $\Theta = Q^{\gamma}$, given by Girsanov’s theorem, with Girsanov density $\exp\left(-\int_0^{T*} \underline{\gamma}_t \underline{dW}_t - \frac{1}{2}\int_0^{T*} \left|\underline{\gamma}_t\right|^2 dt\right)$,

With respect to this new probability measure, dZ / Z becomes:

$$\begin{pmatrix} \dfrac{dZ_{t,T_1}^N}{Z_{t,T_1}^N} \\ \dfrac{dZ_{t,T_2}^R}{Z_{t,T_2}^T} \\ \dfrac{dZ_t^T}{Z_t^T} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2}\left(\sigma_{t,T_1}^{N*}\right)^2 - \alpha_{t,T_1}^{N*} + \sigma_{t,T_1}^{N*}\gamma_t^N\right)dt - \sigma_{t,T_1}^{N*} d\hat{W}_t^N \\ \left(\frac{1}{2}\left(\sigma_{t,T_2}^{R*}\right)^2 - \alpha_{t,T_2}^{R*} + r_t^R - r_t^N + \mu_t - \rho_{R,I}\sigma_{t,T_2}^{R*} + \sigma_{t,T_2}^{R*}\gamma_t^R - \sigma_t^I \gamma_t^I\right)dt + \sigma_t^I d\hat{W}_t^I - \sigma_{t,T_2}^{R*} d\hat{W}_t^R \\ \left(r_t^R - r_t^N + \mu_t - \sigma_t^I \gamma_t^I\right)dt + \sigma_t^I d\hat{W}_t^I \end{pmatrix} \quad (5)$$

and to have Z_*^h driftless, we need that:

$$\begin{pmatrix} \alpha_{t,T_1}^{N*} \\ \alpha_{t,T_2}^{R*} \\ \mu_t \\ \alpha_{t,T_2}^{R*} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\sigma_{t,T_1}^{N*}\right)^2 + \sigma_{t,T_1}^{N*}\gamma_t^N \\ \frac{1}{2}\left(\sigma_{t,T_2}^{R*}\right)^2 + \sigma_{t,T_2}^{R*}\gamma_t^R - \sigma_t^I\gamma_t^I + r_t^R - r_t^N + \mu_t - \rho_{RI}\sigma_t^I\sigma_{t,T_2}^{*R} \\ r_t^R - r_t^R + \sigma_t^I\gamma_t^I \\ \frac{1}{2}\left(\sigma_{t,T_2}^{R*}\right)^2 + \sigma_{t,T_2}^{R*}\gamma_t^R - \rho_{RI}\sigma_{t,T_2}^{*R} \end{pmatrix} \tag{6}$$

(the last equation for α_{t,T_2}^{R*} in the fourth line above, results from substituting μ_t for its expression in the third line.)

In order for $Q = Q^{\gamma}$ to be an EMM simultaneously for bond prices of all maturities, $\underline{\gamma}_t$ needs to be T-independent and this in turn means that, given the (integrated) volatilities $\underline{\sigma}_{t,T}^{h*}$, equation (6) is a condition on the (integrated) drifts, $\int_t^T \alpha_{t,U}^h dU$.

Differentiating both sides with respect to T we obtain a condition on the α's themselves:

$$\begin{pmatrix} \alpha_{t,T}^N \\ \alpha_{t,T}^R \\ \mu_t \end{pmatrix} = \begin{pmatrix} \sigma_{t,T}^N\sigma_{t,T}^{N*} + \sigma_{t,T}^N\gamma_t^N \\ \sigma_{t,T}^R\left(\sigma_{t,T}^{R*} - \rho_{RI}\sigma_t^I\right) + \sigma_{t,T}^R\gamma_t^R \\ r_t^N - r_t^R + \sigma_t^I\gamma_t^I \end{pmatrix}$$

Our HJM-model is therefore determined by:

- specifying the volatilities $\left\{\sigma_{t,T}^N, \sigma_{t,T}^R, \sigma_t^I\right\}$ with respect of the three risk-factors $\left\{W_{t,T}^N, W_{t,T}^R, W_t^I\right\}$ and
- specifying the corresponding market prices of risk $\left\{\gamma_{t,T}^N, \gamma_{t,T}^R, \gamma_t^I\right\}$.

These solutions for the drifts, $\{\alpha, \alpha^{*h}$ and $\mu\}$, allow us to write the following system of equations which is the basis for our estimation:

$$\begin{pmatrix} \frac{d_t P_{t,T}^N}{P_{t,T}^N} \\ \frac{d_t P_{t,T}^R}{P_{t,T}^R} \\ \frac{dP_{t,T}^T}{P_{t,T}^T} \\ \frac{dP_t^T}{P_t^T} \\ \frac{dI^t}{I^t} \end{pmatrix} = \begin{pmatrix} r_t^N dt - \sigma_{t,T}^{N*} d\hat{W}_t^N \\ \left(r_t^R + \rho_{RI}\sigma_t^I\sigma_{t,T}^{*R}\right)dt - \sigma_{t,T}^{R*} d\hat{W}_t^R \\ r_t^N dt + \sigma_t^I d\hat{W}_t^I - \sigma_{t,T}^{R*} d\hat{W}_t^R \\ r_t^N dt + \sigma_t^I d\hat{W}_t^I \\ \left(r_t^N - r_t^R\right)dt + \sigma_t^I d\hat{W}_t^I \end{pmatrix} \tag{7}$$

4.1 Pricing contingent claims

Let X be $F\frac{W}{T}$ - measurable.

- The martingale representation theorem allows to write any discounted claim's price as:

$$\widetilde{X} := \frac{X}{B_T} = X_0 + \int_0^T \left(\delta_t^N d\hat{W}_t^N + \delta_t^R d\hat{W}_t^R + \delta_t^I d\hat{W}_t^I\right) \quad (8)$$

for some $X_0 \in \mathrm{P}$ and $F\frac{\hat{W}}{t}$ adapted processes $\underline{\delta}_t = \left(\delta_t^N, \delta_t^R, \delta_t^I\right)$.

- Using equations (5) and (6), we get:

$$\begin{pmatrix} \frac{dZ_{t,T_1}^N}{Z_{t,T_1}^N} \\ \frac{dZ_{t,T_2}^R}{Z_{t,T_2}^R} \\ \frac{dZ_t^T}{Z_t^T} \end{pmatrix} = \begin{pmatrix} -\sigma_{t,T1}^{N*} d\hat{W}_t^N \\ \sigma_t^I d\hat{W}_t^I - \sigma_{t,T_2}^{R*} d\hat{W}_t^R \\ \sigma_t^I d\hat{W}_t^I \end{pmatrix} \quad (9)$$

and, assuming that the 3×3 matrix Σ

$$\sum = \begin{pmatrix} -\sigma_{t,T_1}^{N*} & 0 & 0 \\ 0 & -\sigma_{t,T_2}^{R*} & \sigma_t^I \\ 0 & 0 & \sigma_t^I \end{pmatrix}$$

is invertible, for all t, it is possible to invert the set of linear relations (9), and with

$$d\hat{W} = \Sigma^{-1} \frac{dZ}{Z}$$

substituting this back into (8), it results:

$$\widetilde{X} = X_0 + \int_0^T \left(\varphi_t^N dZ_{t,T_1}^N + \varphi_t^R dZ_{t,T2} T + \varphi_t^I dZ_t^T\right),$$

with:

$$\begin{pmatrix} \varphi_t^N \\ \varphi_t^R \\ \varphi_t^I \end{pmatrix} = \begin{pmatrix} \frac{-\delta_t^N}{\sigma_{t,T_1}^{N*} Z_{t,T_1}^N} \\ \frac{-\delta_t^R}{\sigma_{t,T_2}^{R*} Z_{t,T_2}^T} \\ \frac{\delta_t^R}{\sigma_{t,T_2}^{R*} Z_{t,T_2}^T} + \frac{\delta_t^I}{\sigma_t^I Z_t^T} \end{pmatrix}$$

$\underline{\varphi}_{t=} \left(\varphi_t^N, \varphi_t^R, \varphi_t^I\right)$, invested in the three assets together with

$$\psi_t = -\left(\varphi_t^N Z_{t,T_1}^N + \varphi_t^R Z_{t,T_2}^T + \varphi_t^I Z_t^T\right) + \int_0^t \left(\varphi_t^N dZ_{t,T_1}^N + \varphi_t^R dZ_{t,T_2}^T + \varphi_t^I dZ_t^T\right)$$

in a saving account, constitute a self-financing trading strategy replicating $\widetilde{X}$ at T .

- The price of our contingent claim, X_0, is obtained by taking expectations with respect to E_Θ, as $\mathrm{E}_{\Theta,t}\left(dZ_{t,*}^h\right) = 0$, by construction.

5. Data treatment/generation

5.1 Data description

The data set includes daily closings of selected euro-benchmark government nominal bonds[12] and French government inflation-linked bonds for the period 09/03/2007 to 26/02/2010, as well as monthly data on the harmonized consumer price index (HICP), covering the same period but with a monthly frequency. Consequently, the data set comprises around 800 daily observations for each tenor corresponding to nominal and real bonds and 36 observations corresponding to the HICP.

5.2 Nominal and real interest rates

Data for nominal bonds (spot rates) was available on 15 different maturities: three and six months, one to 10, 15, 20 and 30 years. Data on linkers (daily prices) corresponded to the five benchmark French HICP-linked bonds.

Zero-coupon (spot) nominal rates were available directly from Bloomberg information service.[13] They are estimated by Bloomberg itself, on the basis of the data on traded nominal bonds, issued by euro area-based sovereign issuers. The data as published by Bloomberg are constant maturity rates.

Data on zero-coupon real rates were not readily available from Bloomberg. Therefore, in order to extract the rates at the desired maturities, we estimated the relevant daily term structures on the basis of the five benchmark euro-denominated French sovereign bonds, linked to the euro area HICP inflation index (excluding tobacco) and published monthly by Eurostat.

The estimation procedure involved cross-sectional fitting of the zero-coupon, Nelson-Siegel (1987) term structure to all daily price observations, available from 09/03/2007 until 26/02/2010.[14] It is a fairly accurate approximation of the current term structure of zero-coupon rates, provided that its shape is not too irregular. The model is still widely used by the market participants (see eg BIS (2005 ([7])).

The starting point is the description of the forward rate curve. Its shape is given by the time to maturity, $T-t$, as well as four parameters: $\beta_0, \beta_1, \beta_2, \kappa,$ according to the following formula:

$$f_{t,T}^R = \beta_0 + \left(\beta_1 + \beta_2 (T-t)\right) e^{-\kappa(T-t)}$$

where $f_{t,T}^R$ stands for the rate, set at t , for borrowing over $[T, T+dt], T > t$ in the "foreign country".

The corresponding spot rate term structure takes the form:

$$r_{t,T}^R = \beta_0 + \left(\beta 1 + \frac{\beta_2}{\kappa}\right)\left(\frac{1-e^{-\kappa(T-t)}}{\kappa(T-t)}\right) - \frac{\beta_2}{\kappa} e^{-\kappa(T-t)}$$

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12 French government treasury bills for maturities up to one year and German government bonds for maturities beyond one year.

13 More precisely, the indices can be found using the Fair Market Curve (FMC) function in Bloomberg, and then choosing curve number F960.

14 Although the Nelson-Siegel model family is known to violate the no-arbitrage assumptions when considered in the time dimension, it must be noticed that the estimations were carried out for a series of cross-sectional observations. In other words, the inter-temporal dynamics of the Nelson-Siegel model did not play any role in the analysis.

By making T approach t the instantaneous short rate results as $r^R(t) = \beta_0 + \beta_1$ and allowing for T to grow unbounded the long-term rate $r^R_{t,\infty}$ becomes equal to $r^R_{t,\infty} = \beta_0$. The remaining parameters govern the location and size of the hump.

These spot rates can be easily transformed into the discount factors:

$$P^R_{t,T} = e^{-r^R_{t,T}(T-t)}$$

and these, in turn, can be used to price financial assets traded on the market, including bonds. However, the specific problem encountered in the present project necessitated the application of a reverse engineering, whereby the observed market prices of the five French HICP inflation-linked bonds served to estimate the unknown parameters. More specifically, the observed bond prices were compared to the theoretical prices given by the formula:

$$B^R_{t,T} = \sum c^R_j \times P^R_{t,j}$$

where c^R_j denote the (real) cash-flows, and $B^R_{t,T}$ stands for the sum of the discounted real cash-flows – ie the “foreign-currency’ price of the bond. The estimation of the parameters was conducted by way of minimizing the sum of the squared errors between the prices of the five French inflation-linkers and their model counterparts. Obviously, given that the Nelson-Siegel model in its original form is static (ie it describes the term structure at a given moment, and not its evolution over time), the estimation procedure needed to be carried out separately for each day in the sample – ie around 800 daily observations.[15]

Then, the (theoretical) real zero-coupon rates were also calculated for the 15 selected maturities, as outlined above. From this set of zero-rates, the (daily) returns are derived as follows:

$$R_{t,T} = (r_{t-1,T} - r_{t,T})(T-t) + r_{t,T}\frac{1}{252} \tag{10}$$

Both the nominal zero-coupon rates published by Bloomberg and the model-derived real zero-coupon rates have in effect constant maturities. Thus, in order to compute the return (either nominal or real) for holding a Z-bond over a one day period, we need the corresponding ($(t-T)-1$day) maturity rate, which is not directly available from the data. For this purpose, we assume that the interest rate of a given maturity $(t-T)$ is also valid for maturity ($t - T - 1$day).[16]

Zero-coupon forward bonds are martingales under the forward measure. To preserve consistency in the empirical part, parameter estimation was conducted using forward rates. For example, the three-month spot and six-month spot nominal rates for a given day were used to calculate the implied 3x3M forward rate for that same day. The same transformation was conducted for the remaining maturities. The resulting dataset comprised the forward nominal and real interest rates, with the following fourteen maturities: (3M, 9M, 1.75Y, 2.75Y, 9.75Y, 14.75Y, 19.75Y, 29.75Y).

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[15] One of the French inflation-linkers, FRTR –3:15% maturing in July 2032, was deliberately left out of the estimation sample, in order to evaluate the out-of-sample performance of the model. The table at the end of this section presents the results of the estimation.

[16] Although several “fine-tuned” alternatives are possible, the impact of this assumption on the final result is negligible.

5.3 Returns on inflation-linked zero-coupon bonds

As explained in Section 2, inflation-linked bonds are traded on a nominal basis, after adjusting for the inflation accrued over a given period:

$$B_{t,T}^{T} = B_{t,T}^{R} \times IR_t = \sum c_j^R \times P_{t,j}^{T} \quad (11)$$

where $P_{t,j}^{T} = P_{t,j}^{R} \times IR_t$ denotes the price, at time t, of an index-linked zero-coupon bond, $P_{t,j}^{R}$ stands for the t-time price of the underlying real zero-coupon bond, and IR_t is the corresponding index ratio for day t.

The price of the synthetic 3M-forward zero-ILB was calculated according to the following formula:

$$P_{t,t+3M,j}^{T} = \frac{P_{t,j}^{T}}{P_{t,3M}^{N}} \quad (12)$$

with $P_{t,t+3M,j}^{T}$ denoting the price, agreed at t, of a 3M-forward index linked zero-coupon bond delivered at $t+3M$ and maturing at j. $P_{t,3M}^{N}$, in turn, is a price of a three-month nominal bill. The return on such synthetic forward zero-coupon bonds was calculated in line with the procedure as outlined in the equation (10).

5.4 Smoothing algorithm

The final step before estimating the variance was to apply the smoothing algorithm, similar to that implemented by Jarrow and Yildirim (2003). The aim of the procedure was to ensure that the obvious outliers (eg resulting from the poor market quotes), which generate noise in the data, are excluded from the analysis. The smoothing algorithm was based on the following formula:

$$\left| \frac{yield - Mean(yield)}{\sigma_{yield}} \right| \leq k \quad (13)$$

where k varies from 3.25 to 2.50, depending on the maturity. The purpose of varying the parameter k was to ensure that the overall data sample is broadly balanced (ie the number of observations is approximately equal) across the maturities.

5.5 The rate of inflation

The monthly rate of inflation was calculated from the euro area HICP inflation index (excluding tobacco), published monthly by Eurostat.[17] In line with Jarrow and Yildirim (2003), the raw index data was transformed into the rate of inflation using the following formula:

$$\frac{dI_t}{I_t} = \ln\left(\frac{\Delta I_t}{I_t}\right) \quad (14)$$

[17] The data can be found eg in Bloomberg using the following mnemonic: CPTFEMU < Index >.

5.6 Estimation procedure

The aim of the procedure was to estimate the following parameters:

Parameter	Definition
α^N	Time decay factor of the nominal return's volatility: $\sigma^N e^{-\alpha^N (T-t)}$.
α^R	Time decay factor of the real return's volatility: $\sigma^R e^{-\alpha^R (T-t)}$.
σ^N	Scale factor for the nominal return's volatility.
σ^R	Scale factor for the real return's volatility.
σ^I	Constant HICP's volatility.
$\rho_{N,R}$	Correlation between Nominal and Real return risk drivers.
$\rho_{N,I}$	Correlation between Nominal return and Inflation risk drivers.
$\rho_{R,I}$	Correlation between Real return and Inflation risk drivers.

The estimation proceeded in several steps. In each case, it involved fitting of the variance/covariance function to the cross section of the observed variances/covariances of the returns on the forward real bonds, forward inflation-linked bonds, forward nominal bonds, as well as inflation. Fitting was performed using nonlinear least squares.

As shown in the table above, volatilities were assumed to be time dependent, but deterministic $\left(\sigma_{t,T}^h = \sigma^h e^{-\alpha^h (T-t)}, h \in \{N,R\}\right)$ and as a consequence the rates of price changes are Gaussian. The parameters of the nominal return process were then estimated using the following equation:

$$Var\left(\frac{dP_{\tau,T}^N}{P_{\tau,T}^N}\right) = \left[\int_\tau^T \sigma^N e^{-\alpha^N (U-\tau)} dU\right]^2 dt = \left\{\sigma^N \left(\frac{1-e^{-\alpha^N}(T,\tau)}{\alpha^N}\right)\right\}^2 dt \qquad (15)$$

with τ denoting the (forward) maturity and $dt = 1/252$ representing the time step. As usual, the variable to be explained (the variance of the forward returns on the nominal bonds) is on the left-hand side, and the only explanatory variable on the right-hand side is the forward maturity. Likewise, the parameters of the real return process were evaluated based on the equation:

$$Var\left(\frac{dP_{\tau,T}^R}{P_{\tau,T}^R}\right) = \left[\int_\tau^T \sigma^R e^{-\alpha^R (U-\tau)} dU\right]^2 dt = \left\{\sigma^R \left(\frac{1-e^{-\alpha^R}(T,\tau)}{\alpha^R}\right)\right\}^2 dt \qquad (16)$$

The four parameters $\alpha^N, \sigma^N, \alpha^R, \sigma^R$ served immediately to evaluate the correlation between the nominal and the real returns, $\rho_{R,N}$. To this end, use was made of the following equation:

$$Cov\left(\frac{dP_\tau^R}{P_\tau^R}, \frac{dP_\tau^N}{P_\tau^N}\right) = \rho_{R,N} \times \sigma^R \times \sigma^N \times \left(\frac{1-e^{-\alpha^R (T-\tau)}}{\alpha^R}\right)\left(\frac{1-e^{-\alpha^N (T-\tau)}}{\alpha^N}\right) dt$$

The next step required the evaluation of the volatility of inflation, σ^I. It was approximated by the sample standard deviation of the rate of inflation (Eurozone HICP) over the period starting in March 2007 and ending in February 2010 (36 observations). With use of this additional parameter, it was possible to estimate the correlation between the nominal returns and the inflation, based on the equation:

$$Cov\left(\frac{dI_\tau}{I_\tau}, \frac{dP_\tau^N}{P_\tau^N}\right) = -\rho_{I,N} \times \sigma^I \times \sigma^N \times \left(\frac{1-e^{-\alpha^N(T-\tau)}}{\alpha^N}\right)$$

Another equation, analogous to the previous one, albeit involving the real rate and the inflation, was used to estimate $\rho_{R,I}$, the correlation between these two processes:

$$Cov\left(\frac{dI_\tau}{I_\tau}, \frac{dP_\tau^R}{P_\tau^R}\right) = -\rho_{I,R} \times \sigma^I \times \sigma^R \times \left(\frac{1-e^{-\alpha^R(T-\tau)}}{\alpha^R}\right)$$

5.7 Calibration based on ZCIIS

An alternative way to calibrate the real part of our HJM model is to recur to the market for inflation derivatives, in particular to the Zero-Coupon Inflation-Indexed Swaps (ZCIIS).

ZCIIS are actively traded in the European, UK and US markets and are the most liquid inflation derivatives. As their prices are model-independent, the term structure of real rates can be easily derived from the nominal term-structure and market inflation swap rates.

On a ZCIIS one party pays inflation on a notional amount N, whereas the other party pays fixed on the same notional. The contract is for settlement at maturity (T) and its value is zero at inception (t). The fixed rate (k) is chosen so as to make the value of the fixed leg equal to that of the inflation leg, when the swap is initially traded at t. Formally:

$$B_t^N \mathrm{E}_\Theta\left\{B_T^{N-1}\left(\frac{I_T}{I_t} - (1+k)^{T-t}\right)\middle| F_t\right\} = 0 \qquad (17)$$

from where it results that:

$$P_{t,T}^R = P_{t,T}^N (1+k)^{T-t} \qquad (18)$$

where k is the quoted ZCIIS, and the corresponding data series is available from Bloomberg.

5.8 Results of estimation

The following tables present the estimated coefficients, together with their standard errors and significance levels for both, the Nelson-Siegel derived real z-bonds and the ZCIIS-derived bonds:

Estimation using Nelson-Siegel-derived zero-coupon real bonds.		
Parameter	**Value**	**St. Error**
α^N	1.9713E – 03	(1.575E – 03)
α^R	6.0379E – 03***	(8.65E – 04)
σ^{N^2}	4.5289E – 05***	(1.94E – 06)
σ^{R^2}	4.4214E – 05***	(1.01E – 06)
σ^{I^2}	2.1052E – 04	*
$\rho_{N,R}$	0.7434***	(7.384E – 03)
$\rho_{N,I}$	0.3780***	(1.566E – 02)
$\rho_{R,I}$	0.2468***	(8.559E – 03)

*** significance at $\alpha = 1\%$

* estimate based on sample variance of inflation

Estimation using ZCIIS-derived zero-coupon real bonds.		
Parameter	**Value**	**St. Error**
α^N	1.9713E – 03	(1.575E – 03)
α^R	11.4361E – 03***	(13.14E – 04)
σ^{N^2}	4.5289E – 05***	(1.94E – 06)
σ^{R^2}	7.0432E – 05***	(2.37E – 06)
σ^{I^2}	2.1052E – 04	*
$\rho_{N,R}$	0.7995***	(8.29E – 03)
$\rho_{N,I}$	0.3780***	(1.566E – 02)
$\rho_{R,I}$	0.1809***	(8.563E – 03)

*** significance at $\alpha = 1\%$

* estimate based on sample variance of inflation

6. Hedging analysis

The three-factor HJM model we have fitted to the market needs now to be validated.

We will do this via a hedging analysis, ie we will price traded inflation-linked bonds and nominal bonds out of the model and compare these model-derived prices with those actually traded in the market. We will do this for the whole time range considered in this study

(09/03/2007–26/02/2010). Traded are coupon-bearing bonds, not zeroes, so we need to do it with actual traded bonds. The procedure to hedge the linker is as follows:[18]

- First build two portfolios: portfolio A, including the linker whose price we are trying to validate and portfolio B, including (in principle) three bonds (two linkers and one nominal bond). Portfolio B requires three different bonds in order to control for the three risk factors in the economy.

- Then calculate the required amounts of each bond in portfolio B ($n_{1,t}$, $n_{2,t}$ and $n_{3.t}$) so that the total investment required to build it at time t matches exactly the cost of buying the linker in portfolio A.

- Then calculate the daily return of each portfolio $\left(R_t^A, R_t^B\right)$ and compute the difference $\left(\varepsilon_t = R_t^A - R_t^B\right)$. If the model is correct, the difference should be indistinguishable from 0.

- Finally validate the model via analysis of residuals.

Due to the fact that we have specified the volatilities as deterministic functions of time, all zero-coupon bonds (nominal and real) are Markov in three state variables: The instantaneous nominal and real rates, r_r^N, r_r^R and the inflation index, I_t.

In particular, the specification of volatilities in the model – $\sigma_{t,T}^h = \sigma^h e^{-\alpha^h (T-t)}$ – translates into r^N and r^R following Ornstein–Uhlenbeck stochastic processes:

$$\begin{pmatrix} dr_t^N \\ dr_t^R \end{pmatrix} = \begin{pmatrix} \alpha^N \left(\theta_t^N - r_t^N\right) dt + \sigma^N d\hat{W}_t^N \\ \alpha^R \left(\theta_t^R - r_t^R\right) dt + \sigma^R d\hat{W}_t^R \end{pmatrix} \tag{19}$$

which in turn determines specific forms for the corresponding zero-coupon bond prices:

$$\begin{pmatrix} P_{t,T}^N \\ P_{t,T}^R \end{pmatrix} = \begin{pmatrix} e^{A_{t,T}^N - b_{t,T}^N r_t^N} \\ e^{A_{t,T}^R - b_{t,T}^R r_t^R} \end{pmatrix} \tag{20}$$

where $A_{t,T}^h, h \in \left(N, R\right)$ are functions of time that turn out not to matter for the hedging exercise, and

$$b_{t,T}^h = \left(\frac{1 - e^{-\alpha^h (T-t)}}{\alpha^h} \right)$$

We first build portfolio B so that it is worth the same as the price of the ILB we are trying to hedge:

$$n_1 B_{t,1}^T + n_2 B_{t,2}^T + n_3 B_{t,3}^N = B_{t,0}^T$$

where $B_{t,k}^f$, $f \in \left(T, N\right)$, $k \in$ (0, 1, 1, 3) stand for the price at t of the corresponding coupon bearing bond k.

[18] As stated in the section describing the index-linked bonds, linkers usually include a par floor, granting that the capital received will at least be equal to 100%. The value of this option is usually considered to be zero, and we treat them similarly, given that it is highly unlikely that it would ever need to be executed.

Prices of traded linkers are actually the product of the "real" bond prices, $B^R_{t,k}$ and the corresponding "exchange rate", I_t/I_k, with I_k standing for the associated base index.

$$B^T_{t,k} = \frac{I_t}{I_k} B^R_{t,k}, \text{ with } B^R_{t,k} = \sum_{j=1}^{n_k} c^k_j P^R_{t,j}$$

where

$$\left(c^k_j \right) = \begin{pmatrix} c^k \quad \forall j < n_k \\ \left(c^k + 1\right) \; for \quad j = n_k \end{pmatrix}$$

In order for portfolio B to hedge portfolio A (the linker) we need:

$$\begin{array}{ccc} B^N_{t,0} & = & n_1 B^T_{t,1} + n_2 B^T_{t,2} + n_3 B^N_{t,3} \\ dB^T_{t,0} & = & n_1 dB^T_{t,1} + n_2 dB^T_{t,2} + n_3 dB^N_{t,3} \\ \frac{\partial \mathrm{B}^{\mathrm{T}}_{\mathrm{t,0}}}{\partial \mathrm{r}^{\mathrm{R}}_{\mathrm{t}}} dr^R_t + \frac{\partial B^T_{t,0}}{\partial I_t} dI_t + \cdots & = & \sum_{i=1}^{2} n_i \frac{\partial B^T_{t,i}}{\partial r^R_t} dr^R_t + n_i \frac{\partial B^T_{t,i}}{\partial I_t} dI_t + n_3 \frac{\partial B^N_{t,3}}{\partial r^R_t} dr^R_t + n_3 \frac{\partial B^N_{t,3}}{\partial I_t} dI_t + \cdots \end{array}$$

where the second equation follows from the strategy being self-financing. The dots in the formulae involve other terms multiplied by dt, which cancel out. The system is solved by gathering all terms associated with each of the two random magnitudes (dI_t and dr^R_t) and making their coefficients equal to zero for each t:

$$\begin{pmatrix} n_{1,t} \\ \\ n_{2,t} \\ \\ n_{3,t} \end{pmatrix} = \begin{pmatrix} \frac{B^T_{t,0}}{B^T_{t,1}} \times \frac{\sum_{j=1}^{n_2} \omega^{(2)}_{t,j} b^R_{t,j} - \sum_{j=1}^{n_0} \omega^{(0)}_{t,j} b^R_{t,j}}{\sum_{j=1}^{n_2} \omega^{(2)}_{t,j} b^R_{t,j} - \sum_{j=1}^{n_1} \omega^{(1)}_{t,j} b^R_{t,j}} \\ \frac{B^T_{t,0}}{B^T_{t,2}} \times \frac{\sum_{j=1}^{n_0} \omega^{(0)}_{t,j} b^R_{t,j} - \sum_{j=1}^{n_1} \omega^{(1)}_{t,j} b^R_{t,j}}{\sum_{j=1}^{n_2} \omega^{(2)}_{t,j} b^R_{t,j} - \sum_{j=1}^{n_1} \omega^{(1)}_{t,j} b^R_{t,j}} \\ \\ 0 \end{pmatrix} \tag{21}$$

where the $\omega^k_{t,j}$'s are:

$$\omega^k_{t,j} = \frac{c^k_j P^R_{t,j}}{B^R_{t,j}}, \quad \sum_j \omega^k_j = 1$$

There are no nominal bonds in the hedging strategy $(n_3 = 0)$, which results from the fact that nominal bonds don't depend directly neither on r^R_t nor on I_t. However, they are correlated with them.

The procedure to hedge the nominal bond is as follows:

- First build two portfolios: portfolio A, including the nominal bond whose price we are trying to validate and portfolio B, including two other bonds (just nominal bonds, as their prices only depend directly on r^N_t, not on r^R_t or I_t). Portfolio B requires just one bond in order to control for the single risk factor r^N_t, but a second bond is required in order to ensure that the total value of portfolio B exactly matches that of portfolio A.

- Calculate the required amounts of each bond in portfolio B ($n_{1,t}$ and $n_{2,t}$) so that the total investment required to build it at time t matches exactly the cost of buying the bond in portfolio A.
- Then calculate the daily return of each portfolio (R_t^A, R_t^B) and compute the difference $\left(\varepsilon_t = R_t^A - R_t^B\right)$ in the same way we did for linkers. If the model is correct, the difference should be undistinguishable from 0.
- Finally validate the model via analysis of residuals.

In order for portfolio B to hedge portfolio A we need:

$$\begin{aligned} B_{t,0}^N &= n_1 B_{t,1}^N + n_2 B_{t,2}^N \\ dB_{t,0}^N &= n_1 dB_{t,1}^N + n_2 dB_{t,2}^N \\ \frac{\partial B_{t,0}^N}{\partial r_t^N} dr_t^N + \cdots &= \sum_{i=1}^{2} n_i \frac{\partial B_{t,i}^N}{\partial r_t^N} dr_t^N + \cdots \end{aligned}$$

where the second equation follows from the strategy being self-financing. The dots in the formulae involve other terms multiplied by dt, which cancel-out. Solving the system for each t in a similar way as before, the result is:

$$\begin{pmatrix} n_{1,t} \\ \\ n_{2,t} \end{pmatrix} = \begin{pmatrix} \dfrac{B_{t,0}^N}{B_{t,1}^N} \times \dfrac{\sum_{j=1}^{n_0} \omega_{t,j}^{(0)} b_{t,j}^N - \sum_{j=1}^{n_2} \omega_{t,j}^{(2)} b_{t,j}^N}{\sum_{j=1}^{n_1} \omega_{t,j}^{(1)} b_{t,j}^N - \sum_{j=1}^{n_2} \omega_{t,j}^{(2)} b_{t,j}^N} \\ \dfrac{B_{t,0}^N}{B_{t,2}^N} \times \dfrac{\sum_{j=1}^{n_1} \omega_{t,j}^{(1)} b_{t,j}^N - \sum_{j=1}^{n_0} \omega_{t,j}^{(0)} b_{t,j}^N}{\sum_{j=1}^{n_1} \omega_{t,j}^{(1)} b_{t,j}^N - \sum_{j=1}^{n_2} \omega_{t,j}^{(2)} b_{t,j}^N} \end{pmatrix} \tag{22}$$

6.1 Hedging results

The hedging exercise was run first run on the five existing benchmark linker bonds employed to estimate the HJM parameters: there are 10 possible combinations of three bonds each out of these five bonds, which are shown in the table below numbered from one to 10. We also included as portfolio 11 the hedging of the single bond left out of the parameter estimation, the OAT *i* 3, 15%2032, to check the model performance on an out-of-sample bond.

We performed this exercise both, for the NS and for the ZCIIS estimated parameters. All in all $11 \times 2 = 22$ portfolios were created.

These hedging portfolios generated 22 series of errors, which constitute the basis for the model-validation analysis. The error analysis was performed on the original series (as generated from the hedging exercise) and also on the same number of “filtered” series, ie series where the errors were filtered in line with the three sigma rule, according to which all the outlier observations exceeding three sample standard deviations were iteratively excluded from the series, until the sample moments (mean and variance) converged to a stable level.[19]

[19] This allowed for smoothing the series and decreasing the dispersion of the sample distribution.

Hedged bond hedging bonds		
1) BTANi 1,25% 2010	OATi 3% 2012	OATi 1,6% 2015
2) BTANi 1,25% 2010	OATi 3% 2012	OATi 2,25% 2020
3) BTANi 1,25% 2010	OATi 3% 2012	OATi 1,8% 2040
4) BTANi 1,25% 2010	OATi 1,6% 2015	OATi 2,25% 2020
5) BTANi 1,25% 2010	OATi 1,6% 2015	OATi 1,8% 2040
6) BTANi 1,25% 2010	OATi 2,25% 2020	OATi 1,8% 2040
7) OATi 3% 2012	OATi 1,6% 2015	OATi 2,25% 2020
8) OATi 3% 2012	OATi 1,6% 2015	OATi 1,8% 2040
9) OATi 3% 2012	OATi 2,25% 2020	OATi 1,8% 2040
10) OATi 1,6% 2015	OATi 2,25% 2020	OATi 1,8% 2040
11) OATi 3,15% 2032	OATi 2,25% 2020	OATi 1,8% 2040

The following chart presents the result of the analysis:

The results clearly show that in all cases but the NS-filtered series from portfolio 11 (the 2020/2040 portfolio hedging the 2032 bond) there is no reason to reject the null hypothesis of zero mean error.[20] This constitutes a strong argument for validating the model: In other words, none of the strategies considered allows for making consistent profits (ie arbitrage opportunities).

[20] Regarding the filtered series, as it is apparent from the above results, the smoothing algorithm did not affect the inference regarding the zero mean error.

Figure 1

ILBs – Null hypothesis: mean error= 0

Portfolio	Estimate	Sample size	Mean (bps)	stdev	Test t-statistic	Probability	Reject Null?
P.1	NS	774	-0.3	7.4	-1.04	30%	no
	NS-filt	734	-0.2	5.6	-0.99	32%	no
	ZC	774	-0.3	23.5	-0.37	72%	no
	ZC-filt	748	-0.2	18.1	-0.28	78%	no
P.2	NS	774	-0.4	10.6	-1.10	27%	no
	NS-filt	739	-0.4	8.1	-1.33	18%	no
	ZC	774	-0.4	21.8	-0.50	62%	no
	ZC-filt	747	-0.3	16.6	-0.51	61%	no
P.3	NS	774	-0.5	53.9	-0.28	78%	no
	NS-filt	752	0.0	44.2	0.01	99%	no
	ZC	774	-0.8	65.6	-0.34	73%	no
	ZC-filt	754	-1.1	54.3	-0.57	57%	no
P.4	NS	774	-1.1	25.7	-1.15	25%	no
	NS-filt	736	-0.7	18.9	-1.06	29%	no
	ZC	774	-0.9	59.3	-0.41	69%	no
	ZC-filt	743	-1.0	41.0	-0.65	52%	no
P.5	NS	774	-0.5	23.8	-0.61	54%	no
	NS-filt	742	-0.4	18.6	-0.56	58%	no
	ZC	774	-0.4	38.4	-0.32	75%	no
	ZC-filt	752	-1.1	29.7	-0.99	32%	no
P.6	NS	774	0.0	43.0	0.03	98%	no
	NS-filt	731	-0.1	29.8	-0.14	89%	no
	ZC	774	-0.1	54.6	-0.04	97%	no
	ZC-filt	754	-1.0	45.3	-0.61	54%	no
P.7	NS	774	-0.4	10.1	-1.11	27%	no
	NS-filt	738	-0.4	7.3	-1.60	11%	no (marginal)
	ZC	774	-0.2	35.5	-0.18	86%	no
	ZC-filt	753	-0.7	26.0	-0.73	47%	no
P.8	NS	774	-0.2	13.6	-0.31	76%	no
	NS-filt	737	0.0	10.0	-0.13	90%	no
	ZC	774	-0.1	27.2	-0.08	93%	no
	ZC-filt	756	-0.4	21.3	-0.51	61%	no
P.9	NS	774	0.4	35.1	0.29	77%	no
	NS-filt	733	0.9	24.3	1.01	31%	no
	ZC	774	0.2	45.4	0.15	88%	no
	ZC-filt	750	-0.2	35.8	-0.13	89%	no
P.10	NS	774	0.4	18.0	0.58	56%	no
	NS-filt	728	0.4	12.4	0.79	43%	no
	ZC	774	0.2	30.2	0.15	88%	no
	ZC-filt	737	-0.5	20.8	-0.68	50%	no
P.11	NS	774	-0.4	11.2	-1.04	30%	no
	NS-filt	710	-0.5	6.4	-2.27	2%	reject at 5%
	ZC	774	-0.6	25.8	-0.63	53%	no
	ZC-filt	741	-0.7	17.4	-1.07	29%	no

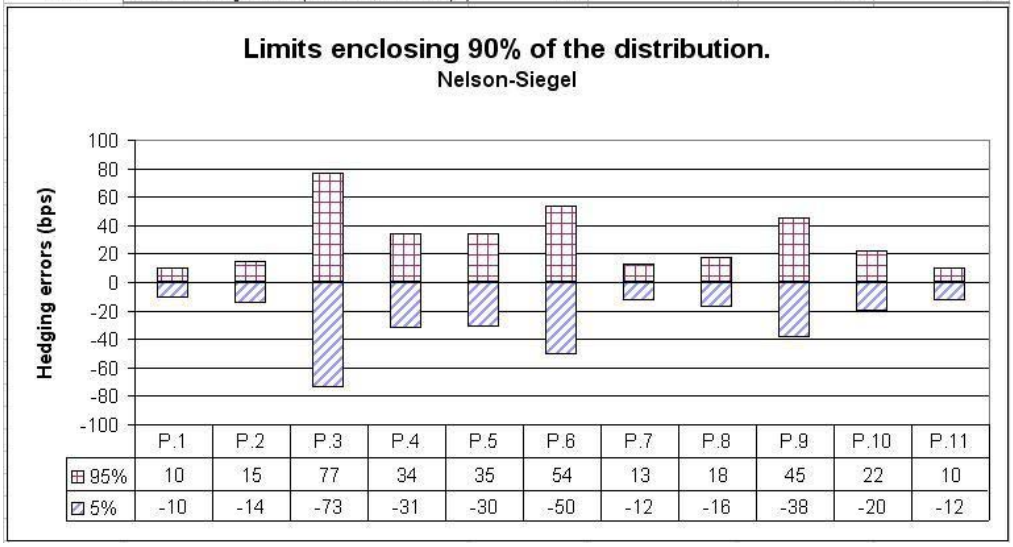
Figure 2 presents a table showing the mean error and corresponding standard deviation per portfolio (both, for the NS and the ZCIIS filtered series), and a chart showing the error range covering 90% of the distribution for the NS-filtered series.[21] The chart permits to have a second assessment on the quality of the HJM-model to represent the economy: six out of 11 hedging portfolios produced errors which stayed inside ±20 bps 90% of the time and two portfolios produced errors which stayed inside ±35 bps 90% of the time. Remarkably, portfolio 11, which hedges the bond 2032 (which wasn't included in the set of bonds to estimate the HJM parameters), produced very small errors.

[21] The table shows that the NS parameter estimation produced much more accurate results than the ZCIIS approach.

Figure 2

ILBs – NS-filtered selection

#	Hedging Portfolios	Errors NS		Errors ZCIIS	
		AVG (bps)	St.Dev.(bps)	AVG (bps)	St.Dev.(bps)
P.1	ILB 2010 hedged with (ILB 2012; ILB 2015)	-0.2	6	-0.2	18
P.2	ILB 2010 hedged with (ILB 2012; ILB 2020)	-0.4	8	-0.3	17
P.3	ILB 2010 hedged with (ILB 2012; ILB 2040)	0.0	44	-1.1	54
P.4	ILB 2010 hedged with (ILB 2015; ILB 2020)	-0.7	19	-1.0	41
P.5	ILB 2010 hedged with (ILB 2015; ILB 2040)	-0.4	19	-1.1	30
P.6	ILB 2010 hedged with (ILB 2020; ILB 2040)	-0.1	30	-1.0	45
P.7	ILB 2012 hedged with (ILB 2015; ILB 2020)	-0.4	7	-0.7	26
P.8	ILB 2012 hedged with (ILB 2015; ILB 2040)	0.0	10	-0.4	21
P.9	ILB 2012 hedged with (ILB 2020; ILB 2040)	0.9	24	-0.2	36
P.10	ILB 2015 hedged with (ILB 2020; ILB 2040)	0.4	12	-0.5	21
P.11	ILB 2032 hedged with (ILB 2020; ILB 2040)	-0.5	6	-0.7	17



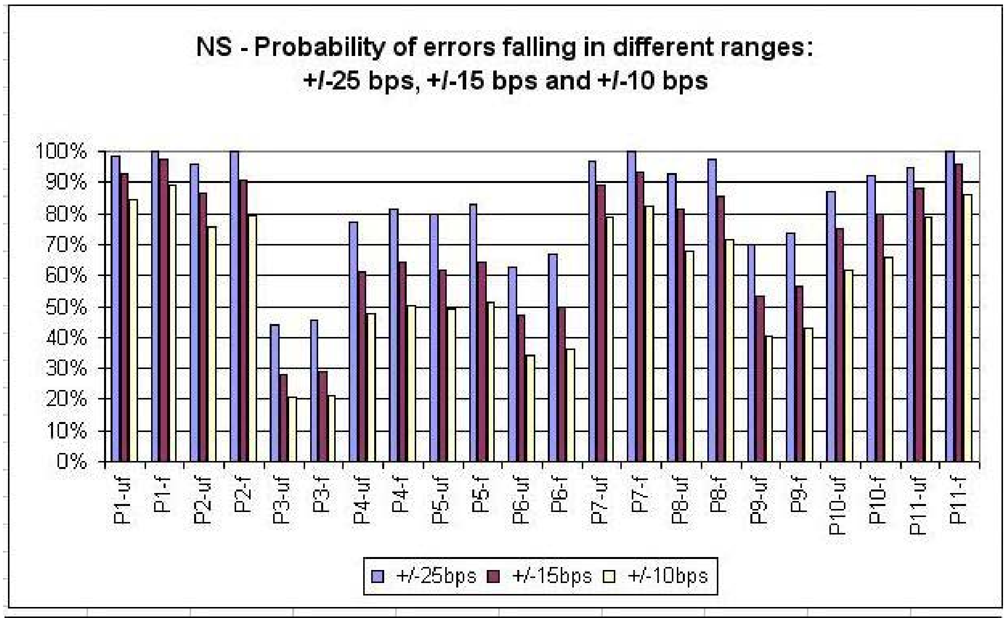
Finally, Figure 3 shows from another angle that the NS approach to build the zero-coupon bonds produced superior results compared to the ZCIIS approach. The probability mass of hedging errors enclosed in specific ranges is larger in the NS approach than in the ZCIIS. The results are presented both, for filtered and unfiltered errors.

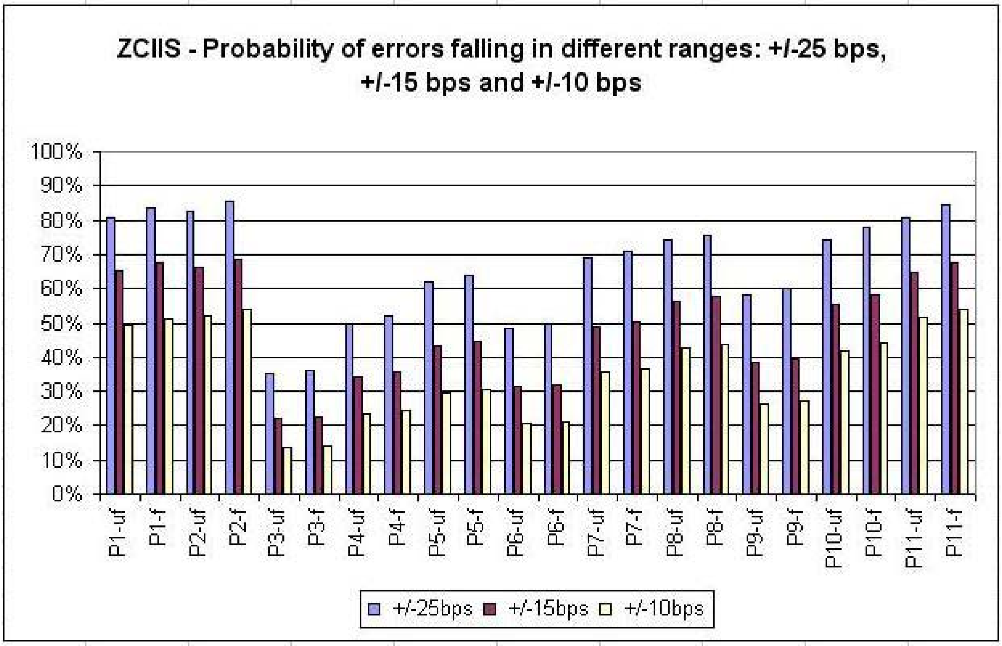
There is still a need to perform a similar hedging analysis on nominal bonds in order to validate the model, which is done below. The hedging analysis was based on 10 portfolios made out of five different synthetic bonds, built to exactly match the maturities of the five benchmark linker bonds and the same methodology used to hedge the ILBs was used to analyze the nominal bonds.

The following charts present the results of the analysis.

Figure 3

ILBs–probability mass





The results show that in all cases there is no reason to reject the null hypothesis of zero mean error.[22] This completes the argument for validating the model: In other words, none of the strategies considered allows for making consistent profits (ie arbitrage opportunities).

22 Regarding the filtered series, as was the case also for ILBs, the smoothing algorithm did not affect the inference regarding the zero mean error.

Figure 4

Nominal bonds-null hypothesis: mean error= 0

Portfolio	Estimate	Sample size	Mean (bps)	stdev	Test t-statistic	Probability	Reject Null?
P.1	Unfiltered	774	-0.2	18.4	-0.28	78%	no
	Filtered	718	0.3	7.9	0.95	34%	no
P.2	Unfiltered	774	-0.3	33.6	-0.28	78%	no
	Filtered	738	0.3	22.3	0.37	71%	no
P.3	Unfiltered	774	-0.1	34.2	-0.12	91%	no
	Filtered	733	0.5	22.7	0.60	55%	no
P.4	Unfiltered	774	0.3	44.7	0.21	83%	no
	Filtered	741	0.2	33.2	0.15	88%	no
P.5	Unfiltered	774	-0.2	16.9	-0.37	71%	no
	Filtered	719	0.2	8.1	0.71	48%	no
P.6	Unfiltered	774	-0.2	16.9	-0.34	74%	no
	Filtered	718	0.3	8.3	0.83	41%	no
P.7	Unfiltered	774	0.0	14.6	0.01	99%	no
	Filtered	743	0.0	9.8	-0.03	97%	no
P.8	Unfiltered	774	0.1	10.9	0.17	86%	no
	Filtered	737	0.1	6.7	0.47	64%	no
P.9	Unfiltered	774	0.2	11.3	0.53	60%	no
	Filtered	756	0.0	9.8	0.08	94%	no
P.10	Unfiltered	774	-0.1	8.5	-0.30	77%	no
	Filtered	745	0.0	6.7	-0.05	96%	no

7. Portfolio selection

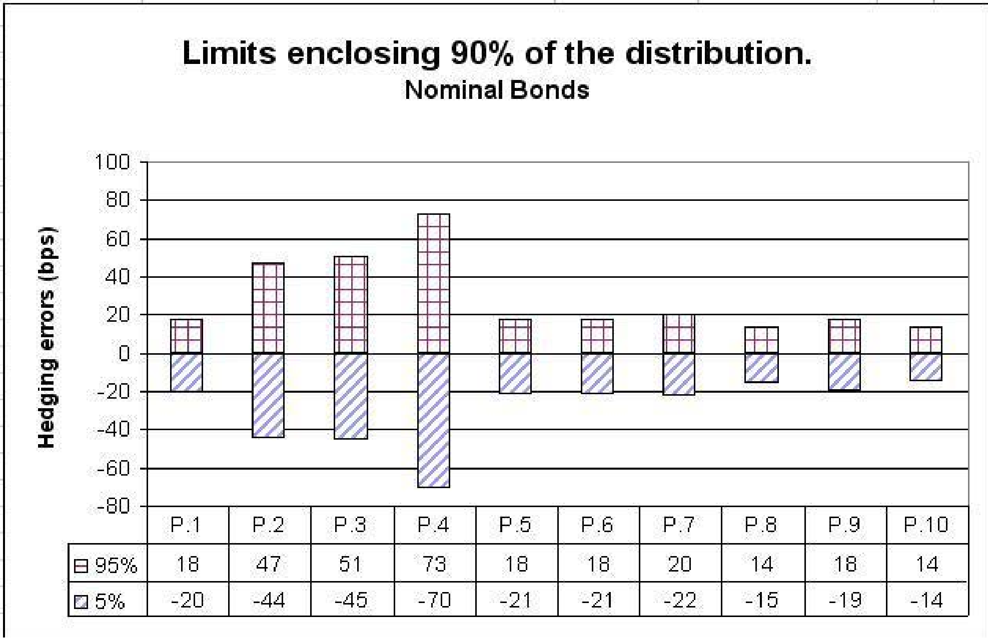
We have built a model to characterize the bond market (including nominal and inflation-linked sovereign bonds) and we are now capable of building a sovereign bond portfolio including both asset classes. As the model is Gaussian, the variance-covariance matrix characterizing all securities in the portfolio is required, as is the corresponding vector of expected returns.

As there are just three risk factors driving the market, all we need is the risk-free rate and three bonds (two linkers and one nominal bond) spanning the whole maturity range.

Figure 5

Nominal bonds–filtered selection

#	Hedging Portfolios	Errors	
		AVG (bps)	St.Dev.(bps)
P.1	2040 hedged with 2020 & 2032	-0.2	18
P.2	2040 hedged with 2020 & 2015	-0.3	34
P.3	2040 hedged with 2020 & 2012	0.1	34
P.4	2040 hedged with 2015 & 2012	0.3	45
P.5	2040 hedged with 2015 & 2032	-0.2	17
P.6	2040 hedged with 2012 & 2032	-0.2	17
P.7	2020 hedged with 2012 & 2032	0.0	15
P.8	2020 hedged with 2015 & 2032	0.1	11
P.9	2020 hedged with 2015 & 2012	0.2	11
P.10	2015 hedged with 2012 & 2032	-0.1	9



Let $R^N_{t,T}dt$ be the return on a nominal bond:

$$R^N_{t,T}dt = r^N_t dt - \sigma^N \left(\frac{1 - e^{-\alpha^N (T-t)}}{\alpha^N} \right) dW^N_t = r^N_t dt - \sigma^N Q^N_{t,T} dW^N_t \qquad (23)$$

where we have defined $Q^N_{t,T} \equiv \left(\frac{1 - e^{-\alpha^N (T-t)}}{\alpha^N} \right)$

Let $R^T_{t,T}dt$ be the return on a ILB:

$$R^T_{t,H}dt = r^N_t dt - \sigma^R Q_{t,HR} dW^R_t + \sigma^I dW^I_t \qquad (24)$$

with $Q^R_{t,H} \equiv \left(\frac{1 - e^{-\alpha^R (H-t)}}{\alpha^R} \right)$

Let $B^h_{t,T} = \sum_{j=1}^{N} c^h_j P^h_{t,j}$ be the price of a traded bond $(h \in \{N, T\})$ where

$$c^h_j = \begin{cases} c^h \forall j < N \\ (c^h + 1) \;\; for \;\; j = N \end{cases}$$

The instantaneous rate of return on a bond (either nominal or real) is:

$$\frac{dB_{t,T}^{h}}{B_{t,T}^{h}} = \sum_{j=1}^{N} \frac{c_j^h P_{t,j}^h}{B_{t,T}^h} \frac{dP_{t,j}^h}{P_{t,j}^h} = \sum_{j=1}^{N} \omega_j^h R_{t,j}^h; \quad h \in \{N,T\}$$

7.1 Variance-covariance

In this subsection the required Variance-covariance matrix is derived. The covariances between the different returns of discount bonds are presented below:

$$Cov\left(R_{t,T_1}^N dt R_{t,T_2}^N dt\right) = \sigma^{N^2} Q_{t,T_1}^N Q_{t,T_2}^N dt$$

$$Cov\left(R_{t,H_1}^T dt R_{t,H_2}^T dt\right) = \sigma^{R^2} Q_{t,H_1}^R Q_{t,H_2}^R dt + \sigma^{I^2} dt - \sigma^R \sigma^I \rho_{R,I} \left(Q_{t,H_1}^R + Q_{t,H_2}^R\right) dt$$

$$Cov\left(R_{t,T}^N dt R_{t,H}^T dt\right) = \sigma^N \sigma^R \rho_{N,R} Q_{t,T}^N Q_{t,H}^R dt - \sigma^N \sigma^I \rho_{N,I} Q_{t,T}^N dt$$

Now, using these expressions together with those for the returns of traded bonds, the covariances between returns of the different traded (coupon-paying bonds) are derived:

$$V\left(\frac{dB_{t,T}^N}{B_{t,T}^N}\right) = \sigma^{N^2} D_n^2 dt$$

$$V\left(\frac{dB_{t,H}^T}{B_{t,H}^T}\right) = \left(\sigma^{R^2} D_T^2 + \sigma^{I^2} - 2\sigma^R \sigma^I \rho_{R,I} D_T\right) dt$$

$$Cov\left(\frac{dB_{t,T}^N}{B_{t,T}^N} \frac{dB_{t,H}^T}{B_{t,H}^T}\right) = \left(\sigma^N \sigma^R \rho_{N,R} D_N D_T - \sigma^N \sigma^I \rho_{N,I} D_N\right) dt$$

$$Cov\left(\frac{dB_{t,H_1}^T}{B_{t,H_1}^T} \frac{dB_{t,H_2}^T}{B_{t,H_2}^T}\right) = \left(\sigma^{R^2} D_{T_1} D_{T_2} + \sigma^{I^2} - \sigma^R \sigma^I \rho_{R,I} \left(D_{T_1} + D_{T_2}\right)\right) dt$$

Based on the sample period of our database (09/03/2007–26/02/2010) these formulae produced the following variance-covariance matrix for the set of the three selected bonds:

Nominal bond coupon 3.85% maturity July/2040

ILB bond coupon 1.25% maturity July/2010

ILB bond coupon 2% maturity July/2040

$$\Omega = \begin{pmatrix} 1.32\% & 0.10\% & 1.19\% \\ 0.10\% & 0.03\% & 0.18\% \\ 1.19\% & 0.18\% & 1.96\% \end{pmatrix}$$

Using this variance-covariance matrix, the portfolio allocation between nominal bonds and linkers results from an optimization exercise between three securities ($B_{t,T}^N$, B_{t,H_1}^T, B_{t,H_2}^T).

$$\min \lambda' \Omega \lambda \quad subject \quad to:$$

$$\lambda' 1 = 1$$
$$\lambda' R = r$$
$$\lambda \geq 0$$

where λ stands for the asset's optimal weights in the portfolio, R for the vector of expected returns and r for the portfolio's expected return.

8. Conclusions

The financial crisis changed the appreciation of different asset classes among public investors leading to a fundamental reassessing of their risks, which in turn reduced the investment universe. As a result, the quest for diversification became even more critical and the case for including inflation linkers in a fixed income portfolio grew stronger.

We first discussed the general case for including linkers in an otherwise traditional fixed income portfolio, to later develop a specific model to characterize the market.

Using French ILB's market prices and zero-coupon inflation indexed swaps, we derived corresponding real zero-coupon bond price curves. Zero (real) coupon prices were derived as it is typically done in the industry, ie by recourse to traded ZCIIS, but also by fitting Nelson-Siegel curves to the daily data. Both methodologies resulted in different parameter estimates, which were later tested in the hedging analysis to validate the model.

We then fitted a three-factor HJM model to characterize the economy, with time-dependent (non-stochastic) volatilities, which consequently resulted on a Gaussian economy.

Some 21 hedging portfolios were built and the statistical characteristics of their errors permitted to validate the model.[23] The validation of the model provided a coherent theoretical background to build a portfolio of bonds which includes linkers as well as nominal bonds.

In the context of this model, the asset returns are normally distributed, so the case for including linkers in a bond portfolio is reduced to the classical CAPM analysis, as assets are characterized by their expected returns and their variance-covariance matrix.

This is the first, to the authors' knowledge, attempt to calibrate the HJM framework using data on European inflation-linked bonds.

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