A technical note on the estimation of the zero coupon yield and forward rate curves of Japanese government securities

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This note covers data description and estimation techniques for estimating zero coupon yield curves and implied forward rate curves using Japanese government securities. The Bank of Japan estimates these curves based on the method developed by Fisher et al (1995).

1. Data description

To estimate the yield curves of risk-free fixed income assets, the following four types of Japanese government securities are used: 10-year and 20-year government bonds (hereinafter 10-yr JGBs and 20-yr JGBs, respectively) and three- and six-month Treasury bills (hereinafter 3m TBs and 6m TBs, respectively). The first two are fixed income bonds with semiannual coupon payments while the latter two are discount securities. In each case, the data required for estimation are: ID number, quote date, redemption date, coupon rate (zero for TBs), and price.

Table 1

Selected features of each security

	10-yr JGBs	**20-yr JGBs**	**3m TBs**	**6m TBs**
Frequency of issuance	Monthly	Quarterly	Monthly	Monthly
Number[1] of issues outstanding in secondary markets	Around 85	Around 35	3	6
Price data for estimation purposes	Small-lot trades,[2] price as of 3 pm on TSE[3]		Inter-dealer price as of 5 pm	

[1] Varies over time. [2] Orders of more than JPY 1 million but less than JPY 10 million, par value. [3] 3 pm is the closing time of the Tokyo Stock Exchange (TSE).

Input data set details:

1. The data set includes all JGBs listed on the TSE and TBs outstanding in the inter-dealer market with one exception: due to the existence of a “redemption fee”,[1] the data for 10-yr JGBs with a remaining maturity of less than half a year are excluded.[2]

2. TB prices are adjusted for withholding tax levied at issuance and repaid at redemption according to the following formula:

   TB price for input data = quoted price $*$ 100 / (100 + withholding tax),
   where withholding tax = (100 – average issuance price) $*$ 0.18 and 0.18 is the withholding tax rate.

[1] The redemption fee (currently JPY 0.09) is gradually incorporated into the JGB price as remaining maturity becomes shorter, especially when it becomes less than half a year.

[2] Since the shortest remaining maturity of 20-yr JGBs is currently around nine years, the “redemption fee” problem applies only to 10-yr JGBs - at least for now.

2. Estimation techniques

Interpolating forward rate curves by smoothing splines

Following Fisher et al (1995), zero coupon yield and forward rate curves are extracted by smoothing the spline with the roughness penalty selected according to generalised cross validation (GCV).

First, the cash flow of each security i is decomposed into the scheduled coupon and principal payments $(c_{1i}, c_{2i}, ..., c_{mi})$ and the number of days to each payment $(t_{1i}, t_{2i}, ..., t_{mi})$ is calculated. m is the number of remaining coupon payments until maturity.

Let $\delta(t_{ji})$ be the discount factor at time t_{ji}. Also let $\tilde{\delta}(t_i)$ and c_i be the column vector of discount factors and payments for security i, respectively. Then, the price of security i can be written as $p_i = c_i' \tilde{\delta}(t_i)$.[3] This decomposition is effected for all securities in the data set ($i = 1, 2, ..., n$), and P is defined as $(p_1, p_2, ..., p_n)'$.

Instantaneous forward rate curves are expressed by linear combination of cubic B-splines as below:

$$f(t) \equiv (\phi_1(t), \phi_2(t), ..., \phi_\kappa(t))(\beta_1, \beta_2, ..., \beta_\kappa)' \equiv \phi(t)\beta \tag{1}$$

where $(\phi_1(t), \phi_2(t), ..., \phi_\kappa(t))$ is a cubic B-spline basis, $(\beta_1, \beta_2, ..., \beta_\kappa)'$ is a column vector of coefficients, and κ is the number of knot points plus 2.

By definition, the discount factor can be written with the above forward rates as:

$$\delta_s(t, \beta) = \exp\left(-\int_0^T \phi(s)\beta ds\right) \tag{2}$$

where T is the largest t_{ji} for all j's and i's.

Let $\Pi(\beta)$ be the vector of prices of securities based on the above interpolating forward rates such that:

$$\Pi(\beta) \equiv (\pi_1(\beta), \pi_2(\beta), ..., \pi_m(\beta)) \tag{3}$$

where $\pi_i(\beta) \equiv c_i' \tilde{\delta}_s(t_i, \beta)$ and $\tilde{\delta}_s(t_i, \beta) \equiv \left(\delta_s(t_{1i}, \beta), \delta_s(t_{2i,} \beta), ..., \delta_s(t_{mi}, \beta)\right)'$

The smoothing spline minimises the following problem for a given λ (stated below) with respect to β:

$$\min_\beta \left[(P - \Pi(\beta))'(P - \Pi(\beta)) + \lambda \int_0^T f''(t)^2 dt \right] \tag{4}$$

The first term of this expression is the sum of the residuals of squares and the second term defines the roughness penalty. λ, a constant, is a weighting parameter of the roughness penalty. The bigger λ becomes, the smoother the estimated forward rate curves look. In the smoothing splines, the number of effective knots is determined automatically to secure a certain degree of smoothness and goodness of fit; at the same time, the minimiser β (denoted as $\beta^*(\lambda)$) is derived once the value of λ is set.

In order to set λ, we have to refer to the shape of the yield curve and the size of residual terms, which is inevitably a subjective operation. GCV works to choose λ in a more objective way. Once the "tuning parameter" (θ) is set by discretion, GCV selects λ under a constant criterion for each estimation time.

We choose the value of λ as the minimiser of the GCV value (γ):

$$\min_\lambda \gamma(\lambda) = \frac{\left(P - \Pi(\beta^*(\lambda))\right)' \left(P - \Pi(\beta^*(\lambda))\right)}{\left(n - \theta tr(A(\lambda))\right)^2} \tag{5}$$

where $A(\lambda) \equiv X(\beta^*(\lambda)) \left(X(\beta^*(\lambda))' X(\beta^*(\lambda)) + \lambda H\right)^{-1} X(\beta^*(\lambda))'$

(= measure of the effective number of parameters)

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[3] (′) denotes the transpose.

$$X(\beta^*(\lambda)) \equiv \left.\frac{\partial \Pi(\beta)}{\partial \beta'}\right|_{\beta=\beta^*(\lambda)} \quad \text{and} \quad H \equiv \int_0^T \phi''(t)'\phi''(t)dt$$

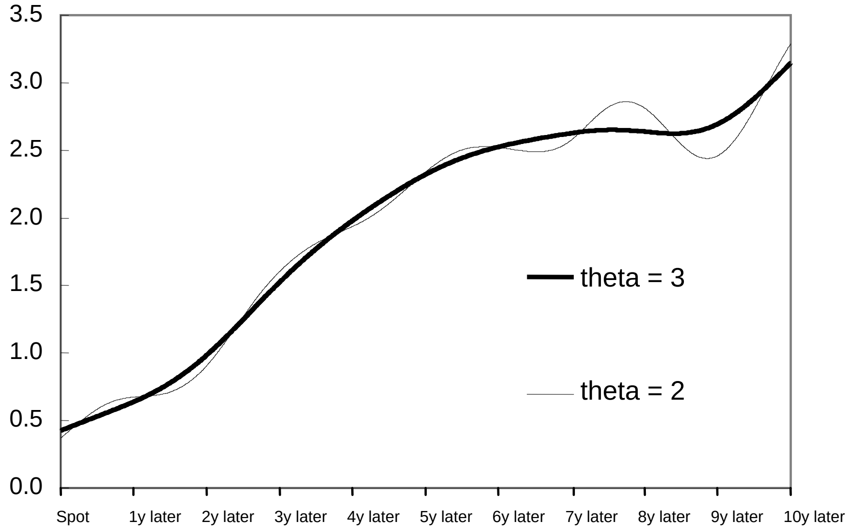
As θ gets bigger, the forward rate curves appear smoother at the expense of goodness of fit. In the original paper, θ is set to be 2 for US data. However, we found it is not always enough for the data set we examined.

Graph 1 shows instantaneous forward rate curves as of 24 December 1997 estimated using the program written by Fisher and Zervos (1996).[4] With $\theta = 2$ the estimated forward rate curve (solid curve) seems too rough, especially between five and 10 years, while with $\theta = 3$ it looks more reasonable (thick curve).[5] After examining samples between April 1997 and January 1998, we decided to set θ (at least provisionally) to be 3.

Graph 1

Estimated instantaneous forward rates with different tuning parameters

As of 24 December 1997; in percentages



Market conventions with respect to Japanese government securities

Some market conventions peculiar to Japan make the estimation procedure and the interpretation of output complicated. We adjusted the original program (see subsections 1 and 2 below).

1. Initial and final coupon payments

Coupon payments are semiannual for JGBs. The actual number of days to the next coupon payment varies, for the two reasons stated below. However, the amount of each coupon payment is fixed at half the coupon rate regardless of the actual number of days, except for initial and final payments.

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4 The estimation procedure described in this paper is implemented using this program with some adjustments (described in the next section). It runs on *Mathematica* version 2.2.

5 In either case, we obtain a humped region between five and 10 years, which implies that JGB prices with a remaining maturity of seven to 10 years are somewhat overvalued. This phenomenon might be attributable to the fact that these JGBs are eligible for settlement as JGB futures.

- Coupon payment and redemption dates for JGBs are basically the 20th of the month, becoming the nearest following business day if the 20th is not a business day.[6] Only the final payment takes account of such a shift in the redemption date.
- 10-yr JGBs are currently issued every month but redemption months are grouped by the rule[7] shown in Table 2. The month of initial coupon payment is determined accordingly. Thus, the number of months from issuance to initial coupon payment may be seven or eight, rather than six, which means an additional payment to the initial payment. In practice, the initial coupon payment is calculated based on the number of days between the issuance day and the 20th of the initial coupon payment month (including the issuance day).

Table 2

Issuance month, redemption month and initial coupon payment month for 10-yr JGBs

Issuance month	**Redemption month**	**Month for initial coupon payment**
Apr, May, Jun	Jun	Dec
Jul, Aug, Sep	Sep	Mar
Oct, Nov, Dec	Dec	Jun
Jan, Feb, Mar	Mar	Sep

Note: This rule has been in effect since 1987.

Similarly, there is a rule, as shown in Table 3, regarding 20-yr JGBs; initial and final coupon payments are treated accordingly.

Table 3

Issuance month, redemption month and initial coupon payment month for 20-yr JGBs

Issuance month	**Redemption month**	**Month for initial coupon payment**
Apr	Sep	Sep
Jul	Sep	Mar
Oct	Mar	Mar
Jan	Mar	Sep

Note: This rule has been in effect since 1996.

6 The date of issuance varies irregularly.

7 This rule is not official, but de facto.

2. Accrued interest

The price data do not include accrued interest, which is calculated using the following formula[8] and added to the price.

Accrued interest = N * coupon rate / 365, where N is the number of days from the last coupon payment to settlement.

3. Business days from the quote day to the settlement day

For both JGBs and TBs, the number of business days from the quote day to the settlement day is now three. Our estimation procedure treats the settlement day as if it were the quote day. Thus, for example, estimated zero coupon rates are, strictly speaking, forward rates whose delivery day is three business days after the quote day. However, since this interval (three business days) is quite short and an appropriate short-term risk-free rate does not exist in the Japanese market, we decided to report the estimated rates as of the quote day without any adjustment.

References

Fisher, M, D Nychka and D Zervos (1995): "Fitting the term structure of interest rates with smoothing splines", Board of Governors of the Federal Reserve System, *Working Paper*, no 95-1.

Fisher, M and D Zervos (1996): "Yield curves", in *Computational economics and finance*, H R Varian (ed), Springer-Verlag New York, Inc, Chapter 12.

Muranaga, J (1997): *Generating yield curves of Japanese government bonds using spline functions* (in Japanese), Institute for Monetary and Economic Studies, Bank of Japan, mimeo.

8 Because of the special treatment for initial payment, N in the formula is replaced with the number of days from issuance to settlement (including the issuance day) for the first payment.