A technical note on the Svensson model as applied to the Swiss term structure

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The purpose of this note is to describe the methodology used by the Swiss National Bank to construct the Swiss government zero coupon curve by applying an estimation based on the Svensson model.

1.   Data and data selection

Spot and forward rates are estimated based on daily observations of the yield to maturity on Swiss government bonds and their coupon payments for bonds with maturities ranging from one to 50 years, and four money market rates on one-, three-, six- and 12-month holdings respectively. Callable bonds and bonds with a residual maturity of less than one year are excluded from the estimation. At the moment (August 2003), the estimation is done with 18 bonds and the money market rates.

2.   The model

The Swiss National Bank uses a model developed by Charles Nelson and Andrew Siegel in 1987 and extended by Svensson. Nelson and Siegel assume that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Let $f(t, t+m)$ denote the instantaneous forward rate with time to settlement m, for a given trade date t. Then the Svensson forward rate function can be written as:

$$f(t, t+m, b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right) \tag{1}$$

where $b = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$ is a vector of parameters (β_0, τ_1 and τ_2 must be positive). The Svensson method is identical to Nelson and Siegel's, but adds the term $\beta_3 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_2}\right)$ to the instantaneous forward rate function. In contrast to the Nelson-Siegel approach, this functional form allows for more than one local extremum along the maturity profile. This can be useful in improving the fit of yield curves.

The spot rate can be derived by integrating the forward rate and then dividing the result by the remaining time to maturity. It is given by:

$$i(t, t+m, b) = \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \left(\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right)\right) + \beta_3 \left(\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right)\right) \tag{2}$$

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[1]   Swiss National Bank, Statistics Section.

The discount function is related to the spot rate by:

$$d(t, t+m, b) = \exp\left(-\frac{i(t, t+m, b)}{100} m\right) \tag{3}$$

This discount function is used to compute the estimated (theoretical) bond price. We can define the price of a coupon bond, similarly to the traditional valuation of an investment, as the sum of discounted future coupon payments c and the present value of the face value paid after m years. It follows that the price of the bond $P(t, t+m)$ on trade day t can then be approximated by:

$$P(t, t+m, b) = \sum_{k=1}^{m} cd(t, t+k, b) + 100 d(t, t+m, b) \tag{4}$$

For coupon bonds, yields to maturity are often quoted. The yield to maturity is the internal rate of return for the coupon bond that makes the present value of the coupon payments and the face value equal to the price of the bond. Thus, the price of the coupon bond can be written as a function of the yield to maturity $y(t, t+m)$:

$$P(t, t+m) = \sum_{k=1}^{m} c \exp\left(-\frac{y(t, t+m)}{100} k\right) + 100 \exp\left(-\frac{y(t, t+m)}{100} m\right) \tag{5}$$

3. The estimation

The discount function is estimated for each trade date by minimising either the sum of squared price errors or the sum of squared yield errors. We choose the parameter so as to minimise yield-to-maturity errors. Minimising price errors sometimes results in fairly large yield errors for bonds and money market rates for short maturities. This is because yields are very sensitive to prices for short maturities.

The estimation is done with the restriction that the forward rate curve (and hence the spot rate curve) should start at the left end (from the overnight rate). This means that the term $(\beta_0 + \beta_1)$ equals the overnight rate.

The optimisation is performed using a numerical non-linear optimisation to maximise a log-likelihood function subject to the constraint on the parameter β_1 (= *overnight rate* – β_0). First, we use the Simplex algorithm to compute starting values and then the Berndt, Hall, Hall and Hausmann (BHHH) algorithm to estimate the final parameters. The optimisation procedure consists of the following steps for both numerical algorithms:

Initialising the parameter $b_t = (\beta_{0t}, \beta_{1t}, \beta_{2t}, \beta_{3t}, \tau_{1t}, \tau_{2t})$ for $t = 1$.

Calculating those spot rates $i(t, t+m, b_t)$ which are necessary in order to calculate the discount functions (equation (2)).

Computing the discount factors $d(t, t+m, b_t)$ for each bond (equation (3)).

Computing the estimated (theoretical) prices for the N different bonds (equation (4)).

Calculating the estimated (theoretical) yields to maturity for all the bonds (equation (5)) using the Newton-Raphson numerical algorithm.

Computing the function $\sum_{i=1}^{N}(y_{it}(t, t+m_i) - \hat{y}_{it}(t, t+m_i, b_t))^2$ (sum of squared yield errors) using first the Simplex algorithm and then the BHHH algorithm in order to determine a new b_{t+1}.

Examining the convergence condition:

$(b_{t+1} - b_t)'(b_{t+1} - b_t) < \alpha$ for an $\alpha > 0$

If the condition does not hold, go back to step 2 with b_{t+1} as a new vector instead of b_t.

The optimisation procedure alternates the parameters of *b* and with them the spot rates so as to minimise the sum of the (squared) differences between observed and calculated yields to maturity. The 95% confidence intervals are computed using the delta method.[2]

References

Nelson, C R and A F Siegel (1987): "Parsimonious modeling of yield curves", *Journal of Business*, 60, pp 473-89.

Svensson, L E O (1994): "Estimating and interpreting forward interest rates: Sweden 1992-4", *NBER Working Paper Series*, no 4871, September.

Table 1

Term structure of interest rates - estimation details

Estimation method	Minimised error	Shortest maturity in estimation	Adjustments for tax distortions	Relevant maturity spectrum
Svensson	Yields	Money market rates: ≥ 1 day bonds: ≥ 1 year	No	1 to 30 years

Table 2

Term structure of interest rates - availability from the EASY database

Method	Estimates available since	Frequency	Spot rates	Forward rates	Parameters	Notation
Svensson	4 January 1988 5 January 1998	Weekly Daily	1 to 10 years 15 years 20 years 30 years (since 5 January 1998)	1 to 10 years 15 years 20 years 30 years	6	Percentages

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[2] The delta method implies that, for the purpose of computing confidence intervals for the instantaneous forward rate, the estimated forward rate $f(m; b)$ for a given time to settlement m is considered to be distributed as a normal variable with mean $f(m; \hat{b})$ and covariance $\frac{\partial f(m, \hat{b})'}{\partial b} \hat{\Sigma} \frac{\partial f(m, \hat{b})}{\partial b}$, where $\hat{b}$, $\hat{\Sigma}$ and $\frac{\partial f(m, \hat{b})}{\partial b}$ denote the estimates of the parameter vector *b*, its covariance matrix and the column vector of partial derivatives for the parameters, respectively. When the BHHH algorithm is used, a natural estimator for the asymptotic covariance matrix for $\hat{\beta}$ is $\hat{\Sigma}_{\hat{\beta}} = \left[\sum_{i=1}^{N} \left(\frac{\partial L_n}{\partial b} \right) \left(\frac{\partial L_n}{\partial b'} \right) \right]^{-1} \Big|_{\hat{b}, \hat{\sigma}^2}$, where *L* is the log-likelihood function and *i* = 1, 2, ..., *N* the number of securities.