Estimation of spot and forward rates from daily observations

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Introduction

Spot and forward interest rates are calculated from daily observations of the yield to maturity on Norwegian government bonds and their coupon payments for bonds with maturities in the range of two to 10 years, and four money market rates on one-, three-, six- and 12-month holdings respectively.

We use money market rates instead of Treasury bill rates since the secondary market for the latter is much less liquid in Norway.

Details of the estimation procedure

We consider two variants of parametric forward interest rate functions $f(m, \beta)$ proposed by Nelson and Siegel (1987) and Svensson (1994), where m denotes the remaining maturity and β the parameter vector to be estimated. The corresponding spot interest rate function can be written as the average of the instantaneous forward rates with settlement between 0 and m:

$$s(m,\beta) = \frac{1}{m}\int_{\tau=0}^{m} f(\tau,\beta)\,d\tau \tag{1)}$$

For a given trading date, let there be n bonds (c_j, m_j, y_j, p_j), $j = 1, ...,n$ represented by their coupons c_j, remaining maturity m_j and observed yield to maturity y_j. p_j denotes the observed price of a given bond. For bonds with annual coupon payments, we index the coupon payments by the sequence $\tau_{jk}, k = 1, ..., K_j$, where K_j denotes the number of coupon payments for bond j. Allowing remaining maturity m_j to be non-integer, we define:

$$\tau_{jk} = m_j - [m_j] + k - 1 \tag{2}$$

$$K_j = [m_j] + 1 \tag{3}$$

where $[m_j]$ denotes the highest integer lower than m_j. The estimated price of a coupon bond $P_j(\beta)$ can be written as the sum of prices of a sequence of zero coupon (discount) bonds related to each coupon payment and the face value of the bond (normalised to 1), each priced with the discount function:

$$d(m_j,\beta) = \exp\left[-\frac{s(m_j,\beta)}{100}m_j\right] \tag{4}$$

Hence :

$$P_j(\beta) = \sum_{k=1}^{K_j} c_j d(\tau_{jk},\beta) + d(\tau_{jK},\beta), \quad j = 1, ...,n \tag{5}$$

We note that we can characterise each bond either by the observed triplet (c_j, m_j, p_j) or by the triplet (c_j, m_j, y_j) replacing the price p_j of the bond with the bond's yield to maturity y_j. From the coupons $c_j, j = 1, ...,n$ and the indexed sequence of payments $\tau_{jk}(m_j)$, k = 1, ..., K_j, we can then use the present value function and estimate a corresponding price P_j of bond j:

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$$P_j\left(c_j, m_j, y_j\right) = \sum_{k=1}^{K_j} \frac{c_j}{\left(1+\frac{y_j}{100}\right)^{\tau_{jk}(m_j)}} + \frac{1}{\left(1+\frac{y_j}{100}\right)^{\tau_j K_j(m_j)}}, \quad j = 1, \ldots\ldots, n \tag{6}$$

Alternatively, when we know the observed price p_j on bond j, we can estimate the yield to maturity (Y_j) by solving for Y_j in:

$$p_j = \sum_{k=1}^{k_j} \frac{c_j}{\left(1+\frac{Y_j}{100}\right)^{\tau_{jk}(m_j)}} + \frac{1}{\left(1+\frac{Y_j}{100}\right)^{\tau_j K_j(m_j)}}, \quad j = 1, \ldots\ldots, n \tag{7}$$

using a standard Newton-Raphson algorithm.

Likewise, this relationship between Y_j and P_j can be used in the parametric case when we derive the discount function from the forward interest rate function $f(m,\beta)$, hence the estimated yield to maturity for bond j denoted $Y_j(\beta)$ can then be computed from the present value function:

$$P_j(\beta) = \sum_{k=1}^{K_j} \frac{c_j}{\left(1+\frac{Y_j(\beta)}{100}\right)^{\tau_{jk}}} + \frac{1}{\left(1+\frac{Y_j(\beta)}{100}\right)^{\tau_j K_j}}, \quad j = 1, \ldots\ldots, n \tag{8}$$

also using a standard Newton-Raphson algorithm.

The observed yield to maturity y_j is assumed to differ from the estimated yield to maturity $Y_j(\beta)$ by a normally distributed error term $\varepsilon_j \sim \text{Niid}(0,\sigma_\varepsilon), \forall j$:

$$y_j = Y_j(\beta) + \varepsilon_j, \quad j = 1, \ldots\ldots, n \tag{9}$$

We use the method proposed in Svensson (1994) and estimate the following forward rate function, with parameters $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$. This relationship is also denoted as the extended Nelson-Siegel forward rate function:

$$f\left(m,\beta\right) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right) \tag{10}$$

It can be shown that the corresponding spot interest rate function can be expressed as:

$$s\left(m,\beta\right) = \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right)\right] + \beta_3 \left[\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right)\right] \tag{11}$$

The parameters in the forward rate function β are estimated by solving the following maximum likelihood estimation problem:

$$\hat{\beta} : \underset{\beta}{Max} \left[-\frac{n}{2} \ln\left(2\pi\sigma_\varepsilon\right) - \frac{1}{2} \sum_{j=1}^{n} \left(\frac{y_j - Y_j(\beta)}{\sigma_\varepsilon}\right)^2\right] \tag{12}$$

inserting the following MLE : $\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(y_j - Y_j(\beta))^2}$ for σ_ε. An alternative to minimising the sum of squared yield errors would be to minimise the sum of squared price errors $\sum_{j=1}^{n}\left(p_j - P_j(\beta)\right)^2$ instead.

As pointed out by Svensson (1994), however, minimising price errors sometimes results in fairly large yield errors for bonds and bills with short maturities while minimising yield errors gives a substantially better fit for short maturities, and the two procedures seem to perform equally well for long maturities. This is because prices are very insensitive to yields for short maturities.

On the other hand, minimising yield errors entails, as we have seen, an extra Newton-Raphson iteration where we solve for the yield Y_j in the price function. This could potentially cause some convergence problems to occur at certain data points.

The continually compounded spot and forward interest rates which are derived from the equations above for a given $\hat{\beta}$ are finally transformed into annually compounded interest rates, ie:

$$
\begin{aligned}
s_a(m,\hat{\beta}) &= 100\left(\exp\left[\frac{s(m,\hat{\beta})}{100}\right]-1\right) \\
f_a(m,\hat{\beta}) &= 100\left(\exp\left[\frac{f(m,\hat{\beta})}{100}\right]-1\right)
\end{aligned}
\tag{13}
$$

References

Nelson, C R and A F Siegel (1987): “Parsimonious modeling of yield curves”, *Journal of Business*, 60, pp 473-89.

Svensson, L E O (1994): “Estimating and interpreting forward interest rates: Sweden 1992-4”, International Monetary Fund, *IMF Working Paper*, 1994/114.