Technical note on the estimation of forward and zero coupon yield curves as applied to Italian euromarket rates

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1. Estimation of the nominal yield curve: data and methodology

The nominal yield curve is estimated from Libor and swap rates, with maturity dates of one to 12 months for Libor rates and two to 10 years for swap yields, downloaded daily from Reuters. Rates are quotes in the London market provided by the British Bankers' Association and Intercapital Brokers respectively.[1] The underlying assumption is that the price (par value) of these securities equals the present values of their future cash flows (ie coupon payments and final redemption payment at maturity).

At the Bank of Italy, we have a fairly long tradition of estimating zero coupon rate yield curves and have experimented with several methodologies and models. In the middle of the 1980s, we started zero coupon yield curve estimation by using the CIR (1985) one-factor model for the short rate, estimated on a cross section of government bond prices (Barone and Cesari (1986)[2]); before that, a cubic splines interpolation was in place as a routine device to gauge the term structure of interest rates. The CIR model application was later updated (Barone et al (1989)) and then the CIR model extended to a two-factor model for the short rate (Majnoni (1993)), along the lines of Longstaff and Schwartz (1992). Drudi and Violi (1997) have tried to efficiently combine cross-section and time series information in estimating parameters for a two-factor model of the term structure, in which a stochastic central tendency rate is introduced as a second factor determining the shape of the yield curve.

More recently, we have been considering the Nelson-Siegel approach, as a viable alternative to the general equilibrium model-based yield curve estimation, because of its relatively low implementation and running cost in building a forward yield curve on a daily basis.

2. Functional specification of the discount function: Nelson-Siegel vs Svensson approach

Forward rates and yield to maturity are estimated using the methodology suggested in Nelson and Siegel (1987), subsequently extended in Svensson (1994). The modelling strategy is based on the following functional form for the discount function:

$d(\tau) = \exp(-y(\tau)\tau)$

with

$$y(\tau) \equiv \beta_0 + \beta_1 \left[\frac{1-\exp\left(-\frac{\tau}{\tau_1}\right)}{\left(\frac{\tau}{\tau_1}\right)} \right] + \beta_2 \left[\frac{1-\exp\left(-\frac{\tau}{\tau_1}\right)}{\left(\frac{\tau}{\tau_1}\right)} - \exp\left(-\frac{\tau}{\tau_1}\right) \right] + \beta_3 \left[\frac{1-\exp\left(-\frac{\tau}{\tau_2}\right)}{\left(\frac{\tau}{\tau_1}\right)} - \exp\left(-\frac{\tau}{\tau_2}\right) \right] \tag{1}$$

where τ represents time to maturity, $y(\tau)$ the yield to maturity and vectors (β_0, β_1, β_2, β_3, τ_1, τ_2) the parameters to be estimated, with (β_0, τ_1, τ_2) > 0.

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1 Reuters RIC pages: FRBD/H and ICAQ/T respectively.

2 See also Barone et al (1991).

The spot yield function, $y(\tau)$, and forward rate function, $f(\tau)$, are related by the equation:

$$y(\tau) = \int_0^\tau \frac{f(s)}{\tau} ds \tag{2}$$

Replacing (1) into (2) and differentiating, one obtains the closed-form expression for the forward yield curve:

$$f(\tau) = \beta_0 + \left(\beta_1 + \beta_2 \frac{\tau}{\tau_1}\right)\left[\exp\left(-\frac{\tau}{\tau_1}\right)\right] + \beta_3 \frac{\tau}{\tau_2}\left[\exp\left(-\frac{\tau}{\tau_2}\right)\right] \tag{3}$$

where β_0 represents the (instantaneous) asymptotic rate and $(\beta_0 + \beta_1)$ the instantaneous spot rate. Restricting β_3 equal to zero in (3), one obtains the Nelson-Siegel (1987) forward rate function. This function is consistent with a forward rate process fulfilling a second-order differential equation with two identical roots. Such a restriction limits to only one local minimum (or maximum) the maturity profile, according to the sign of β_2. When β_3 differs from zero, eg Svensson extension, more than one local maximum or minimum is allowed, hence increasing flexibility in fitting the yield curves.

Estimation requires prior specification of a price, P_i, for the i-th security, obtained by discounting the cash flow profile, $\{C^j\}_i$, for a given time to maturity, $\{\tau^i\}$. This is carried out on a daily sample of n securities whose price is modelled as the sum of their discounted cash flows:

$$P_i(b) = \sum_{j=1}^{k_i} C_i^j d(\tau_i^j; b)$$

$$b \equiv (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2) \tag{4}$$

$$\forall i = 1, \ldots, n$$

where k_i stands for the time to maturity for the i-th security.

The econometric implementation leads to the introduction of a pricing error process, ε_i:

$$P_i^* = P_i(b) + \varepsilon_i$$

$$\forall i = 1, \ldots, n \tag{5}$$

where P^* indicates the market price of the security and ε_i is assumed to be a white noise process. The objective function minimises the squared deviation between the actual and the theoretical price, weighted by a value related to the inverse of its duration, Φ_i:

$$\underset{b}{MIN} \sum_{i=1}^{n} \varepsilon_i^2 \Phi_i^*$$

$$\Phi_i^* \equiv 1/\Phi_i \Big/ \sum_{i=1}^{n} 1/\Phi_i \tag{6}$$

$$\Phi_i \equiv \frac{\partial P_i^*}{\partial y} \frac{1+y}{P_i^*}$$

Criterion (6) is implemented by means of a non-linear least squares algorithm (TSP command LSQ) to derive the parameters' estimates. The Nelson-Siegel parsimonious parametrisation has been preferred to Svensson's extended version for practical reasons. Often, the Svensson extension seems to be less robust at the shortest end of the yield curve. In our experience, the Svensson approach offers little, if any, practical advantage in improving the precision of the estimates, in the terms of both pricing errors and information criteria (for instance, Akaike or Schwarz-Bayes). With the Nelson-Siegel specification, simulated yield curves normally show average pricing errors of some 4-5 basis points, equivalent to 1-2 basis points in terms of yield to maturity. Parameter significance tests, with the covariance matrix corrected for heteroskedasticity, are almost always passed. In comparing daily pricing errors over time across maturities, we have found some evidence of autocorrelated residuals, pointing to regression residuals which are not always "white". In addition, larger, duration-adjusted, pricing errors often seem to show up more often at the shorter end of the curve; the Svensson extension does not provide a remedy for these latter shortcomings.

References

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