Portfolio optimization and long-term dependence[1]

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Introduction

It is a widespread practice to use daily or monthly data to design portfolios with investment horizons equal or greater than a year. The computation of the annualized mean return is carried out via traditional interest rate compounding – an assumption free procedure –, while scaling volatility is commonly fulfilled by relying on the serial independence of returns' assumption, which results in the celebrated square-root-of-time rule.

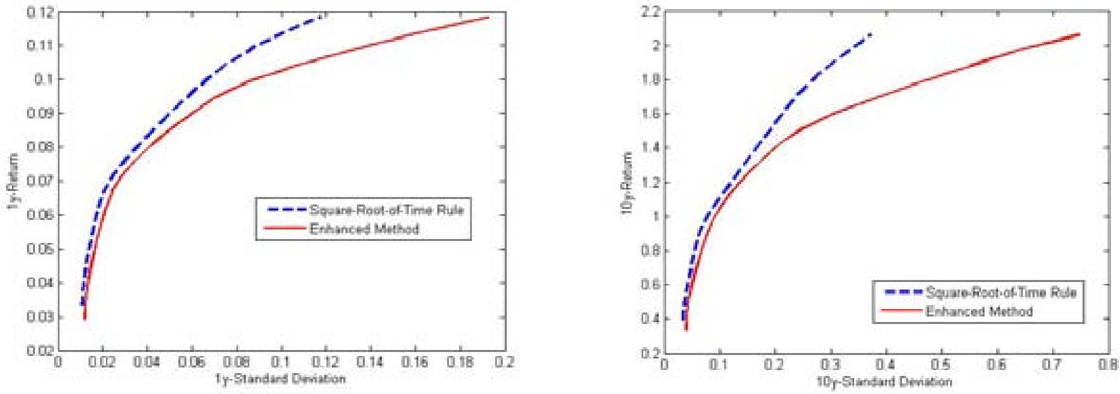
While it is a well-recognized fact that the serial independence assumption for asset returns is unrealistic at best, the convenience and robustness of the computation of the annual volatility for portfolio optimization based on the square-root-of-time rule remains largely uncontested.

As expected, the greater the departure from the serial independence assumption, the larger the error resulting from this volatility scaling procedure. Based on a global set of risk factors, we compare a standard mean-variance portfolio optimization (eg square-root-of-time rule reliant) with an enhanced mean-variance method for avoiding the serial independence assumption. Differences between the resulting efficient frontiers are remarkable, and seem to increase as the investment horizon widens (Figure 1).

Figure 1

Efficient frontiers for the standard and enhanced methods

1-year | 10-year



Source: authors' calculations.

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Because this type of error lurks beneath customary asset allocation procedures, including the prominent Black-Litterman (1992) model, the main objective of this paper is to challenge the square-root-of-time rule as a proper volatility scaling method within the mean-variance framework, and to present a robust alternative.

In order to fulfill the stated objective, this paper estimates financial assets' long-run dynamic. The impact of long-term serial dependence on asset returns is assessed for a wide set of markets and instruments, with a sample which covers the most recent market turmoil. The estimation relies on a revised and adjusted version of the classic rescaled range analysis methodology (*R/S*) first introduced by Hurst (1951) and subsequently enhanced by Mandelbrot and Wallis (1969a and 1969b).

Similar to Hurst's results in geophysics and to financial literature (Malevergne and Sornette, 2006; Los, 2005; Daníelsson and Zigrand, 2005), the results confirm that numerous individual risk factors exhibit significant long-term dependence, thus invalidating the square-root-of-time rule. Interestingly, most previous findings related to long-term dependence in financial time series are still supported, even after the most recent period of market crisis.

The results also demonstrate that some major asset allocation issues could be explained to some extent by the inability of the square-root-of-time rule to properly scale up volatility in the presence of serial long-term dependence. Some of these issues are: (i) excessive risk taking in long-term portfolios (Valdés, 2010; Reveiz et al. 2010; Pastor and Stambaugh, 2009; Schotman et al. 2008); (ii) a tendency to hold a disproportionate level of investments within the domestic market –home bias– (Solnik, 2003; Winkelmann, 2003b); (iii) a reluctance to hold foreign currency-denominated assets (Lane and Shambaugh, 2007; Davis, 2005); and (iv) the presence of extreme portfolio weights or "corner solutions" (Zimmermann et al. 2003; He and Litterman, 1999).

This paper consists of six chapters; this introduction is the first one. The second chapter presents a brief examination of the square-root-of-time rule and its use for scaling high-frequency volatility (eg daily) to low-frequency volatility (eg annual). The third describes and develops the classic rescaled range analysis (*R/S*) methodology for detecting and assessing the presence of long-term serial dependence of returns. The fourth chapter exhibits the results of applying an adjusted version of *R/S* to selected risk factors. The fifth analyzes the consequences of the results for portfolio optimization. Finally, the last chapter highlights and discusses some relevant remarks.

1. The square-root-of-time rule

The square-root-of-time rule consists in multiplying the standard deviation calculated from a *d*-frequency (eg daily) time series by the square-root of *n*, where *n* is the number of *d* units to scale standard deviation up. For example, if σ_d is the standard deviation of a *d*-frequency time series, to scale volatility to an *a*-frequency, where $a = dn$, σ_d should be multiplied by the square-root of *n*, as follows:

$$\sigma_a = \sigma_{dn} = \sigma_d \sqrt[2]{n} = \sigma_d n^{0.5} \qquad \text{F1}$$

The value of this rule is evident for market practitioners: as acknowledged by Dowd et al. (2001), obtaining time series suitable –long enough– to make reliable volatility estimations for monthly or annual frequencies is rather difficult. Besides, even if such time series do exist, questions about the relevance of far-in-the-past data may arise.

Perhaps the most celebrated application of the square-root-of-time rule has to do with Value at Risk (VaR) estimation. According to the technical standards originally established by the Basel Committee on Banking Supervision (BIS, 1995), the VaR must be calculated for at least a ten-day holding period. VaR estimations could be based on shorter holding periods (eg using daily time series), but the ten-day holding period VaR should be attained by means of scaling up to ten days by the square-root-of-time.[4]

Discussing Bachelier's (1900) contribution to the construction of the random-walk or Brownian motion model, Mandelbrot (1963) described it as follows: if $Z(t)$ is the price of a stock at the end of time period t, successive differences of the form $Z(t+T) - Z(t)$ are (i) independent, (ii) Gaussian or normally distributed, (iii) random variables (iv) with zero mean and (v) variance proportional to the differencing interval T.

These assumptions have been notably challenged by mere observation of financial markets, and rejected using traditional significance tests. Nevertheless, methodologies and practices based on the Brownian motion still endure; one of such lasting practices is volatility scaling *via* the square-root-of-time rule, which is the most important prediction of the Brownian motion model (Sornette, 2003).

The assumption underlying the square-root-of-time rule is independence. Under this assumption past behavior of the variable is irrelevant. This is also known as the weak form of the Efficient Market Hypothesis (EMH), and it is the core hypothesis of the martingale model for asset pricing, which states that the current price is the best forecast for future price (Campbell et al., 1997).

Under the independence assumption the probability distribution of changes in the same variable for two or more periods is the sum of the probability distribution; when two independent normal distributions are added, the result is a normal distribution in which the mean is the sum of means and the variance is the sum of variances (Hull, 2003).

Accordingly, if the probability distribution of changes of an independent variable (Ω) has an *A-B* range (Figure 2, left panel), the resulting range at the end of two periods will be proportional to twice *A-B*, and for three periods it will be proportional to three times *A-B*; it is irrelevant whether the probability distribution (Ω) is Gaussian or not.

If the distribution is Gaussian, the *A-B* range can be conveniently characterized by the variance. Hence, if the distribution of Ω can be defined as $\Omega \sim N(0,1)$, where N stands for normally distributed, zero is the mean and 1 the variance, after three periods the distribution of the possible values of the – independent – variable corresponds to $\Omega \sim N(0,1+1+1)$ or $\Omega \sim N(0,3)$.

Alternatively, the *A-B* range can be characterized by a different dispersion metric: standard deviation. However, because standard deviation corresponds to the square-root of variance, it is not additive; therefore, the three-period distribution of possible values of the – independent – variable corresponds to $\Omega \sim N\left(0, \sqrt[2]{1+1+1}\right)$ or $\Omega \sim N\left(0, \sqrt[2]{3}\right)$. This is the origin of the square-root-of-time-rule.

In the absence of independence this rule is no longer valid. As the right panel of Figure 2 reveals, if we let a return above the mean lead to a different (ϕ) more disperse distribution (*C-D*>*A-B*) – which is an example of dependence – then it is impossible to affirm neither that the resulting range at the end of two periods is going to be proportional to twice *A-B*, nor twice *C-D*. This impossibility applies even if the distributions (ϕ and Ω) are strictly Gaussian, and it would cause any standard rule to scale range, variance or standard deviation to falter.

Moreover, the presence of long-term dependence not only invalidates any use of the square-root-of-time rule, but helps explain the slow convergence of the distribution of financial assets' returns towards normality, even for low-frequency (eg monthly, quarterly) data (Malevergne and Sornette, 2006).

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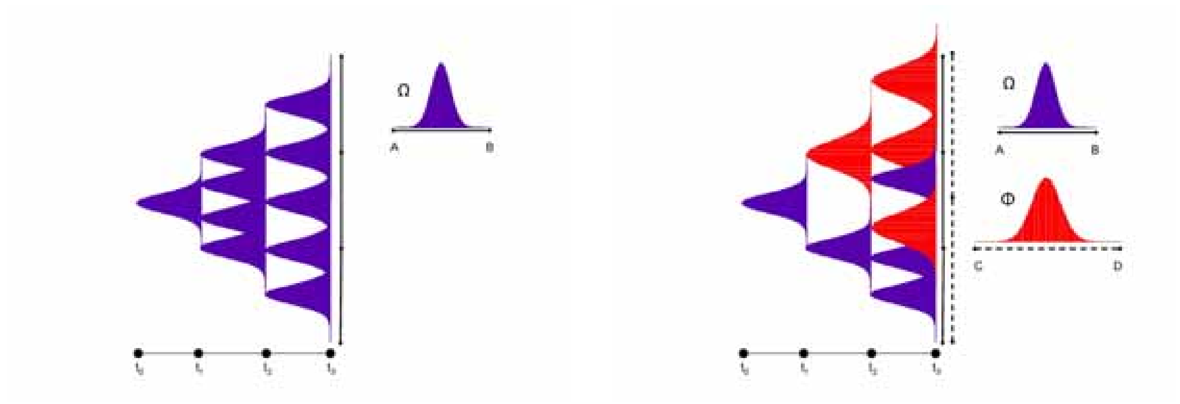
4 Technical caveats to the usage of the square-root-of-time rule were recently introduced (BIS, 2009).

Figure 2

Independence and the square-root-of-time rule

Independent process | Process with memory



Source: authors' design.

While asset returns' independence has been accepted as one of the core foundations in economics and finance since Bachelier (1900), contradictory evidence also dates back to the dawn of the 20th century (Mitchell, 1927; Mills, 1927; Working, 1931; Cowles and Jones, 1937). However, it was complex natural phenomena which forced physicists to deal with the absence of independence. Geophysics, not economics or finance, was the source of methodologies to identify and measure long-term dependence.

2. Rescaled range analysis (*R/S*)

Long-term dependence detection and assessment for time series began with hydrology (Mandelbrot and Wallis, 1969a), when the British physicist H.E. Hurst (1880–1978) was appointed to design a water reservoir on the Nile River. The first problem Hurst had to deal with was to determine the optimal storage capacity of the reservoir; that is, restricted to a budgetary constraint, design a dam high enough to allow for fluctuations in the water supply whilst maintaining a constant flow of water below the dam.

Deciding on the optimal storage capacity depended on the inflows of the river, which were customarily assumed to be random and independent by hydraulic engineers at that time. Nonetheless, when checking the Nile's historical records (622 B.C.–1469 B.C.) Hurst discovered that flows could not be described as random and independent: data exhibited persistence, where years of high (low) discharges were followed by years of high (low) discharges, thus describing cycles but without an obvious periodicity.

Hurst concluded that (i) evidence contradicted the long-established independence assumption and (ii) that the absence of significant autocorrelation proved standard econometric tests to be ineffective (Peters, 1994). Thus, since the absence of independence vindicated caring about the size and sequence of flows, Hurst developed a methodology capable of capturing and assessing the type of dependence he had documented.

Hurst's methodological development was based on Einstein's (1905) work about particle movement, which Scottish botanist Robert Brown (1828 and 1829) had already depicted as inexplicable, irregular and independent. Einstein originally formulated that the distance or average displacement (R) covered by a particle suspended in a fluid per unit of time (n) followed $R = n^{0,5}$; this is analogous to the square-root-of-time rule.

Unlike Brown and Einstein, Hurst's primary objective was a broad formula, capable of describing the distance covered by any random variable with respect to time. Hurst found his observations of several time series were well represented by $R \sim cxn^{H}$, where H corresponds to the way that distance (R) behaves with respect to time.

Hurst defined that the metric for the distance covered per unit of time or sample (*n*) would be given by the range R_n [F2], where $x_1, x_2, x_3..x_n$ correspond to the change of the random variable within the sample, and $\overline{x_n}$ is the average of these changes. Range R_n is standardized by the standard deviation of the sample for that period (S_n), which results in the rescaled range for the n sample $(R/S)_n$ [F2].

$$\left(R/S\right)_n = R_n/S_n = \frac{\left[\max_{1\le k\le n}\left(\Sigma_{j=1}^{k}\left(x_j - \overline{x_n}\right)\right) - \min_{1\le k\le n}\left(\Sigma_{j=1}^{k}\left(x_j - \overline{x_n}\right)\right)\right]}{S_n} \qquad \text{F2}$$

Hurst found that the behavior of this rescaled range [F2] adequately fitted the dynamic of numerous time series from natural phenomena, where the adjustment could be represented as follows [F3]:

$$\left(R/S\right)_n \sim cxn^{H} \qquad \text{F3}$$

Paraphrasing Peters (1992), Hurst's novel methodology measures the cumulative deviation from the mean for various periods of time and examines how the range of this deviation scales over time. $\hat{H}$, the estimated exponent that measures the way distance (*R*) behaves with respect to time, takes values within the 0 and 1 interval ($0 < \hat{H} \le 1$), where $\hat{H}=0.5$ corresponds to Einstein's and Brown's independency case.

Mandelbrot and Wallis (1969a and 1969b) proposed to plot Hurst's function [F3] for several sample sizes (*n*) in a double logarithmic scale, which served to obtain $\hat{H}$ through a least squares regression. $\hat{H}$ would be the slope of the estimated equation [F4]; this procedure is known as the rescaled range analysis $\left(R/S\right)$.

$$Log\left(R/S\right)_n = Log(c) + HLog(n) \qquad \text{F4}$$

According to Mandelbrot (1965) the application of *R/S* to random series with stationary and independent increases, such as those characterized by Brown (1828 and 1829) and Einstein (1905), results in $\hat{H}=0.5$, even if the distribution of the stochastic process is not Gaussian, in which case $\hat{H}$ asymptotically converges to 0.5 ($\hat{H} \approx 0.5$).

As said by Sun et al. (2007), in the $\hat{H}=0.5$ and $\hat{H} \approx 0.5$ cases the process has no memory – is independent – hence the next period's expected result has the same probability of being lower or higher than the current result. Applied to financial time series this is akin to assuming that the process followed by asset returns is similar to coin tossing, where the probability of heads (rise in the price) or tails (fall in the price) is the same (½), and is independent of every other toss; this is precisely the theoretical base of the Capital Asset Pricing Model (CAPM), the Arbitrage Pricing Theory (APT), the Black & Scholes model and the Modern Portfolio Theory (MPT).

When $\hat{H}$ takes values between 0.5 and 1 ($0.5 < \hat{H} \le 1$) evidence suggests a persistent behavior; therefore, one should expect the result in the next period to be similar to the current one (Sun et al., 2007). According to Menkens (2007) this means that increments are positively correlated: if an increment is positive, succeeding increments are most likely to be positive than negative. In other words, each event has influence on future events; therefore there is dependence or memory in the process. Moreover, as $\hat{H}$ becomes closer to one (1) the range of possible future values of the variable will be wider than the range of purely random variables; Peters (1996) argues that the presence of persistency is a signal that today's behavior does not influence near future only, but distant future as well.

On the other hand, when $\hat{H}$ takes values below 0.5 ($0 \leq \hat{H} < 0.5$) there is a signal that suggests an antipersistent behavior of the variable. This means, as said by Sun et al. (2007), that a positive (negative) return is more likely followed by negative (positive) ones; hence, as stated by Mandelbrot and Wallis (1969a), this behavior causes the values of the variable to tend to compensate with each other, avoiding time series' overshooting. Applied to financial market series, Menkens (2007) affirms that this kind of continuously compensating behavior would suggest a constant overreaction of the market, one that would drive it to a permanent adjustment process. Similarly, Peters (1996) links this behavior to the well-known "mean-reversion" process.

Hurst's methodology and results[5] were gathered, corrected and reinterpreted by Mandelbrot (1972) and Mandelbrot and Wallis (1969a and 1969b). Based on random simulation models they verified that (i) Hurst's conclusions were correct, but his calculations were imprecise; (ii) their corrected version of *R/S* is robust to detect and measure dependence, even in the presence of significant excess skewness or kurtosis;[6] (iii) their corrected version of *R/S* is asymptotically robust to short-term dependency (eg autoregressive and moving average processes); (iv) asymptotically $\hat{H}=0.5$ for independent processes, even in the absence of Gaussian processes; and (v) in contrast to other methodologies, *R/S* can detect non-periodic cycles.

Shortcomings of Mandelbrot's (1972) and Mandelbrot and Wallis' (1969a and 1969b) developments regarding the presence of significant long-term dependence in financial time series were depicted by Lo (1991). He introduced modified rescaled range methodology (*mR/S*) as an effort to establish whether *R/S* results are due to the presence of genuine long-term dependence, or to some sort of short-term memory.

Despite considering comparative results of both *R/S* and *mR/S* as inconclusive, Los (2003) states that evidence documented by Peters (1994) shifts the balance of proof in the direction of the existence of the long-term dependence in financial assets' time series. Peters (1994) works on long-term dependence in capital markets discarded autoregressive processes (AR), moving average (MA) and autoregressive moving average (ARMA) as sources of the persistence effect or long-term memory that is captured by the *R/S*, while generalized autoregressive conditional heteroskedasticity (GARCH) processes showed a marginal persistence effect only.[7]

Although literature about short-term dependence in asset returns is abundant, that on long-term dependence is rather scarce, whereas *R/S* is a popular and robust methodology. As exhibited in Figure B1 (Annex B), evidence on *R/S* application to currencies, stock indexes, fixed income securities and commodities supports the long-term dependence hypothesis, as

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5 Hurst (1956) studied 76 natural phenomena. $\hat{H}$ was significantly different from 0.5, and was close to 0.73 ($\sigma = 0.092$). Hurst found no evidence of significant autocorrelation in the first lags, which led him to reject short-term dependence as the source of this phenomenon; neither could he find a slow and gradual decay with increasing lags, which supported his rejection for long-term dependence in the traditional sense of Campbell et al. (1997).

6 Mandelbrot and Wallis (1969a) were the first to recognize *R/S* as non-parametric, even in the presence of extreme skewness or with infinite variance. León and Vivas (2010), Martin et al. (2003), Willinger et al. (1999) and Peters (1996 and 1994) verified this statement.

7 Moreover, since the purpose of this paper is not to establish the source of dependence, either short-term or long-term, but to detect and measure its impact on financial asset returns' long-run dynamic, Lo's (1991) criticism is rather irrelevant.

well as Peters' (1996) statement regarding the difficulty of finding antipersistent financial time series.

Evidence of significant antipersistence has been documented for energy prices; Weron and Przybylowicz (2000) explain such findings as resulting from energy's particularities (eg market regulation, storage problems, transmission, distribution). Reveiz (2002) documents similar findings for currencies floating within a band that introduces non-linear features to foreign exchange trading.

Peters (1996 and 1989) concluded that asset returns do not follow a pure random walk, but exhibit some degree of persistence ($0.5 < \hat{H} \leq 1$); Peters named this type of tainted random walk "biased random walk". When asset returns follow a biased random walk they trend in one direction until some exogenous event occurs to change their bias. The presence of persistency, according to Peters, is evidence that new events are not immediately reflected in prices, but are manifested as an enduring bias on returns; this contradicts the EMH.

Some explanations for financial assets' return persistence are found in human behavior, since the latter contradicts the rationality assumption in several ways; for example: (i) investors' choices are not independent, and they are characterized by non-linear and imitative behavior (LeBaron and Yamamoto, 2007; Sornette, 2003); (ii) investors resist changing their perception until a new credible trend is established (Singh and Dey, 2002; Peters, 1996); and (iii) investors do not react to new information in a continuous manner, but rather in a discrete and cumulative way (Singh and Dey, 2002).

Other explanations for financial assets' return persistence have to do with the importance of economic fundamentals (Nawrocki, 1995; Lo, 1991; Peters, 1989) and the use of privileged information (Menkens, 2007). Alternatively, some authors (Bouchaud et al., 2008; Lillo and Farmer, 2004), based on the persistence of the number and volume of buying and selling orders in transactional systems, conclude that markets' liquidity makes instantaneous trading impossible, leading to transaction splitting, and decision clustering, resulting in market prices which fully reflect information not immediately, but incrementally.

3. Estimated Hurst exponent ($\hat{H}$) for major risk factors

Estimating the Hurst exponent ($\hat{H}$) requires the implementation of the algorithm described in the Appendix, and the design of significance tests for evaluating the null hypothesis of independence.

Confidence intervals and significance tests

One of the main difficulties of *R/S* methodology is the selection of an ad-hoc optimal size of periods (n) to calculate $(R/S)_n$. In the literature there is consensus about *R/S* not being reliable for reduced periods because estimations may become unstable and biased (Cannon et al., 1997; Peters, 1994; Ambrose et al., 1993). However, there is no consensus about an optimal minimum size of periods (n_{min}).[8]

The same issue arises with the choice of optimal maximum period size (n_{max}). Cannon et al. (1997) and Peters (1996) recognize that the stability of $\hat{H}$ diminishes when using extended

[8] Cannon et al. (1997) estimate optimal minimum size of periods to be $n_{min} \geq 2^8$ (≥ 256 observations) to achieve standard deviations below 0.05; Mandelbrot and Wallis (1969a) use 20 observations; Wallis and Matalas (1970) point out that the window must have at least 50 observations, unless series are of considerable length; Peters (1994) acknowledges that financial series are not long enough to discard reduced windows, and suggests at least 10 observations; Nawrocki (1995) argues that minimum number of observations should be large enough to minimize the effect of short-term dependence.

periods. Therefore, Cannon et al. advise dismissing the use of data windows where estimations are made on a few segments of the time series.

Given the absence of consensus on the optimal period size, all calculations were made using a minimum size of 32 observations ($n_{min} \geq 2^5$). This choice not only recognizes the intricacy of finding extended time series (Peters, 1994), but also results in reduced standard errors of the estimators in the sense of Cannon et al. (1997), and guarantees that the effect of conventional short-term serial dependence (eg autocorrelation) for a daily-frequency series is minimized (Nawrocki, 1995).

The maximum period size constraint (n_{max}) consists of restricting time series to be divided into at least ten contiguous non-overlapping segments; in this way, estimations based on a narrow number of samples and unstable estimators are avoided.

Concerning significance tests for $\hat{H}$, two well-documented issues have to be taken into account (León and Vivas, 2010; Ellis, 2007; Couillard and Davison, 2005; Peters, 1994). First, there is a positive bias in the estimation – overestimation – of *H* resulting from finite time series and a minimum size of periods below approximately 1,000 observations. Second, $\hat{H}$ distributes like a normal regardless of the empirical distribution of the random variables.

Regarding the first issue, the estimation bias resulting in the overestimation of $\hat{H}$ can be conveniently assessed. Several assessment methods for estimating such bias have been documented, but this work focuses on the single most well-known. First proposed by Anis and Lloyd (1976), subsequently revised by Peters (1994), and recently verified and applied by León and Vivas (2010), Ellis (2007) and Couillard and Davison (2005), the chosen method consists of a functional approximation for estimating the expected value of $(R/S)_n$ when the random variable is independent and of finite length. This method yields the expected Hurst exponent corresponding to an independent random variable, which will be noted as $\dot{H}$, and is based on the following calculation of the expected value of $(R/S)_n$:

$$E\left(R/S\right)_n = \frac{n-\frac{1}{2}}{n}\frac{1}{\sqrt{n\pi/2}}\sum_{i=1}^{n-1}\sqrt{\frac{n-i}{i}} \qquad \text{F5}$$

Any divergence of $\hat{H}$ from $\dot{H}$ would signal the presence of long-term memory in the time series. However, as customary in statistical inference, it is critical to develop appropriate statistical tests to distinguish between significant and non-significant deviations from the long-term independence null hypothesis.

The significance test used is similar to those proposed by Ellis (2007) and Couillard and Davison (2005). Because $\hat{H}$'s distribution is established to be normal, even for random variables that are not, a conventional *t*-statistic test may be implemented. Let $\hat{H}$ be the *R/S*'s estimated value of the Hurst exponent, $\hat{\mu}(\dot{H})$ and $\hat{\sigma}(\dot{H})$ the expected value and standard deviation of the expected Hurst exponent corresponding to an independent random variable ($\dot{H}$); the significance test would be as follows:[9]

$$t = \frac{\hat{H} - \hat{\mu}(\dot{H})}{\hat{\sigma}(\dot{H})} \qquad \text{F6}$$

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[9] Let *N* be the length of time series, due to $\hat{H}$ distributing like a normal the ordinary choice for $\hat{\sigma}(\dot{H})$ is $\approx 1/N^{1/2}$ as in Peters (1994). According to Couillard and Davison (2005), this choice corresponds to an infinite length time series, and yields easy and frequent rejections of the independence null hypothesis. They propose $\approx 1/eN^{1/3}$, which is the authors' choice.

As usual, if t is higher than ±1.96 it is possible to reject the null hypothesis of long-term independence with a 95% confidence level. The sign of t reveals the type of dependence: if it is positive (negative) there is evidence of persistence (antipersistence).

For convenience, given that $\dot{H}$ is the estimated Hurst exponent for random, independent and finite time series of length N, the spread between $\dot{H}$ and 0.5 corresponds to the bias estimation resulting from using finite time series and the choice of the size of periods (n). Subtracting such spread from the Hurst exponent estimated using *R/S*, namely $\hat{H}$, results in an adjusted estimated Hurst exponent, which will be noted as $\breve{H}$:

$$\breve{H} = \hat{H} - (\dot{H} - 0.5) \qquad \text{F7}$$

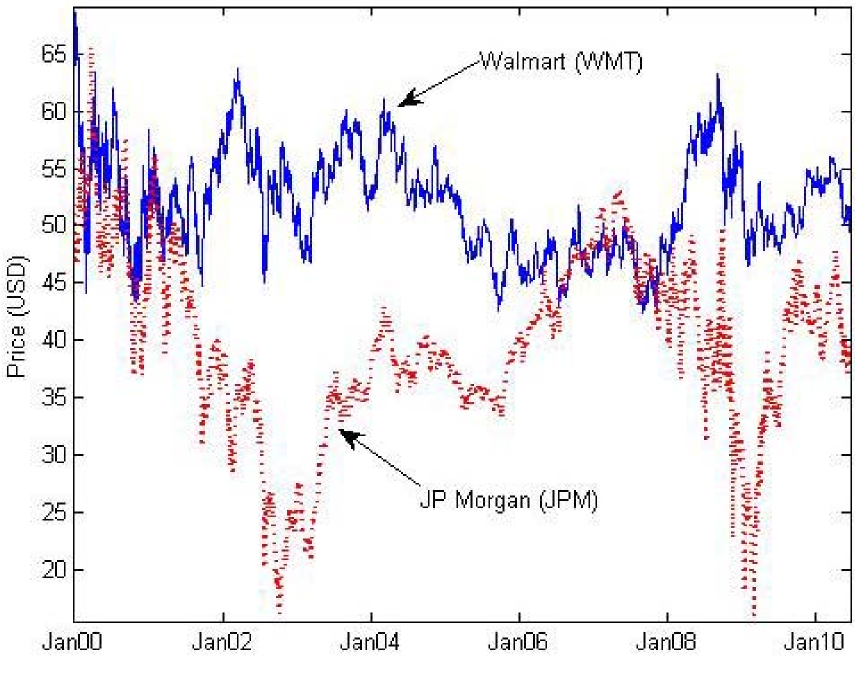
This adjusted estimated Hurst exponent ($\breve{H}$) is essential since it allows a practical and unbiased volatility scaling as will be presented in the following sections. Unlike prior literature on the estimation of the Hurst exponent in Finance and Economics, adjusting for estimation bias allows for practical applications such as portfolio theory and risk.

Estimated values of Hurst exponent ($\hat{H}$)

Figure 3 exhibits the Walmart and JP Morgan price-series from January 1st 2000 to June 25th 2010. Walmart's exhibits a narrower range in which prices fluctuate, where returns appear to compensate each other, while JP Morgan's appear to persist over time; since both share the same dollar scale, it is somewhat apparent that JP Morgan's time series are more persistent than Walmart's.

Figure 3

Daily prices for Walmart and JP Morgan



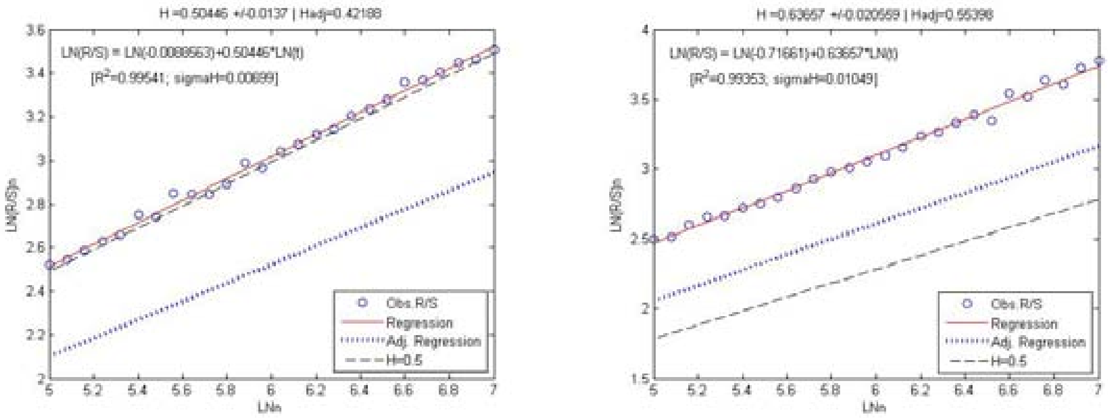
Source: authors' calculations.

Figure 4 exhibits the graphical result of applying *R/S* to both series' returns. Walmart exhibits an estimated Hurst exponent slightly above 0.5 ($\hat{H}_{WMT} = 0.504$), which would be a signal of non-significant persistence, while JP Morgan's $\hat{H}$ clearly diverges from 0.5 ($\hat{H}_{JPM} = 0.637$). Nevertheless, after acknowledging the positive estimation bias, the adjusted estimated Hurst exponent reveals that Walmart's time series is in fact significantly antipersistent ($\breve{H}_{WMT} = 0.422$; $t_{WMT} = -2.96$), whereas JP Morgan's remains significantly persistent ($\breve{H}_{JPM} = 0.554; t_{JPM} = 2.04$).

Figure 4

Walmart and JP Morgan (adjusted and unadjusted Hurst exponent)

Walmart (WMT) JP Morgan (JPM)



Source: authors' calculations.

Figure 5 exhibits the adjusted estimated Hurst exponent ($\breve{H}$) for individual risk factors (small dots) pertaining to different markets (eg developed and emerging) and instruments (fixed income, equity and commodities). As before, if the adjusted estimated Hurst exponent ($\breve{H}$) is greater (lower) than 0.5 there exists evidence of persistence (antipersistence), where the area between the vertical lines corresponds to the 95% confidence interval in which the independence hull hypothesis cannot be rejected.

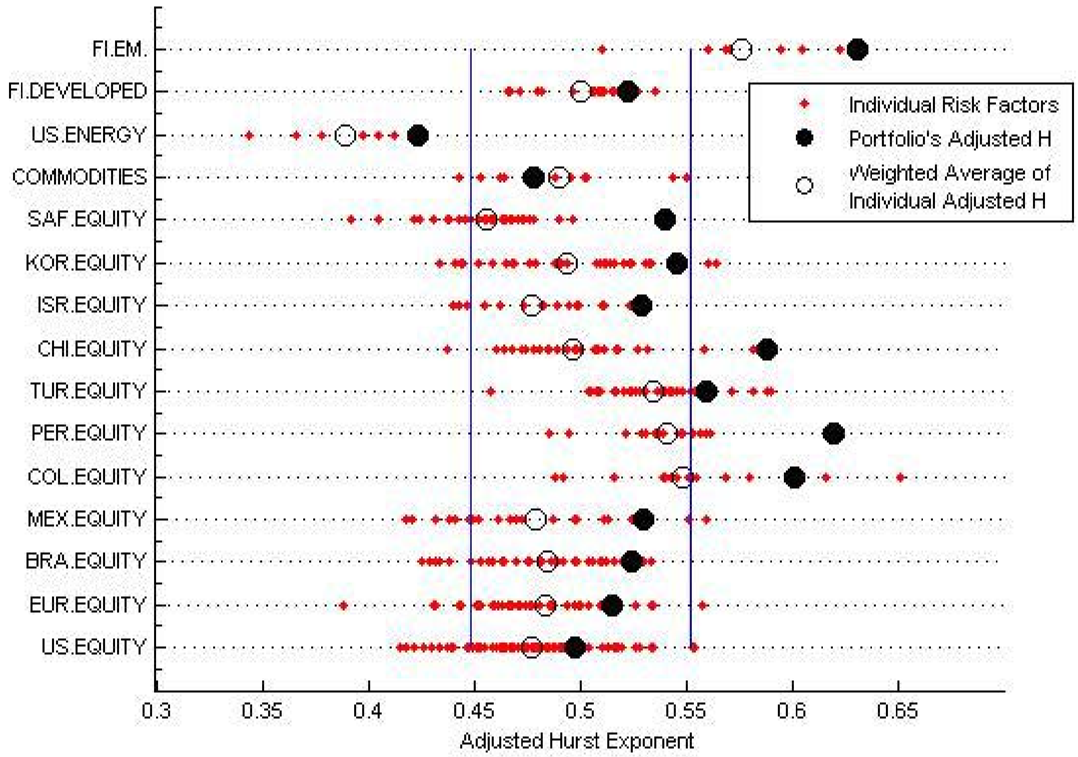
Individual risk factors across markets and instruments display different degrees of dependence, where persistence is typical of emerging markets' fixed income instruments (FI.EM) and of less-developed equity markets (eg Colombia, Turkey and Peru). Developed equity markets (eg US and EUR) and liquid emerging markets (eg Brazil, Mexico) show less incidence of persistent individual risk factors, even with several cases of antipersistence. These findings support Cajueiro and Tabak's (2008) comparison between developed and emerging markets. Results also correspond to the findings of Weron and Przybylowicz (2000) in relation to significant antipersistence of energy prices, but contradict Peters' (1996) affirmation about the difficulty of finding financial time series with antipersistent returns.

Regarding persistence at the portfolio level, Figure 5 displays the adjusted estimated Hurst exponent ($\breve{H}$) for an equally weighted portfolio of the individual risk factors (filled circles) and the equally weighted average of the individual risk factors' adjusted estimated Hurst exponent (empty circles). It is remarkable that the portfolios' adjusted estimated Hurst exponent tends to be higher than the weighted average of the individual exponents, which would indicate that the diversification effect does not apply to serial dependence as it does to variance or standard deviation.

It is also noteworthy that for emerging fixed income and equity markets the portfolios' adjusted estimated Hurst exponent ($\breve{H}$) is significantly higher than the weighted average of the individual exponents. Because aggregating risk factors should result in specific or idiosyncratic risk diversification, this could indicate that the remaining systemic risk is relatively more important for emerging than for developed markets; this could be the result of poor diversification opportunities within a small and illiquid market, or of the generalized impact of systemic shocks and the corresponding changes in risk appetite and liquidity in those markets.

Figure 5

Adjusted estimated Hurst exponent ($\breve{H}$)



Source: authors' calculations.

The markets included are: FI.EM. = Emerging Markets' Fixed Income (EMBI Global of Brazil, Mexico, Colombia, Peru, South Africa, Turkey and Chile); FI.DEVELOPED = Developed Markets' Fixed Income (as in Table 2); US.ENERGY = Off-peak day ahead electricity for several U.S. regions; COMMODITIES = oil, gold, copper, wheat, corn, cotton, aluminum, sugar, coffee, cocoa, rice, soy; and a market-capitalization representative set of securities from the equity markets of the United States (U.S.), Europe (EUR), Brazil (BRA), Mexico (MEX), Colombia (COL), Peru (PER), Turkey (TUR), Chile (CHI), Israel (ISR), Korea (KOR) and South Africa (SAF). All estimations were based on January 1st 2000-June 25th 2010 time series, except US.ENERGY (January 1st 2002–June 25th 2010).

4. Portfolio optimization under long-term dependence

The most far-reaching consequences of long-term dependence or memory in financial asset returns were pointed out by Lo (1991). He recognized that the long-term dependence conveys the invalidity of modern finance's milestones, where the most hard-hit would be the optimal consumption/savings and portfolio decisions, as well as the pricing of derivatives based on martingale methods.

Volatility scaling, investment decisions and portfolio optimization

Conventional portfolio optimization uses high-frequency data and customary procedures for return and volatility scaling in order to obtain allocations for low-frequency horizons. Let $\hat{\mu}_d$ and $\hat{\sigma}_d$ be the estimated high-frequency (eg daily) continuously compounded expected return and standard deviation, $\hat{\mu}_a$ and $\hat{\sigma}_a$ the estimated low-frequency (eg annual) continuously compounded expected return and standard deviation, and p the number of days-in-a-year convention. The standard procedure for asset allocation typically involves the following expected return [F8] and volatility escalation [F9]:

$$\hat{\mu}_a \sum_{t=1}^{p} \hat{\mu}_{d(t)} \qquad \text{F8}$$

$$\hat{\sigma}_a = \hat{\sigma}_d p^{0.5} \qquad \text{F9}$$

The standard procedure to scale returns up (eg from daily to annual) is assumption-free, and consists of interest compounding calculations. However, conventional volatility scaling inexorably involves the serial independence assumption.

If asset returns exhibit no serial dependence, using the square-root-of-time rule is adequate. Nevertheless, in the absence of independence some assets' volatility may increase with the time horizon, while others' may decrease; even if all assets' volatility increases, it may not increase at the same pace. Thus, Holton (1992) highlights the importance of considering volatility and the investment horizon as risk's first and second dimensions.

In the presence of long-term dependence, scaling returns up as in [F8] remains unchanged. However, for estimating volatility the scaling procedure should be generalized as follows:

$$\hat{\sigma}_a = \hat{\sigma}_d p^{\breve{H}} \qquad \text{F10}$$

Additionally, because mean-variance portfolio optimization involves working with the covariance matrix, the latter should be scaled up properly. Under the random-walk assumption, low-frequency covariance between two assets, *i* and *j*, corresponds to the arithmetic sum of high-frequency covariances (Winkelmann, 2003a); thus the relative variance between assets remains unrelated to the investment horizon.

Nevertheless, in the presence of dependence, either $\breve{H}_i \neq 0.5$ or $\breve{H}_j \neq 0.5$, as an extension to the volatility scaling procedure [F10], the *d*-frequency covariance between assets *i* and *j* ($\hat{\sigma}^2_{\{(i,j),d\}}$) should be scaled up to the *a*-frequency covariance ($\hat{\sigma}^2_{\{(i,j),a\}}$) as in Greene and Fielitz (1979) [F11]; this recognizes that memory in financial time series causes relative variance between assets to vary with the investment horizon.

$$\hat{\sigma}^2_{\{(i,j),a\}} = \left(p^{\breve{H}_i + \breve{H}_j}\right)\left(\hat{\sigma}^2_{\{(i,j),d\}}\right) \qquad \text{F11}$$

Long-term dependence inclusive portfolio optimization

In order to illustrate the impact of including long-term dependence adjustments to the covariance matrix scaling for asset allocation, a long-term portfolio optimization exercise is implemented based on the two methods for scaling volatility: (i) the square-root-of-time rule [F9] conventional method, and (ii) the method proposed by the authors [F10 and F11].

The square-root-of-time rule-based method begins by estimating the first two moments of the distribution of the risk factors and the covariance matrix from daily data. Afterwards, a traditional mean-variance optimization is employed, and the expected return and standard deviation of the resulting portfolios are customarily scaled up; since the square-root-of-time rule assumes volatilities' time-consistency, the portfolio weights remain the same regardless of the investment horizon.

The second method also begins by estimating the first two moments of the distribution and the covariance matrix from daily data. Next, because risk factors' dependence causes portfolio weights to vary according to the investment horizon, the standard deviation and covariance matrix scaling for long-term dependence effects [F10 and F11] takes place before optimizing.

Table 1 presents the set of risk factors to be considered in the portfolio optimization procedure. Consistent with the literature on strategic asset allocation, which points out that currency risk hedging is inappropriate for long-term portfolios (Solnik et al., 2003; Froot, 1993), all risk factors were included in their original currency.

According to Table 1, long-term dependence is significant for the two emerging market risk factors considered, namely equity and fixed income indexes, which – again – validates the findings of Cajueiro and Tabak (2008). Regarding commodities, divergence between $\breve{H}$ and 0.5 is rather low, with minor signals of antipersistence for metals and crude oil; agriculture and livestock commodities' $\breve{H}$ matches the independence assumption.

Table 1

Adjusted Hurst exponent for selected risk factors

Market	**Description**		**Abbreviation**	**Mean return**	**Standard deviation**	**Adjusted $\hat{H}$**	***t*-stat**
Commodities	Precious metals		PREC.MET.	0.03%	1.07%	0.47	(1.15)
	Industrial metals		IND.MET.	0.02%	1.38%	0.48	(1.00)
	Agriculture & livestock		AGR.&L.S.	-0.01%	0.91%	0.50	(0.11)
	Crude oil		CRUDE.OIL	0.04%	2.24%	0.48	(0.92)
Equity	Developed markets		EQ.DEV.	0.01%	1.01%	0.51	0.51
	Emerging markets		EQ.EM	0.02%	1.26%	0.59	**3.70**
Fixed income	Emerging markets		EMBI	0.04%	0.74%	0.59	**3.86**
	U.S. Treasury	1-5Y	US.T 1-5Y	0.02%	0.15%	0.53	1.14
		5-10Y	US.T 5-10Y	0.03%	0.36%	0.52	0.77
		10+Y	US.T 10+Y	0.03%	0.60%	0.51	0.26
	U.S. corporate AAA-AA	1-5Y	US.CORP 1-5Y	0.02%	0.17%	0.52	0.72
		5-10Y	US.CORP 5-10Y	0.03%	0.37%	0.50	0.05
		10+Y	US.CORP 10+Y	0.03%	0.55%	0.48	(0.94)
	U.S. mortgages AAA		US.MRTG	0.03%	0.21%	0.50	(0.20)
	GER treasury	1-5Y	GER.T 1-5Y	0.02%	0.12%	0.54	1.69
		5-10Y	GER.T 5-10Y	0.03%	0.27%	0.52	0.84
		10+Y	GER.T 10+Y	0.03%	0.53%	0.48	(0.87)
	JAP treasury	1-5Y	JAP.T 1-5Y	0.01%	0.09%	0.51	0.42
		5-10Y	JAP.T 5-10Y	0.02%	0.24%	0.50	(0.21)
		10+Y	JAP.T 10+Y	0.02%	0.41%	0.50	(0.05)
	U.K. treasury	1-5Y	UK.T 1-5Y	0.02%	0.14%	0.55	**2.10**
		5-10Y	UK.T 5-10Y	0.03%	0.30%	0.54	1.55
		10+Y	UK.T 10+Y	0.03%	0.52%	0.49	(0.28)

Calculations based on daily time series (January 1st 1995–June 25th 2010. Significant (95%) *t*-stats are highlighted. All denominated in their original currency. Precious metals, industrial metals, agriculture & livestock and crude oil correspond to S&P indexes; equity corresponds to MSCI indexes; Emerging market fixed income index is JP Morgan's EMBI; all other fixed income indexes correspond to Merrill Lynch (Bank of America) indexes.

Sources: All indexes provided by Bloomberg. Calculations are the authors'.

Developed markets' fixed income risk factors show low levels of persistence, except for short-term treasuries from U.K. and Germany, and medium-term treasuries from the U.K.; it is noteworthy that long-term fixed instruments consistently tend to exhibit lower persistence than short-term ones. Concerning developed markets' equity, the findings of Cajueiro and Tabak (2008), Menkens (2007), Couillard and Davison (2004), Ambrose et al. (1993) and

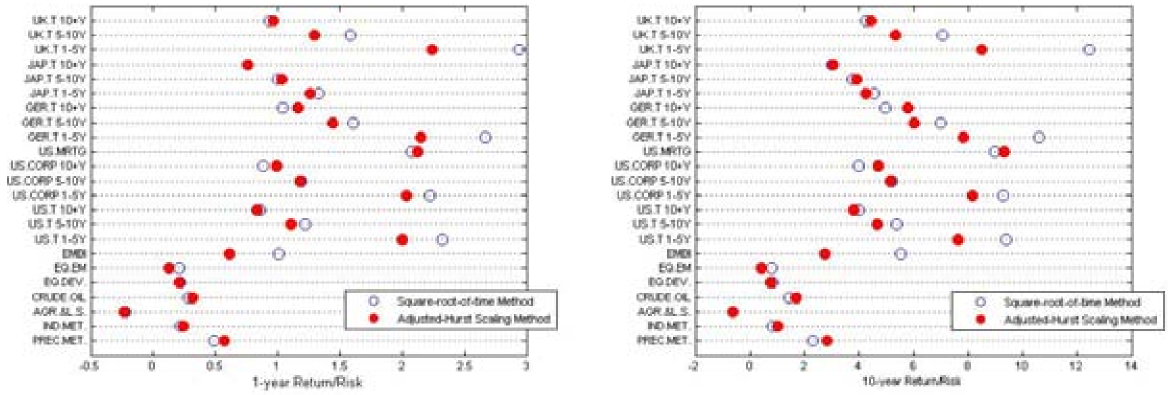
Lo (1991) are verified: there is no evidence of significant long-term dependence. Therefore, Peters' (1992) findings about long-term dependence in developed equity markets are contradicted; the reader should be aware that Peters and other authors in Table 1 do not adjust results for estimation bias.

Interestingly, contrary to conventional wisdom, fixed income instruments' mean returns significantly outperformed equity's for the time series under analysis; thus, it is likely that resulting efficient portfolios will disregard equity *vis-à-vis* academic basics. This supports recent concerns regarding the existence of a *natural hedge* from stocks in the long run and of a positive equity risk premium (Valdés, 2010; Arnott, 2009).

Figure 6

Return/risk ratio for the standard and enhanced methods

1-year 10-year

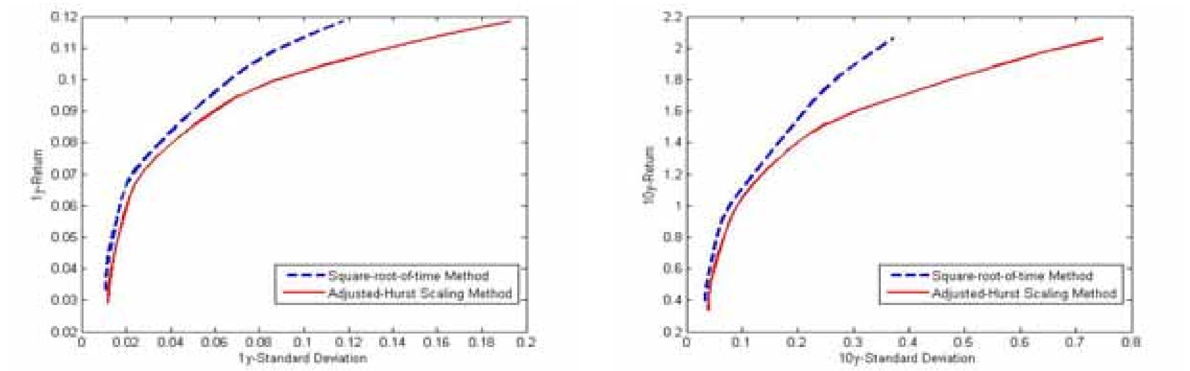


Source: authors' calculations.

Figure 7

Efficient frontiers for the standard and enhanced methods

1-year 10-year



Source: authors' calculations.

Using the adjusted estimated Hurst exponent (Table 2), Figure 6 exhibits the risk/return ratios for both scaling methods for 1-year and 10-year investment horizons. Relative return/risk ratios between methods clearly differ for almost all risk factors. Once dependence is taken into account, extreme differences between return/risk ratios due to concealed riskiness resulting from serial dependence are moderated; hence, it is plausible that adjusting for long-term persistence helps mitigate the well-known tendency of mean-variance optimization to provide extreme weights or corner solutions. The results shown in Figure 6 coincide with Greene and Fielitz's (1979) concern about how return/risk performance measures (eg Sharpe, Treynor and Jensen ratios) are affected by the differencing interval assumption in presence of long-term dependence.

Figure 7 exhibits the efficient frontiers for both scaling methods for 1-year and 10-year investment horizons. As expected, the standard method obtains a strictly dominating frontier with higher levels of return for each level of risk.

Table 2

1-year horizon efficient frontier weights

Panel a. – Square-root-of-time method

Port. #	Return / risk	Commodities [0.482; 0.19]	Emerging markets		Developed markets						
			Equity [0.585; 0.21]	EMBI [0.589; 1.00]	Equity [0.512; 0.22]	U.S. Treas. [0.517; 1.47]	U.S. corp. [0.499; 1.43]	U.S. mortg. [0.495; 2.07]	GER treas. [0.513; 1.77]	JAP treas. [0.501; 1.03]	U.K. treas. [0.526; 1.82]
1	3.0	0.9%	0.9%	0.0%	0.9%	11.0%	0.0%	0.0%	18.7%	58.4%	9.1%
2	3.5	0.7%	1.0%	0.3%	1.2%	15.0%	0.0%	0.0%	19.5%	39.3%	22.9%
3	3.6	1.0%	0.8%	1.3%	1.3%	11.8%	4.9%	1.4%	20.2%	21.4%	35.9%
4	3.5	1.2%	0.8%	2.0%	1.4%	6.9%	9.5%	4.4%	20.5%	6.8%	46.5%
5	3.3	2.1%	0.0%	6.2%	0.0%	0.0%	3.1%	19.3%	0.0%	1.8%	67.4%
6	2.5	3.5%	0.0%	16.8%	0.0%	0.0%	0.0%	26.3%	8.0%	0.0%	45.4%
7	2.0	5.0%	0.0%	25.1%	0.0%	0.0%	0.0%	31.9%	11.7%	0.0%	26.3%
8	1.6	6.5%	0.0%	37.8%	0.0%	2.5%	0.0%	7.4%	24.4%	0.0%	21.4%
9	1.4	6.4%	0.0%	53.5%	0.0%	2.1%	0.0%	0.0%	38.0%	0.0%	0.0%
10	1.0	0.0%	0.0%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Panel b. – Adjusted Hurst scaling method

Port. #	Return / risk	Commodities [0.482; 0.22]	Emerging markets		Developed markets						
			Equity [0.585; 0.13]	EMBI [0.589; 0.62]	Equity [0.512; 0.21]	U.S. Treas. [0.517; 1.32]	U.S. corp. [0.499; 1.40]	U.S. mortg. [0.495; 2.13]	GER treas. [0.513; 1.59]	JAP treas. [0.501; 1.02]	U.K. treas. [0.526; 1.50]
1	2.4	1.2%	0.5%	0.0%	1.1%	11.9%	0.0%	0.0%	14.8%	66.0%	4.6%
2	2.9	1.2%	0.5%	0.0%	1.3%	5.8%	4.4%	7.5%	16.4%	48.2%	14.9%
3	3.0	1.9%	0.4%	0.0%	1.4%	0.0%	8.6%	16.0%	17.3%	29.5%	24.7%
4	3.0	2.4%	0.6%	0.0%	1.7%	0.0%	10.6%	20.4%	20.1%	13.2%	31.1%
5	2.8	4.1%	0.0%	2.1%	0.4%	0.0%	0.0%	41.7%	0.0%	3.7%	47.9%
6	2.2	7.1%	0.0%	6.6%	0.0%	0.0%	0.0%	49.6%	19.1%	0.0%	17.6%
7	1.7	10.7%	0.0%	12.9%	0.0%	0.0%	0.0%	34.1%	42.3%	0.0%	0.0%
8	1.3	13.5%	0.0%	21.9%	0.0%	2.4%	0.0%	0.0%	62.2%	0.0%	0.0%
9	1.0	8.1%	0.0%	49.9%	0.0%	0.0%	0.0%	0.0%	41.9%	0.0%	0.0%
10	0.6	0.0%	0.0%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

The figures in brackets indicate the average adjusted Hurst exponent and the average return/risk ratio, respectively, for each risk factor.

Source: authors' calculations.

Strict dominance of the traditional method's efficient frontier occurs because relative return/risk ratios do not change with the time horizon; adjusting for long-term dependence causes efficient portfolio weights associated with high (low) persistence risk factors to decrease (increase) as the horizon increases. This becomes evident when observing portfolio weights obtained by each method along the 1-year horizon frontier (Table 2). Each frontier consists of ten portfolios, from the lowest risk to the highest return; the average adjusted exponent $(\breve{H})$ and average return/risk ratio for each category of risk factors are also reported in brackets.

Table 3

10-year horizon efficient frontier weights

Panel a. – Square-root-of-time method

Port. #	**Return / risk**	**Commodities** [0.482; 0.99]	**Emerging markets**		**Developed markets**						
			Equity [0.585; 0.79]	**EMBI** [0.589; 5.54]	**Equity** [0.512; 0.82]	**U.S. Treas.** [0.517; 6.25]	**U.S. corp.** [0.499; 6.16]	**U.S. mortg.** [0.495; 9.00]	**GER treas.** [0.513; 7.52]	**JAP treas.** [0.501; 3.79]	**U.K. treas.** [0.526; 7.91]
1	10.9	0.9%	0.9%	0.0%	0.9%	11.0%	0.0%	0.0%	18.7%	58.4%	9.1%
2	13.4	0.7%	1.0%	0.3%	1.2%	15.0%	0.0%	0.0%	19.5%	39.3%	22.9%
3	14.3	1.0%	0.8%	1.3%	1.3%	11.8%	4.9%	1.4%	20.2%	21.4%	35.9%
4	14.6	1.2%	0.8%	2.0%	1.4%	6.9%	9.5%	4.4%	20.5%	6.8%	46.5%
5	14.1	2.1%	0.0%	6.2%	0.0%	0.0%	3.1%	19.3%	0.0%	1.8%	67.4%
6	11.3	3.5%	0.0%	16.8%	0.0%	0.0%	0.0%	26.3%	8.0%	0.0%	45.4%
7	9.3	5.0%	0.0%	25.1%	0.0%	0.0%	0.0%	31.9%	11.7%	0.0%	26.3%
8	8.0	6.5%	0.0%	37.8%	0.0%	2.5%	0.0%	7.4%	24.4%	0.0%	21.4%
9	7.2	6.4%	0.0%	53.5%	0.0%	2.1%	0.0%	0.0%	38.0%	0.0%	0.0%
10	5.5	0.0%	0.0%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Panel b. – Adjusted Hurst scaling method

Port. #	**Return / risk**	**Commodities** [0.482; 1.23]	**Emerging markets**		**Developed markets**						
			Equity [0.585; 0.41]	**EMBI** [0.589; 2.76]	**Equity** [0.512; 0.75]	**U.S. Treas.** [0.517; 5.38]	**U.S. corp.** [0.499; 6.01]	**U.S. mortg.** [0.495; 9.33]	**GER treas.** [0.513; 6.54]	**JAP treas.** [0.501; 3.74]	**U.K treas.** [0.526; 6.09]
1	8.5	1.3%	0.3%	0.0%	1.1%	11.9%	0.0%	0.2%	13.2%	68.7%	3.2%
2	11.5	1.9%	0.3%	0.0%	1.2%	0.0%	5.2%	16.6%	12.5%	47.9%	14.5%
3	12.0	3.1%	0.2%	0.2%	1.4%	0.0%	4.9%	28.8%	11.7%	25.1%	24.5%
4	11.7	4.6%	0.0%	0.9%	1.3%	0.0%	2.1%	43.9%	2.4%	8.3%	36.5%
5	10.3	7.4%	0.0%	3.7%	0.0%	0.0%	0.0%	53.8%	17.3%	0.0%	18.0%
6	8.3	11.4%	0.0%	8.2%	0.0%	0.0%	0.0%	38.2%	42.3%	0.0%	0.0%
7	6.8	15.6%	0.0%	13.8%	0.0%	1.9%	0.0%	2.0%	66.7%	0.0%	0.0%
8	5.2	9.9%	0.0%	34.2%	0.0%	0.0%	0.0%	0.0%	55.9%	0.0%	0.0%
9	3.8	12.9%	0.0%	57.7%	0.0%	0.0%	0.0%	0.0%	29.4%	0.0%	0.0%
10	2.8	0.0%	0.0%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

The figures in brackets indicate the average adjusted Hurst exponent and the average return/risk ratio, respectively, for each risk factor.

Source: authors' calculations.

Relative overweighting of persistent risk factors (eg emerging markets' fixed income – EMBI) is evident for the conventional method. When dependence is taken into account, this overweighting diminishes in favor of near-independent or antipersistent risk factors, such as Japanese and German treasuries, U.S. mortgages or commodities. The main difference between risk factors across panels consists in lower divergences in the return/risk ratios, which are conveniently obtained from proper volatility scaling. This difference explains persistent risk factors' relative overweighting when using traditional covariance scaling. This analysis is validated for the ten-year horizon too (Table 3).

Table 3 confirms the adjustment of extreme divergences between risk factor's return/risk ratios when using the proposed procedure, and the persistent risk factors' relative overweighting due to traditional covariance scaling. Moreover, against basic financial principles or intuition, but as an obvious consequence of the square-root-of-time rule, traditional 1-year and 10-year efficient frontiers do not differ from each other; this emphasizes that customarily use of mean-variance optimization disregards the investment horizon as a meaningful factor of the asset allocation process.

Table 4 presents a summary of the weights allocated according to each investment horizon. Three cases are depicted: (i) because using the square-root-of-time rule makes the allocation independent from the investment horizon, X-year corresponds to the weight at any horizon; (ii) the 1-year horizon; (iii) the 10-year horizon. Figure 8 presents a graphical summary of the weights assigned to the efficient frontier for the three cases.

Table 4

X-year, 1-year and 10-year horizon efficient frontier weights

Summary (mean and maximum)

	Investment horizon	**Commodities [0.482; 0.99]**	**Emerging markets**		**Developed markets**						
			Equity [0.585]	**EMBI [0.589]**	**Equity [0.512]**	**U.S. Treas. [0.517]**	**U.S. corp. [0.499]**	**U.S. mortg. [0.495]**	**GER treas. [0.513]**	**JAP treas. [0.501]**	**U.K. treas. [0.526]**
Mean	X-y	2.9%	0.3%	25.4%	0.5%	4.6%	1.9%	8.9%	16.8%	11.2%	27.4%
	1-y	5.6%	0.2%	20.2%	0.6%	1.6%	2.5%	17.3%	24.0%	14.3%	13.8%
	10-y	7.9%	0.1%	22.9%	0.4%	0.9%	1.3%	18.8%	25.4%	13.2%	9.3%
Max	X-y	7.1%	1.0%	100.0%	1.4%	15.0%	12.1%	31.9%	38.0%	58.4%	67.4%
	1-y	13.5%	0.6%	100.0%	1.7%	11.9%	10.6%	51.8%	62.2%	66.0%	47.9%
	10-y	15.9%	0.3%	100.0%	1.6%	11.9%	5.5%	55.4%	66.7%	68.7%	36.5%

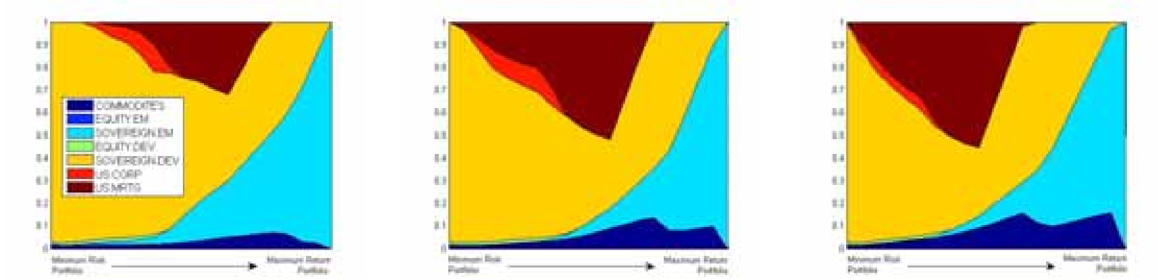
The figures in brackets indicate the average adjusted Hurst exponent for each risk factor.

Source: authors' calculations.

Figure 8

Square-root-of-time and adjusted Hurst methods for 1-year and 10-year weights

H=0.5 (*n*-year) | H=$\breve{H}$ (1-year) | H=$\breve{H}$ (10-year)



Source: authors' calculations.

5. Final remarks

Much attention has been given to financial asset returns' short-term dependence. In this sense many models are readily available to improve the estimation of the variance and, to a lesser degree, covariance inputs for short-term portfolio construction.

Less emphasis has been given to long-term dependence of returns. Akin to the financial literature, this paper shows that (i) significant long-term dependence is common in asset return time series; (ii) significant persistence is prevalent for emerging fixed income markets, and fairly frequent for emerging equity markets – mainly the less liquid ones; (iii) independence is representative of developed fixed income and equity markets, and somewhat recurrent for liquid emerging equity markets; (iv) U.S. energy markets exhibit significant antipersistence.

Interestingly, this document's support for prior evidence includes data from the most recent and severe episode of widespread financial disruption. Divergence with documented literature is circumscribed to our findings of recurrent antipersistence for developed equity markets, as well as a few liquid emerging markets.

This paper's long-term dependence assessment relies on rescaled range analysis (*R/S*), a popular and robust methodology designed for geophysics but extensively used in financial literature. Well-known issues of *R/S* such as the optimal minimum and maximum size of periods were surmounted vis-à-vis some previous studies, resulting in reduced estimators' standard errors and minimal interference of short-term serial dependence in the results.

Ahead of *R/S* financial literature, we used the spread between the estimated Hurst exponent ($\hat{H}$) and the expected Hurst exponent for independent and finite time series ($\dot{H}$) to estimate an adjusted Hurst exponent ($\breve{H}$). Under a generalized version of the conventional volatility and covariance scaling procedure, we suggest using this adjusted measure of long-term dependence for practical purposes, where long-term mean-variance portfolio optimization is a natural choice to begin with.

Comparing the efficient portfolio weights resulting from customary mean-variance optimization (eg independency assumption reliant) and the suggested enhanced procedure shows that the former tends to overweight persistent risk factors. Once long-term dependence is considered via the proposed covariance scaling procedure, the return per unit of risk of persistent (antipersistent) risk factors is adjusted downwards (upwards), decreasing (increasing) the weight of high (low) persistence risk factors as the investment horizon increases. Our results provide evidence of the significance of weight differences for 1-year and 10-year investment horizons and of how these differences reveal that adjusted efficient frontiers may be less optimistic than conventional ones.

Long-term dependence recognition conveys various practical advantages, especially for long-term institutional investors, such as central banks, pension funds and sovereign wealth managers. First, because the proposed scaling procedure exposes concealed riskiness resulting from persistence, extreme relative return/risk ratio differences due to inappropriate risk scaling are moderated, avoiding to some extent excessive risk taking in long-term portfolios and mitigating the presence of extreme portfolio weights.

Second, evidence of significant persistence in small and illiquid capital markets provides proof of masked risks within their securities. Such underestimation of local instruments' long-term risk could explain two well-known facts of those capital markets: (i) the tendency to hold a disproportionate level of investments within the domestic market or "home bias"; and (ii) the reluctance to hold foreign currency-denominated assets. Recognizing long-term dependence would make local – persistent – instruments from small and illiquid markets less attractive within the mean-variance asset allocation framework, and developed markets' – independent or antipersistent – instruments more attractive.

Given these insights, we are currently considering three research topics: firstly, to study the contribution of individual risk factors to portfolio's persistence. The initial results presented here show that persistence at the portfolio level can be significantly higher than the weighted persistence of individual assets, especially for small and illiquid markets, thereby reinforcing the case for international diversification.

Secondly, akin to upside and downside risk concepts, we also envision a methodology capable of differentiating upside from downside persistence. This is a key issue because persistence may be an asset's desirable (undesirable) feature if its price is expected to rise (fall) in the future (eg a persistent bond may be attractive on the verge of monetary expansion). In the meantime, we suggest considering the market environment and investors' views in order to decide on the convenience of underweighting persistent risk factors. Alternatively, including optimization constraints such as a threshold for maximum drawdown (Reveiz and León, 2010) may capture investors' natural inclination (reluctance) to hold upside (downside) persistent risk factors.

Finally, because Black-Litterman portfolio optimization is heavily reliant on the serial long-term independence assumption via traditional volatility scaling and the starting global CAPM equilibrium, our agenda also includes designing long-term dependence adjustments to this celebrated approach. A forthcoming paper by one of the authors will present how Colombia's foreign reserve management approach already incorporates the adjustment suggested here.

Appendix A

- For a time series of N returns, having k independent (non overlapping[10]) windows or samples of size n, divide the original series in such way that $n \times k = N$.
- Estimate the arithmetic mean of each k-segment ($\hat{\mu}_k$) of size n.
- Obtain the difference between each i-return and the mean of each k segment ($\hat{\mu}_k$).

$$Y_{i,k} = x_{i,k} - \hat{\mu}_k$$

- Calculate accumulative differences for each k segment.

$$D_{i,k} = \sum_{i=1}^{n} Y_{i,k}$$

- Calculate range ($R_{n,k}$) of the $D_{i,k}$ series.

$$R_{n,k} = \max\left(D_{1,k}, \ldots, D_{i,k}, \ldots D_{n,k}\right) - \min\left(D_{1,k}, \ldots, D_{i,k}, \ldots D_{n,k}\right)$$

- Estimate standard deviation for each k segment ($S_{n,k}$).

$$S_{n,k} = \sqrt{\frac{1}{(n-1)} \sum_{i=1}^{n} \left(x_{i,k} - \hat{\mu}_k\right)^2}$$

- Calculate rescaled range for each segment k.

$$\left(R/S\right)_{n,k} = R_{n,k}/S_{n,k}$$

- Calculate average rescaled range for k segments of size n.

$$\left(R/S\right)_n = \frac{1}{k} \sum_{i=1}^{k} \left(R/S\right)_{n,k}$$

$(R/S)_n$ corresponds to the average standardized distance covered per unit of time n.

The previous procedure must be done for different values of k, where $k_j = n_{min} \ldots n_{max}$, and where n_{min} y n_{max} corresponds to the minimum and maximum of the chosen window to calculate the rescaled range. Thus, we have j values of $(R/S)_n$, where $n_j = \frac{N}{k_j}$.

Finally, using n and $(R/S)_n$ values we estimate the ordinary least squares regression proposed by Mandelbrot and Wallis (1969a y 1969b), where H corresponds to the estimated Hurst exponent:

$$Log\left(R/S\right)_n = Log(c) + HLog(n)$$

[10] For a discussion regarding the use of overlapping and non-overlapping segments, please refer to Nawrocki (1995) and Ellis (2007).

Appendix B

Table B1

Literature on R/S-estimated Hurst exponent

Author	Time series	Period (frequency)	H	Fit	
Peters (1992)	S&P500 – USA	01/1950 – 06/1988 (M)	0.780	N/A	N/A
Ambrose et al. (1993)	S&P500 – USA	07/1962 – 12/1988 (D)	0.531	1.380	‡
	S&P500 – USA	01/1950 – 07/1988 (M)	0.622	1.490	‡
Sierra (2007)	IPC – MEX.	01/1999 – 02/2006 (D)	0.557	0.990	§
	DJIA – USA	06/1999 – 05/2006 (D)	0.504	0.988	§
Palomas (2002)	IPC – MEX.	01/1988 – 09/2001 (D)	0.584	0.995	§
	IPC – MEX.	01/1983 – 05/2001 (M)	0.713	0.976	§
	DJIA – USA	01/1950 – 08/2001 (M)	0.658	0.994	§
	S&P500 – USA		0.686	0.993	§
Qian and Rasheed (2004)	DJIA – USA	11/1969 – 12/1973 (D)	0.650	N/A	N/A
Bilel and Nadhem (2009)	S&P500 – USA	03/1990 – 09/2008 (M)	0.525	1.400	‡
	S&PTSX – CAN.		0.541	1.465	‡
	CAC40 – FR.		0.537	2.088	‡
	DAX100 – GER.		0.541	1.644	‡
	MIB – ITALY		0.505	1.644	‡
	NIKKEI225 – JAP.		0.551	2.635	‡
	FTSE 100 – U.K.		0.511	2.420	‡
Cajueiro and Tabak (2008)	NIKKEI225 – JAP.	01/1999 – 12/2005 (D)	0.547	0.038	†
	MERVAL – ARG.		0.584	0.040	†
	BOVESPA – BRA.		0.612	0.040	†
	SENSEX – INDIA		0.591	0.040	†
	KOSPY – S.KOR.		0.551	0.039	†
	IPSA – CHILE		0.594	0.040	†
	IPC – MEX.		0.557	0.039	†
	IGBVL – PERU		0.656	0.042	†
	ISE – TURKEY		0.538	0.036	†
	TA-100 – ISRAEL		0.584	0.041	†
	FTSE100 – U.K.		0.521	0.039	†
	S&P500 – USA		0.519	0.037	†
Jagric et al. (2005)	PX50 – CZ.REP.	09/1993 – 07/2004 (D)	0.645	0.018	†
	BUX – HUNG.	01/1991 – 06/2004 (D)	0.626	0.015	†
	WSE – POLAND	03/1994 – 08/2004 (D)	0.569	0.018	†
	RTS – RUSSIA	09/1995 – 08/2004 (D)	0.648	0.020	†
	SAX – SLKIA	07/1995 – 07/2004 (D)	0.525	0.020	†
	SBI – SLVNIA	01/1993 – 07/2004 (D)	0.656	0.017	†
McKenzie (2001)	Australian Stock Exch.	04/1876 – 03/1996 (M)	0.571	2.027	¤
			0.622	1.850	¤

Table B1 (cont)

Literature on R/S-estimated Hurst exponent

Author	Time series	Period (frequency)	H	Fit	
Alptekin (2008)	Gold – Istambul Gold Exchange	01/2003 – 03/2008 (D)	0.600	2.100	‡
Corazza et al. (1997)	Corn Futures – CBOT	01/1981 – 10/1991 (D)	0.760	N/A	N/A
	Oats Futures – CBOT		0.700	N/A	N/A
	Soybean Futures – CBOT		0.740	N/A	N/A
	Soybean oil futures – CBOT		0.800	N/A	N/A
	Wheat futures – CBOT		0.650	N/A	N/A
Erzgraber et al.(2008)	Energy (NordPool) – Norway	01/1999 – 01/2007 (D)	0.270	N/A	N/A
Weron and Przandbandlowicz (2000)	Energy (CalPX) California	03/1998 – 01/2000 (H)	0.439	N/A	N/A
	Energy (SWEP) – Switzerland	03/1998 – 03/2000 (D)	0.529	N/A	N/A
Batten et al. (1999)	DMK/USD	01/1976 – 09/1998 (D)	0.623	2.248	¤
	CHF/USD		0.610	2.053	¤
	JPY/USD		0.609	1.954	¤
	GBP/USD		0.590	1.487	¤
Sierra (2007)	MXN/USD	01/1995 – 02/2006 (D)	0.526	0.994	§
	USD/EUR	06/1999 – 05/2006 (D)	0.559	0.995	§
Da Silva et al. (2007)	BRL/USD	01/1995 – 08/2006 (D)	0.630	3.260	¤
Souza et al. (2008)	DEM/USD	05/1986 – 12/1998 (D)	0.580	0.026	†
	3M future DEM/USD		0.571	0.026	†
	FRF/USD		0.576	0.026	†
	GBP/USD		0.567	0.026	†
	ITL/USD		0.598	0.026	†
Peters (1992)	30Y Treas. – USA	01/1950 – 06/1988 (M)	0.670	N/A	N/A

Frequencies correspond to the following convention: hourly (H), daily (D), monthly (M).

§ Corresponds to the R^2 of the regression [F4].

† Corresponds to the standard error of estimated H.

‡ Corresponds to Lo's Vq statistic (1991).

¤ Corresponds to the *t*-statistic by Couillard and Davison (2005).

Source: authors'design.

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