A simplified credit risk model for supervisory purposes in emerging markets[1]

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1. Introduction[3]

Currently, the mainstream methodologies that are most widely used to measure credit risk can be divided into two broad categories: mark to market models and default models. The differences between these paradigms rest first on the scope of the losses considered. Whereas in default models an obligor can be in only one of two states, default and non-default, so that losses are exclusively those resulting from debtor defaults, mark to market models also consider losses resulting from a change of value of the loans due to credit quality migration. Further differences arise from the functional forms assumed for the underlying probability distributions, and the way in which these are related to obtain the loan portfolio's loss distribution. For example,[4] in CreditMetrics™, which is a mark to market methodology, the key component is the transition matrix related to a rating system, which provides the probabilistic mechanism that models the quality migration of loans. This determines the losses due to obligor defaults, and the changes in the market value of the loans in the portfolio due to quality migration through a Monte Carlo simulation process, to finally obtain the loss distribution for the portfolio. Whereas the transition matrix, the changes of value, the loss-given-default of the loans, and the migration covariances are theoretically estimated from statistical data and market information, the simulation process relies heavily on a normality assumption around the transition probabilities and Merton's[5] asset value model to establish a relation between credit quality and asset value of the debtor firms, and to determine the joint migration behaviour of the loans in the portfolio.

KMV's[6] methodology is also based on Merton's model[7] and defines a *distance to default*, which is the difference between the value of a company's assets and a certain liability threshold, such that if this quantity is negative, the company is bankrupt and will therefore default on its obligations. For standardisation purposes, this distance to default is measured as a multiple of the standard deviation of the value of the firm's assets. KMV has accumulated a large database, which it uses to estimate default probabilities and correlations, as well as the loss distributions due to debtor default and quality migration. For a specific company, this probability is approximated by the *expected default frequencies*, ie the ratio of the number of companies with the same distance to default that actually defaulted to the total number of companies with the same distance to default in the database. Being a mark to market methodology, it differs significantly from CreditMetrics™ in that it relies on EDFs for each debtor rather than average transition rates as estimated from the historical data produced by the rating agencies. There are also considerable differences in the assumptions and the functional forms utilised.

CreditRisk$^+$ is a default model[8] in which the cornerstone of the methodology is the set of individual default probabilities of the loans in the portfolio. A basic assumption is that the default probabilities are

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1 An earlier version of this model was published in English in Economia, Societa' e Istituzioni. See Márquez (2002). The model presented here is an updated version with significant differences compared with the original and several new results.

2 Bank of Mexico. The views expressed are those of the author and do not necessarily reflect those of the Bank of Mexico.

3 A good detailed review of the different approaches is presented by Crouhy et al (2000).

4 CreditMetrics™ is a spin-off from the JP Morgan Risk Management systems development group.

5 The reader unfamiliar with the methodology is referred to Section 8 of the CreditMetrics™ technical document and Merton (1974).

6 This is the proprietary methodology of KMV corporation.

7 See Kealhofer (1998, 1999).

8 CreditRisk$^+$ is marketed by Credit Suisse Financial Products.

always small, so that the number of defaults in the portfolio can be approximated according to a Poisson probability distribution. In its more general version, where default probabilities can change over time, it is further assumed that these probabilities are entirely driven by a weighted sum of *K risk factors*, each distributed according to an independent Gamma distribution. The weights of the risk factors differ depending on the individual rating of the obligor and, conditional on these risk factors, individual obligor defaults are assumed to be independent Bernoulli trials. In the general case, default correlation is *implicit* in the covariation behaviour of the risk factors, and *the Poisson assumption* leads to a negative binomial for the distribution of the number of defaults. Having obtained the distribution of the number of defaults in the portfolio, proceeding in the typical actuarial fashion by selecting a unit of loss and given the recovery rates for the individual loans, these are then grouped into *buckets of equal loss-given-default*, and the *probability generating function* of the loss distribution is obtained. From here it is necessary to resort to a numerical recursion procedure to obtain the loss distribution.

Another popular default methodology is Credit Portfolio View,[9] which is a discrete multiperiod model. Apart from the fact that it is conceived from the beginning as a dynamic model, the highlight of the methodology is the determination of default probabilities, which are *logit functions* of indices of macroeconomic variables. The portfolio is segmented according to geographical location and economic activity of the debtors, and the indices for each segment are linear functions of the associated macroeconomic variables for the segment. In turn, each macroeconomic variable is assumed to obey a second-order univariate, autoregressive process, and due to cross-correlations in the error terms of the linear models for the indices and the autoregressive expressions of the underlying macroeconomic variables, the parameters of both are estimated simultaneously from a system of equations. Credit Portfolio View also resorts to simulation on transition matrices to obtain the loss distribution.

All of the above methodologies have contributed greatly to the understanding of the key issues in credit risk modelling and it is now accepted that all models are converging to produce comparable results. Research by Finger (1998), Crouhy et al (2000) and Gordy (2000) discusses how under certain parametric equivalents the mainstream methodologies such as CreditMetrics™ and CreditRisk+ can be mapped into each other. It is important to note that the emphasis in all of these methodologies is on producing a distribution of losses which is as realistic as possible. Although one can hardly argue against this principle, the computational effort required can be impractical for certain users, such as regulators, who have to oversee the whole financial system and not just one individual bank. Furthermore, the development of management tools such as simple rules for establishing capital adequacy, identifying segments of excessive credit risk concentration and setting single obligor limits to loans that are explicitly related to the risk profile of the portfolio is not directly addressed.

The model presented here assumes that the default probabilities of the loans and their covariances are given. From here, a default model is developed which obtains an explicit functional form for the loss distribution, assuming that it can be characterised by two parameters: the mean and the variance. Given a specific mean-variance distribution of losses, not necessarily normal, it is possible to obtain the value-at-risk (VaR) for the portfolio as the expected loss plus a certain multiple of the standard deviation of losses. This leads to a lower bound on the bank's capitalisation ratio and the resulting inequality establishes capital adequacy. The model is developed in a way which explicitly measures the concentration of the loan portfolio. We can see that the *Herfindahl-Hirschman index* emerges naturally as a measure of concentration, providing a precise quantification of how concentration contributes to the overall credit risk of the portfolio. Two new properties of the index are obtained that relate single obligor limits to concentration along different segments of the portfolio so as to ensure capital adequacy. Furthermore, the research shows how correlation affects concentration and this leads to the definition of a *risk concentration measure*. Finally, it is shown that the model can be implemented with limited information on the actual composition of bank loan portfolios, which is a crucial factor for regulators inasmuch as their capacity to obtain up to date and timely information from banks is limited.

Examples of numerical exercises performed to date on real loan portfolios are shown, and are seen to provide results comparable to those obtained using other methodologies, at a considerable reduction in computational effort. Finally, since all the relevant elements for measuring default credit risk are

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9 This product is offered by McKinsey, the consulting firm. The classic reference is Wilson (1997a,b).

explicitly parameterised, the shortcomings of available information can be compensated by a judicious use of assumptions on the values of the relevant parameters. The computational efficiency of the model results in rapid feedback on the implications and sensitivity of the risk profile of a loan portfolio to changes in the parameters.[10] Since the measurement of concentration is at the heart of the model, we begin with a discussion of this topic.

2. The concentration issue

Loan concentration has long been identified as an important source of risk for banks and loan portfolios. Judging from current technical literature on credit risk, as far as concentration goes the establishment of a generally accepted paradigm has remained elusive in spite of the importance of the problem.[11] The more formal approaches, which look to portfolio theory,[12] have been mainly concerned with optimal diversification of portfolios of traded fixed income assets where information compatible with traditional Markowitz (1959) type models can be obtained in a cost-effective manner. It must be pointed out however, that traditional portfolio theory approaches deal with the concentration issue indirectly, since the preoccupation is the allocation of assets through the well known mean-variance trade-off, but a clear measure of concentration and its relation to risk has never been made explicit. Kealhofer (1998) has an interesting discussion of the issue from the point of view of diversification. First he states that "there has been no method for actually measuring the amount of diversification in a debt portfolio", and that "ex ante, no method has existed which could quantify concentrations"; concentrations have only been detected ex post. He then argues that "measuring the diversification of a portfolio means specifying the range and likelihood of possible losses associated with the portfolio". He goes on to provide a definition that allows the comparison of diversification of two portfolios as:

> "*Portfolio A is better diversified than portfolio B if the probability of loss exceeding a given percent is smaller for A than for B, and both portfolios have the same expected loss*".

Thus, when dealing with portfolios of traditional bank loans, no formal methodology for measuring concentration seems to have emerged. As pointed out by Altman and Saunders (1998), the concentration measurement issue has mainly been dealt with through subjective analysis. Typically, banks and other agents apply a scoring technique based on the opinion of a group of experts about the degree of concentration observed along and across different segments of a portfolio, as regards some classification criterion, in order to obtain an indicator of loan concentration. Generally, the number obtained is of more value in cardinal or hierarchical terms than it is as a direct measure of risk that can quickly be translated into potential losses or value-at-risk.[13]

The approach adopted in the following analysis does not solve all the aforementioned problems, but it does provide a theoretical framework that might allow, ex ante, the detection of risk concentration. The proposed risk concentration measure is consistent with Kealhofer's notion as previously stated. Example 6.2 illustrates how the risk concentration measure can be used to detect the more risky segments of a loan portfolio.

3. Value-at-risk, concentration and the "single obligor limit": the simplest case

Traditionally, banks deal with concentration risk by placing a limit on the maximum amount that can be loaned to a single debtor, along the different dimensions where concentration can occur, ie industry, geographical region, product, country, etc. Normally, the "single obligor limit" is expressed as a

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[10] Due to the closed form expression for value-at-risk, it is also possible to perform analytical exercises.

[11] See Caouette et al (1998), Chapters 17 and 18. See also Kealhofer (1998).

[12] See, for example, Bennet (1984).

[13] See, for example, Moody's Investor Services (1991) and the Coopers and Lybrand (1993) report.

proportion δ of the capital K of the bank. However, when discussing loan concentration, one normally addresses the issue of *how much of the total loans outstanding is concentrated in an individual or group*. Thus, whatever the virtues of setting limits as a percentage of capital, this does not give much information as to the actual concentration of loans in the portfolio. To see this, note that, at least theoretically, a bank could have only one loan that respects the limit but have a totally concentrated portfolio. On the other hand, the bank can have a million uncorrelated loans of exactly the same size, in which case the portfolio would be completely diversified, regardless of whether each loan respects the limit or not. Thus, one can have highly concentrated portfolios as well as highly diversified portfolios that respect the constraint in terms of capital.[14] We will therefore part with tradition, since for the purpose at hand it is better to think of concentration in terms of proportions of the total value of the loan portfolio, and fix limits accordingly. Throughout this paper, individual limits on loans will be expressed as proportions θ of the total value of the loan portfolio V. Furthermore, no generality is lost since δ *and* θ *are linearly related*, so the results are not altered. To see this, let f_k denote the value of the kth of N loans, and analyse the *single obligor limit* as represented by the following constraint:

$$f_k \leq \delta K = \delta \frac{K}{V} \cdot V = \delta\psi V = \theta V; \quad k = 0,1,2,3,\ldots,N \tag{3.1}$$

where $\psi = \frac{K}{V}$ is *the capitalisation ratio*. Thus, $\theta = \delta\psi$,[15] and the single obligor limit will be expressed as:

$$f_k \leq \theta V \quad k = 1,2,\ldots,N$$

If all loans have the same default probability p, and assuming independence, one can define N binary random loss variables x_i as:

$$x_i = \begin{cases} f_i & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Clearly $E(x_i) = pf_i$ and $Variance(x_i) = p(1-p)f_i^2$. Since the variables are independent:

a. $$\mu = E\left(\sum_{i=1}^{N} x_i\right) = \sum_{i=1}^{N} pf_i = pV; \quad \text{where } V = \sum_{i=1}^{N} f_i$$

b. $$\sigma^2 = Variance\left(\sum_{i=1}^{N} x_i\right) = \sum_{i=1}^{N} Variance\,(x_i) = p\,(1-p)\sum_{i=1}^{N} f_i^2$$

Since the distribution of loans (f_i) is totally arbitrary, it is difficult to know the exact distribution of $\sum_{i=1}^{N} x_i$. For the moment, assume that it can be approximated by the normal distribution,[16] so that:

$$VAR_\alpha = \mu + z_\alpha \sigma = pV + z_\alpha \sqrt{p(1-p)\sum_{i=1}^{N} f_i^2} \tag{3.2}$$

If $VAR_\alpha \leq K$, after a little algebra one arrives at the following expression:

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14 For example, if loans are constrained not to exceed 12% of capital, this can be done with only one loan in the portfolio, in which case concentration is maximum. On the other hand, if the portfolio has a thousand loans all representing 12% of capital, it would be a highly diversified portfolio.

15 Note that if there is only one loan in the portfolio, then it is necessarily true that $f_i = V$ so that $\psi\delta = \theta = 1$, which in turn implies that the portfolio is totally concentrated in one loan.

16 See, for example, DeGroot (1988, p 263).

$$H(F)=\frac{\sum_{i=1}^{N} f_i^2}{\left(\sum_{i=1}^{N} f_i\right)^2} \leq \frac{(\psi-p)^2}{z_\alpha^2 p(1-p)} = \Theta(p,\psi,\alpha) \tag{3.3}$$

In this expression, portfolio concentration is measured by:

$$Concentration = H(F) = \frac{\sum_{i=1}^{N} f_i^2}{\left(\sum_{i=1}^{N} f_i\right)^2}$$

Readers familiar with the literature of industrial organisation will have recognised that the above measure is the *Herfindahl-Hirschman* concentration index.[17]

4. Analysis of the capital adequacy inequality

The first observation is that, with the obvious limitations, it seems that portfolio concentration risk can be managed using a very general measure of concentration other than the single obligor limit. Next, it is interesting to note that capital adequacy as represented by the capitalisation ratio ψ requires that

$$\psi \geq p + z_\alpha \sqrt{p(1-p)H(F)} \tag{4.1}$$

This inequality relates capital adequacy to the probability of default, the confidence level used for value-at-risk, and the concentration index. It also shows that there is a direct relation between the Herfindahl index and the variance of losses. Since the index takes on values between the reciprocal of the number of loans N and one, where high concentration is present the variance of losses will vary between $\sqrt{p(1-p)/N}$ and $\sqrt{p(1-p)}$, depending on $H(F)$. Furthermore, note that the role played by $H(F)$ in the above is totally consistent with Kealhofer's definition of concentration since it is obvious from (4.1) that the lower the value of $H(F)$, the lower the probability of loss exceeding a specified level, for the same expected loss.

In what follows, we can see that everything behaves as it should. The following theorem summarises the main implications for risk managers of the previous analysis. These results are introduced early because they remain basically unchanged throughout all future generalisations.

Theorem 4.1

The bound $\Theta(p,\psi,\alpha)$ on the concentration measure has the following properties:

$\Theta(p,\psi,\alpha)$ *varies in direct proportion to the capitalisation ratio* ψ *and inversely to the default probability* p *and the value-at-risk confidence level* z_α.

If the concentration measure exceeds the bound (ie $H(F) > \Theta(p,\psi,\alpha)$), then the capital of the bank is at risk for the given confidence level.

If the default probability p exceeds the capitalisation ratio ψ, then the capital of the bank is at risk for any confidence level, regardless of the concentration of the loan portfolio.

If $\Theta(p,\psi,\alpha) > 1$, no degree of concentration of the loan portfolio places the capital of the bank at risk.

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17 See, for example, Shy (1995) or Tirole (1995).

Proof

Point one is obvious from the form of $\Theta(p,\psi,\alpha$. The second point is easily verified, ie: if $H(F) > \Theta(p,\psi,\alpha)$ then,

$$VAR_\alpha = \left(p + z_\alpha\sqrt{\theta(F)pq}\right)V > \left(p + z_\alpha\sqrt{\Theta pq}\right)V = \left(p + \frac{z_\alpha\sqrt{pq}(\psi - p)}{z_\alpha\sqrt{pq}}\right)V = K$$

Point three follows directly from 4.1:

$$VAR_\alpha \leq K \Leftrightarrow \psi \geq p + z_\alpha\sqrt{p(1-p)H(F)}$$

Point three is also verified easily. If $p > \psi$, then 4.1 is violated:

$$VAR_\alpha = \left(p + z_\alpha\sqrt{H(F)pq}\right)V > \left(\psi + z_\alpha\sqrt{H(F)pq}\right)V = K + z_\alpha V\sqrt{H(F)pq} > K$$

As for point four, it is well known that $H(F) \leq 1$ for any arbitrary F.[18]

Capital adequacy Theorem 4.1 provides some useful rules for the risk manager and for the regulator. First, one can determine capital adequacy because one obtains precise measures of the adjustments in the capitalisation ratio required by variations in the default rates and/or the concentration of the loan portfolio. Furthermore, depending on the amount of control that banks have on the default ratio and loan concentration, adjustments in the default probability and the concentration of the loan portfolio necessary to maintain capital adequacy can also be calculated. Thus, if the concentration of the loan portfolio exceeds the bound at the desired confidence level, inequality (3.2) provides a convenient means of fine-tuning the adjustments required in ψ, p and $H(F)$ so that credit risk does not place the capital of the bank in jeopardy. Also interesting is that if the default rate of the portfolio exceeds the capitalisation ratio, the risk manager and the financial authorities are alerted that the banks' capital is at risk regardless of the concentration of the loan portfolio and the confidence level adopted.

5. A closer look at the Herfindahl index

One of the main features of the approach taken is that a measure of loan concentration as it relates to risk arises naturally. The Herfindahl-Hirschman index (HHI) has been extensively studied in relation to industrial concentration, and it is known to have several important properties. Thus, it is known that the index takes values between the reciprocal of N and one,[19] and that it behaves well in terms of "*the five properties of inequality measures*".[20] We now investigate how the HHI relates to the intuitive notion that concentration is related to the minimum number of obligors where credit is more concentrated. A better understanding of the relation between the single obligor limit and the concentration index has important risk management and regulatory implications.

In order to examine how concentration relates to the notion that more credit in fewer hands means more concentration, it must be consistent with the notion that maximum concentration occurs when all credit is held by a single obligor and the minimum is when all debtors owe the same amount. Formally:

a. The maximum concentration occurs when, for some i, one has that:

$$f_j = \begin{cases} V & \text{for } j = i \\ 0 & \text{for } j \neq i; \quad j = 1,2,\ldots,N \end{cases}$$

[18] See Encaoua and Jacquemin (1980).

[19] A simple normalisation is possible, from which we can easily see that $\phi(F)$ as defined below satisfies $0 \leq \phi \leq 1$.

$$\phi(F) = \frac{N - \frac{1}{H(F)}}{N - 1}$$

[20] See Cowell (1995) and Encaoua and Jacquemin (1980).

ie $F^{\max} = Ve^i$, where $e^i \in E^N$ is the ith unit vector.

b. The minimum concentration occurs when $f_i = \frac{V}{N}$ for $i = 1,2,\ldots,N$

Concentration has to do with numbers, and the HHI has several interesting *numbers-related properties*. The best known is *Adelman's* "*numbers-equivalent*",[21] which for loan concentration states that its inverse can be interpreted as "*the minimum number of loans of equal size that would result in a specific value of the index*". It is now shown that *the value of the index is maximised under the single obligor limit, when all credit is concentrated in the minimum number of obligors, and each obligor holds credit up to the limit.* The theorem establishes the relation between the single obligor limit and the Herfindahl-Hirschman measure of concentration, and in so doing, it shows that Adelman's numbers-equivalent is in fact the maximum concentration possible, when loans are constrained by a certain limit.[22] In what follows, we let F denote the vector of loans $f_k \geq 0$ for $k = 1,2,\ldots,N$. Without loss of generality, we also assume that the elements of this vector have been sorted in decreasing order:

$f_1 \geq f_2 \geq \ldots \geq f_N$

We can also assume that $V = \sum_{k=1}^{N} f_k = 1$. The following proposition is an important basic property of the index.

Proposition 5.1

Assume $F = (f_k)$ is such that $f_i \geq f_{i+1} \geq 0$ for $i = 1,2,3,\ldots,N-1$ and $\sum_{i=1}^{N} f_i = 1$. Then:

a. For f_i, f_j such that $1 \leq i \leq j$; $f_j > 0$ and $\varepsilon > 0$ such that $f_j - \varepsilon > 0$ define the vector $F' = (f_k')$ to be

$$f_k' = \begin{cases} f_k; & k = 1,2,\ldots,N; \quad k \neq i,j \\ f_i + \varepsilon; & k = i \\ f_j - \varepsilon; & k = j \end{cases}$$

then $H(F') > H(F)$.

b. If $f_i > f_j$ and $0 < \varepsilon \leq f_i - f_j$, then the vector $F'' = (f_k'')$ defined as:

$$f_k'' = \begin{cases} f_k; & k = 1,2,\ldots,N; \quad k \neq i,j \\ f_i - \varepsilon; & k = i \\ f_j + \varepsilon; & k = j \end{cases}$$

has the property $H(F'') < H(F)$.

Proof

To prove (a), simply note that:

$$H(F') - H(F) = \sum_{k=1}^{N} \left[(f_k')^2 - f_k^2 \right] = 2\varepsilon \left[(f_i - f_j) + \varepsilon \right] > 0$$

The Proof of (b) is similar: $H(F'') - H(F) = 2\varepsilon \lfloor (f_i - f_j) - \varepsilon \rfloor \leq 0$ since $\varepsilon \leq (f_i - f_j)$

(note that $\varepsilon > f_i - f_j$ implies case (a)).

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21 See Adelman (1969) and Kelly Jr (1981). For other interesting numbers-related properties, see Weinstock (1984).

22 Although the result conforms to intuition, no formal proof has been detected by the authors in the more frequent references, such as Sleuwaegen et al (1989), Weinstock or Encaoua and Jaquemin op cit.

The proposition states that if some element f_k is increased at the expense of decreasing a smaller element f_j, the concentration index will increase. If on the other hand, an element is increased at the expense of a larger element, then the concentration index will decrease. To continue with the analysis, it is now shown that if all credit is concentrated in the minimum number of debtors, while subject to the constraint $f_k \le \theta V$, then $H(F) \le \theta$.

Proposition 5.2

Let $\theta \in (0, 1)$ and $n = \left\lfloor \frac{1}{\theta} \right\rfloor$ be the integer part of $\frac{1}{\theta}$. Let $\varepsilon \in [0, 1)$ be such that $\theta = \frac{1-\varepsilon}{n}$. Then, for the distribution,

$$f_k = \begin{cases} \theta; & k = 1, 2, \ldots, n \\ \varepsilon; & k = n+1 \\ 0 & else \end{cases}$$

we have that $H(F) \le \theta$.

Proof

Note that $\sum f_k = n\theta + \varepsilon = 1$ and therefore:

$H(F) = n\theta^2 + \varepsilon^2 = n\theta^2 + (1 - n\theta)^2 = n(n + 1)\theta^2 - 2n\theta + 1$

For $H(F) = \theta$ one must solve the quadratic equation,

$$(n + 1)n\theta^2 - 2n\theta + 1 = \theta \quad \text{ie} \quad n(n + 1)\theta^2 - (1 + 2n)\theta + 1 = 0 \tag{5.1}$$

It is simple to verify that (5.1) has the following two solutions:

$\theta_1 = \frac{1}{n}$ and $\theta_2 = \frac{1}{n+1}$

This means that if θ^{-1} is an integer, then $H(F) = \theta$. Thus, examine what happens in the interval $\left(\frac{1}{n+1}, \frac{1}{n} \right)$. To do this, let

$$\theta(\lambda) = \lambda \left(\frac{1}{n} \right) + (1-\lambda) \frac{1}{n+1} = \frac{n+\lambda}{n(n+1)} \quad \text{with } \lambda \in (0, 1)$$

Substituting $\theta(\lambda)$ in the left-hand side of (5.1), one obtains:

$$n(n+1) \left(\frac{n+\lambda}{n(n+1)} \right)^2 - (1+2n) \frac{n+\lambda}{n(n+1)} + 1 = \frac{1}{n(n+1)} \left\{ n^2 + 2\lambda n + \lambda^2 - n(1+2n) - \lambda(1+2n) + n^2 + n \right\}$$
$$= \frac{\lambda(\lambda - 1)}{n(n+1)} < 0 \quad \forall \lambda \in (0, 1)$$

It is now shown that if all loans respect the single obligor limit $f_k \le \theta V$, then $H(F) \le \theta$ and the distribution of loans of the previous proposition maximises the value of the index under the single obligor constraint.

Theorem 5.3

Let $F = (f_k)$ be such that:

$$f_k = \begin{cases} \theta; & k = 1,2,\ldots,n \\ \varepsilon; & k = n+1 \\ 0; & k = n+2,\ldots,N \end{cases}$$

with θ, $\varepsilon \geq 0$; $\varepsilon < \theta$ and $\sum f_k = 1$. Then F maximises $H(F)$ for all F such that $f_k \leq \theta \ \forall \ k$ and $H(F) \leq \theta$.

Proof[23]

Proposition 5.2 states that $H(F) \leq \theta$ for this distribution. Necessarily, $n = \left[\frac{1}{\theta}\right]$ and $\varepsilon \geq 0$ are such that $\theta = \frac{1-\varepsilon}{n}$ in order to have $\sum f_k = 1$. Furthermore, any vector with $f'_k = \theta + \delta$; $\delta > 0$ would violate the constraint $f_k \leq \theta \ \forall \ k$. Therefore, the only possibility of altering the distribution of loans would be to decrease some element $f_k = \theta$ or $f_{n+1} = \varepsilon$ by some quantity $\delta > 0$. But then proposition 5.1(b) states that $H(F') < H(F) \leq \theta$.

This result has important implications for risk management and regulation since de facto it states that by placing a limit on individual loans as a propotion of the value of the portfolio, one is also placing a limit on concentration as measured by the HHI by the same amount θ. Therefore, it is simple to check for capital adequacy by

$$\theta \leq \frac{(\psi - p)^2}{z_\alpha^2 p(1-p)} = \Theta(p, \psi, \alpha) \tag{5.2}$$

Alternatively, from (4.1), one can obtain the capital adequacy relation in terms of the single obligor limit (2.6), that is:

$$\psi \geq p + z_\alpha \sqrt{p(1-p)\theta} \tag{5.3}$$

Thus, (5.2) provides a very simple means to check for capital adequacy, without performing complicated calculations. Although crude, simply take θ to be the ratio of the largest loan to the total value of the loan portfolio and the observed default rate as an ex post proxy of default probability and substitute these values in the right-hand side of (5.1). Since Theorem (5.1) guarantees $H(F) \leq \theta$, if the inequality holds it is a good sign that the bank is adequately capitalised.

It should be realised however, that this condition is sufficient but not necessary. As will be shown in the following theorem, if one chooses to explicitly constrain the portfolio to satisfy $H(F) \leq \theta$, it is possible to have specific loans that as a proportion of the total value of the portfolio represent a quantity larger than θ. Intuitively, granting a very large loan while satisfying the constraint on the index is only possible at the expense of the other loans in the portfolio so that in the optimum, the portfolio is composed only of one large loan and all others are small and of equal size.

Theorem 5.4

If $H(F) \leq \theta$ then:

$$f_i \leq \frac{1}{N}\left(1 + \sqrt{(N\theta - 1)(N-1)}\right) < \sqrt{\theta} \quad \text{for } i = 1,2,3,\ldots,N$$

Proof

The idea behind the proof is that under the constraint $H(F) \leq \theta$, a very large loan is only possible at the expense of all the other loans, which must become progressively smaller and of equal size. So, given the constraint $H(F) \leq \theta$, let us maximise the largest element f_1. Suppose $f_1 = a$ is the largest loan possible, then necessarily $f_2 = f_3 = \ldots = f_N = b$; for some $b > 0$; $b < a$. To see this, consider any other distribution with $f_i > f_j$ and $1 < i < j$. Then there exists $\varepsilon > 0$ such that $f'_i = f_i - \varepsilon > f'_j = f_j + \varepsilon > 0$. Proposition 5.1 then states that $H(F') < H(F) \leq \theta$. Now, by continuity of the index on each f_i and

[23] This proof and the one for the next theorem are different from the original proofs in Márquez (2002). They are due to Fausto Membrillo and are more intuitive and elegant than the original.

because of Theorem 5.3, there exists $\varepsilon' > 0$ such that any loan distribution F'' with $f_1'' = f_1 + \varepsilon'$ and $f_j'' = f_j' - \varepsilon' \geq 0$ satisfies $H(F') < H(F'') \leq \theta$, which contradicts the assumption that F is a distribution where f_1 is a maximum. Therefore, if $f_1 = a$, for some $a > 0$, the loan distribution which maximises f_1, subject to the constraint $H(F) \leq \theta$, can be represented as

$$f_k = \begin{cases} a; & k = 1 \\ b; & k = 2, 3, \ldots, N \end{cases}$$

and $a > b$; therefore:

$$H(F) = a^2 + (N - 1)b^2 \leq \theta \tag{5.4}$$

Furthermore $a + (N - 1)b = V$. Solving for b:

$$b = \frac{1-a}{N-1}$$

Substituting b in (5.4) one obtains:

$$a^2 + (N-1)\left(\frac{a-1}{N-1}\right)^2 \leq \theta$$

This leads to the following quadratic equation:

$$Na^2 - 2a + \lfloor 1 - \theta(N - 1) \rfloor \leq 0 \tag{5.5}$$

Equating to zero, the solution of (5.5), yields:

$$a = \frac{1}{N}\left[1 + \sqrt{(N\theta - 1)(N-1)}\right]$$

Note that $a \to \sqrt{\theta}$ when $N \to \infty$, and it is simple to obtain the last inequality:

$$\begin{aligned}
(\theta-1)^2 > 0 \Leftrightarrow \theta^2 - 2\theta + 1 > 0 \Leftrightarrow \theta^2 + 2\theta + 1 > 4\theta \Leftrightarrow (\theta+1)^2 > 4\theta \Leftrightarrow \theta + 1 > 2\sqrt{\theta} \\
\Leftrightarrow -N(\theta+1) < -2\sqrt{\theta}N \Leftrightarrow N^2\theta - N(\theta+1) + 1 < N^2\theta - 2\sqrt{\theta}N + 1 \\
\Leftrightarrow N^2\theta - N\theta - N + 1 < \left(\sqrt{\theta}N - 1\right)^2 \Leftrightarrow (N\theta-1)(N-1) < \left(\sqrt{\theta}N - 1\right)^2 \\
\Leftrightarrow \sqrt{(N\theta-1)(N-1)} < \sqrt{\theta}N - 1 \Leftrightarrow \frac{1}{N}\left(1 + \sqrt{(N\theta-1)(N-1)}\right) < \sqrt{\theta}
\end{aligned}$$

Having a good concentration index is desirable from the regulatory point of view, since it facilitates comparisons of loan concentration between different institutions, and leads to an assessment of concentration risk for the financial system as a whole. For the risk manager of an individual bank, besides measuring his own risk, it provides benchmarks for setting business strategy and goals, and allows comparisons with the competition. The HHI seems particularly well suited for the task, since besides measuring concentration it is directly related to risk, and provides a quick means to check capital adequacy. In the following section it will be seen that the concept is robust under much more general conditions.

5.1 A numerical example

In order to illustrate the results obtained so far, consider the following example taken from the CreditRisk$^+$ manual:

Table 5.1

No of loans	Rating							Total
	A	**B**	**C**	**D**	**E**	**F**	**G**	
1	$4,728	$5,528	$3,138	$5,320	$1,800	$1,933	$358	$22,805
2	$7,728	$5,848	$3,204	$5,765	$5,042	$2,317	$1,090	$30,994
3			$4,831	$20,239	$15,411	$2,411	$2,652	$45,544
4			$4,912			$2,598	$4,929	$12,439
5			$5,435				$6,467	$11,902
6							$6,480	$6,480
Total	$12,456	$11,376	$21,520	$31,324	$22,253	$9,259	$21,976	$130,164

Default probabilities for the loans are taken from the following table:

Table 5.2

Rating	**Default prob**
A	1.65
B	3.00
C	5.00
D	7.50
E	10.00
F	15.00
G	30.00

For this first example let the default probability for the loans be the weighted average of the probabilities of Table 5.2; that is 10.89%. The HHI for the portfolio is 6.61%. Assuming normality and choosing a 5% confidence level, $z_\alpha = 1.96$ and one obtains:

$$\psi \geq p + z_\alpha \sqrt{p(1-p)H(F)} = 0.1089 + 1.96\sqrt{0.1089 \times 0.8911 \times 0.0661} = 0.2658$$

Then the bank's economic capital must be at least:

$VaR_{0.05} = 0.2658 \times V = 0.2658 \times \$130{,}164.00 = \$34{,}602.79$

Suppose economic capital is $35,000, then the capitalisation ratio is:

$$\psi = \frac{K}{V} = \frac{35{,}000}{130{,}164} = 0.2689$$

Since $0.2689 > 0.2658$, the bank exhibits capital adequacy. Now, under 3.2, the maximum concentration that the portfolio can assume is:

$$\frac{(\psi - p)^2}{z_\alpha^2 p(1-p)} = \frac{(0.2689 - 0.1089)^2}{1.96^2 \times 0.1089 \times 0.8911} = 0.0687$$

Since $H(F) = 6.61\%$, the portfolio is not excessively concentrated.

Since the maximum value of the index is 6.87%, no loan in the portfolio should exceed:

$f^* = \sqrt{0.0687} \times V = 0.2621 \times 130{,}164 = 34{,}107.88$

Table 5.1 shows that the largest loan is the $20,239 D-loan, which is smaller than the aforementioned amount. It is interesting to note that the single obligor limit would be violated. According to Theorem 5.2, loans should not exceed:

$f_i \leq 0.0687 \times 130{,}164 = \$8{,}942.27$

There are two loans in the portfolio that are greater than this amount: the $20,239 D-loan and the $15,411 E-loan, confirming that the condition is sufficient but not necessary. Finally, we can see that the largest loan in the portfolio is within the bounds provided by Theorem 5.2, ie $8,942.27 ≤ $20,239 ≤ $34,107.88.

6. Accounting for default correlation and different default probabilities

The results obtained so far rely on the following assumptions:

a. The loss distribution can be characterised by its mean and variance.

b. Default probabilities are homogeneous and independent from each other, for all loans along the dimension where loan concentration can occur.

c. There is only one dimension of possible loan concentration.

d. Nothing is recovered from defaulting loans.

In this section the model is generalised by relaxing the second and third assumptions. We first examine the case where default probabilities can be different and are correlated.

6.1 A general model

Assume that the portfolio loss distribution can be characterised by its mean and its variance and that the vector of default probabilities π and the covariance matrix M are given exogenously. Proceeding along the same lines of the previous analysis, the VaR to capital inequality is now:

$$VAR_\alpha = \pi^T F + z_\alpha \sqrt{F^T M F} \leq K \tag{6.1}$$

Since M is positive definite, it is well known that there exists a matrix Q such that,

$$M = Q \Lambda Q^T \tag{6.2}$$

where Λ is the diagonal matrix of characteristic values of M, and Q is an *orthogonal matrix of the eigenvectors of* M, with the property that $Q^{-1} = Q^T$.[24] Let $S = Q\sqrt{\Lambda}\,Q^T$, where $\sqrt{\Lambda}$ is the diagonal matrix of the square roots of the eigenvalues of M, so that $M = S^T S$. Now change the variable to $G = SF$ so that $F^T M F = G^T G$. This change of variable effectively *rescales* F in terms of the matrix S which in turn is representative of the "*square root*" of the covariance matrix M. It is well known that this is equivalent to rescaling the loans in the portfolio according to the covariances of the default probabilities between the loans, so that loans with higher loss covariances will increase in size, while the opposite will happen to loans with smaller loss covariances. Although much credit in few hands is potentially dangerous, it is even more dangerous when *too much risk is concentrated in a particular group of debtors*, as suggested by the rescaling of the loan portfolio in terms of S. Thus, at a given moment a numerically highly diversified portfolio of small loans that exhibit large variances and are highly correlated may be riskier than a numerically small portfolio of large loans that are uncorrelated and have low default probabilities. In the next section, the discussion is taken a step further.

[24] Any intermediate text on matrix theory can be consulted. See, for example, Strang (1988), or Mirsky (1990).

To continue with the development of the model, multiplying and dividing F^TMF by F^TF, and dividing by $V = 1^TF$, the following capital adequacy relation, relative to the value of the loan portfolio, is obtained:

$$\psi \geq \bar{p} + z_\alpha \sqrt{\frac{F^T MF}{F^T F} H(F)} = \bar{p} + z_\alpha \sigma \sqrt{H(F)} \tag{6.3}$$

where

$$\sigma^2 = \frac{F^T MF}{F^T F} = R(F,M) = \textit{Rayleigh's quotient} \tag{6.4}$$

is a measure of the standard deviation of losses and

$$\bar{p} = \frac{\pi^T F}{V} \tag{6.5}$$

is the expected loss of the portfolio relative to its value which is nothing more than the weighted average of default probabilities. Proceeding in the usual way, and applying Theorem 5.1, one obtains a limit on concentration and single obligor limits as:

$$H(F) \leq \theta \leq \left(\frac{\psi - \bar{p}}{z_\alpha \sigma}\right)^2 \tag{6.6}$$

Note that relations (6.3) and (6.6) have the same structure as those obtained for the simple cases of equal default probabilities and independent loans. In this general case, Rayleigh's quotient measures the variance of losses. One can verify that this reduces to the case of equal default probabilities for all loans and uncorrelated defaults, and that all the results of Theorem 4.1 are still true under this generalisation.

Note that the total variance of losses $\sigma\sqrt{H(F)}$ is decomposed into the variation-covariation effect, represented by σ, and concentration $H(F)$. This emphasises the fact that resizing the loan vector through the covariance matrix M implies that concentration in the number of loans is not necessarily a good measure of risk concentration.

6.2 A measure of risk concentration

In order to investigate how correlation affects concentration and increases risk, consider the special case when all loans have the same default probability p and each pair of loans is similarly correlated through ρ. Then, the covariance of defaults between any two loans (i, j) is:

$$\sigma_{ij} = \sigma_i \sigma_j \rho_{ij} = \sqrt{p_i(1-p_i)}\sqrt{p_j(1-p_j)}\,\rho_{ij} = p(1-p)\rho \quad \forall\ i, j \tag{6.7}$$

In this case the covariance matrix has the following structure:

$$M = p \cdot (1-p) \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix} \tag{6.8}$$

It is convenient to represent this as:

$$M = p(1-p)\{\rho \mathbf{1}\mathbf{1}^T + (1-\rho)I\}, \tag{6.9}$$

"**1**" is the "sum vector" ie $\mathbf{1}^T = (1,1,1,...,1)$ and "I" is the identity matrix.

Thus, the variance of losses of the portfolio is:

$$F^TMF = p(1-p)\{\rho(\mathbf{1}^TF)^2 + (1-\rho)F^TF\}$$

Proceeding in the usual way, and noting that $V = \mathbf{1}^TF$, this leads to a VaR of:

$$VaR = V\left\{p + z_\alpha \sqrt{p \cdot (1-p)}\sqrt{\rho + (1-\rho)H(F)}\right\} \tag{6.10}$$

In this expression, loss variance is decomposed into two distinct elements. The first is the Bernoulli variance $p(1 - p)$, while concentration is captured by:

$$H' = \rho + (1 - \rho)H(F) \tag{6.11}$$

Note that under positive correlation, H' can be interpreted as a convex combination between the HHI of a totally concentrated portfolio ($H(.) = 1$) and the HHI of the portfolio $H(F)$. Clearly, H' increases with ρ and for $\rho = 0$ we have $H' = H(F)$; whereas $H' = 1$ if $\rho = 1$. In other words, if all the loans of a portfolio are perfectly and positively correlated, in terms of risk they behave as a single loan. In general, one can say that the correlated portfolio behaves exactly the same as an uncorrelated portfolio, whose concentration index is H', instead of $H(F)$. Thus, H' could be considered a *correlation-adjusted concentration index*.

Furthermore, (6.11) can be used to compute such an index for any given portfolio by computing p and ρ such that:

$$p(1 - p)\cdot H' = p\cdot(1 - p)\cdot[\rho + (1 - \rho)H(F)] = R(M,F)\cdot H(F) \tag{6.12}$$

Letting $p = \dfrac{\pi^T F}{V}$, solving for ρ gives:

$$\rho = \frac{\left(\dfrac{R(M,F)}{p\cdot(1-p)} - 1\right)}{\left(\dfrac{1}{H(F)} - 1\right)} = \frac{\left[R(M,F) - p\cdot(1-p)\right]H(F)}{p\cdot(1-p)\left[1 - H(F)\right]} \tag{6.13}$$

The expression provides an equivalent correlation measure which summarises how loan defaults are pairwise correlated within the portfolio.

Example 6.1

Consider the loan portfolio of the previous examples. The correlation matrix used in this exercise is as shown in Appendix A, and is segmented into three groups:

$$M = \begin{bmatrix} M_1 & C_{1,2} & C_{1,3} \\ C_{2,1} & M_2 & C_{2,3} \\ C_{3,1} & C_{3,2} & M_3 \end{bmatrix}$$

Assuming normality and a 5% confidence level, VaR is:

$$VaR_{0.05} = \pi^T F + z_{0.05}\sqrt{F^T MF} = 14{,}179 + 1.96(21{,}176) = \$55{,}683$$

From previous examples we know that $p = 0.1089$, $H(F) = 0.0661$, and computation yields:

$$\sigma = \sqrt{\frac{F^T MF}{F^T F}} = \sqrt{0.4006} = 0.6329$$

Thus, capital adequacy requires:

$$\psi > \overline{p} + z_\alpha \sigma \sqrt{H(F)} = 0.4278$$

Assume $K = \$60{,}000$, so that $\psi = \dfrac{60{,}000}{130{,}164} = 0.4610$. Relation (1.5) provides single obligor limits:

$$\theta \le \left(\frac{\psi - \overline{p}}{z_\alpha \sigma}\right)^2 = \left(\frac{0.4610 - 0.1089}{1.96(0.6329)}\right)^2 = 0.0805$$

That is:

$$f_i \le 0.0805 \times \$130{,}164 = \$10{,}482$$

From Table 5.1 we can see that there are only two loans that exceed the limit.

Let us now examine the impact of correlation on concentration. From (6.13):

$$\rho = \frac{[0.4006 - 0.0978] \times 0.0661}{0.0978 \times [1 - 0.0661]} = 0.2191$$

From (6.11), the risk concentration index is:

$H' = 0.2191 + (1 - 0.2191) \times 0.0661 = 0.2707$

Beside the fact that the portfolio of this example is a pretty bad one, if one adds 22% correlation to the high default probability of 10.89% one obtains unexpected losses of $\sigma\sqrt{H(F)} = 0.1627$, as opposed to $\sqrt{p(1-p)H(F)} = 0.0801$ if the loans were independent. Thus, the 22% equivalent correlation doubles the standard deviation of losses over the uncorrelated case. It is also interesting to compare the risk concentration index of $H' = 27.07\%$, which is four times greater than $H(F) = 6.61\%$. In terms of capital adequacy, the correlated portfolio requires a capitalisation ratio $\psi \geq 43\%$, which is substantially greater than the 27% required if the loans were independent.

6.3 Dealing with different dimensions of concentration

Generally, banks partition loan portfolios into subportfolios or "buckets" according to some practical criterion which is somehow related to the way in which they do business. For the purpose of credit risk in general and concentration in particular, it may be desirable to adopt a different criterion. As mentioned initially, one of the most difficult problems is to *determine ex ante* potentially dangerous *dimensions of concentration*, and these may have nothing to do with the organisational structure of the bank. The model permits a totally arbitrary segmentation of the portfolio, in order to determine the segments where concentration is potentially riskier. This permits the differentiation of limits for each segment, as well as differentiation in the allocation of capital.

6.3.1 The analysis of individual segments

Suppose that F is arbitrarily partitioned into h segments, $F^T = (F_1,\ldots,F_h)$, where F_i is a vector whose elements are the amounts outstanding of the loans in group i. Now partition the default probability vector and the associated covariance matrix accordingly:

a. $\pi = (\pi_i)$; where π'' is the vector of default probabilities of segment i; $i = 1,2,3,\ldots,h$

b. The covariance matrix is partitioned as:

$$M = \begin{bmatrix} M_1 & C_{12} & \cdots & C_{1h} \\ C_{21} & M_2 & \cdots & C_{2nh} \\ \vdots & \vdots & \ddots & \vdots \\ C_{h1} & C_{h2} & \cdots & M_h \end{bmatrix}$$

Each diagonal block M_i is the covariance matrix of defaults for the loans in segment i and has dimension $(N_i \times N_i)$, where N_i is the number of loans in the segment. Matrices C_{ij} contain the covariances of the defaults between the loans of segments i and j. Let $V_i = \sum_{j \in F_i} f_j$ be *the value of the portfolio of segment* i, and $\sum_{i=1}^{h} V_i = V$. Let $K_i = \gamma_i K$, where γ_i is the proportion of capital allocated to segment i; $\gamma_i \in [0,1]\ \forall i; \sum_{i=1}^{h} \gamma_i = 1$. Note that when analysing individual segments, only correlations between defaults of the loans in segment i with loans of the other groups should be considered, while correlations of other groups between themselves are irrelevant. Thus, from M construct matrices S_i with the following structure:

$$S_i = \frac{1}{2}\begin{bmatrix} 0 & \cdots & C_{1i} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{i1} & \cdots & 2M_i & \cdots & C_{ih} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & C_{hi} & \cdots & 0 \end{bmatrix} \tag{6.14}$$

Note that $\sum_i S_i = M$. When integrating the analysis of individual segments into the overall portfolio, it is important that the relative weights of each segment in the overall portfolio do not distort the results for the portfolio as a whole. An additivity property is necessary so that addition of over individual segments is consistent for the portfolio. Let

$$\phi = \frac{\sqrt{F^T M F}}{\sum_{i=1}^{h} \sqrt{F^T S_i F}} \tag{6.15}$$

In what follows, we will see that this constant permits the summation of the individual VaR_i. Proceeding in the usual way, the value-at-risk inequality for each segment is:

$$\nu_i = \pi_i^T F_i + z_\alpha \phi \sqrt{F^T S_i F} \le K_i = \gamma_i K \quad \text{for } i = 1, 2, \ldots, h \tag{6.16}$$

where $\gamma_i \ge 0$ and $\sum_i \gamma_i = 1$. It is easily verified that $\sum_i \nu_i = VaR_\alpha = \pi^T F + z_\alpha \sqrt{F^T MF}$.

Dividing by V_i, leads to capital adequacy for each individual segment:

$$\psi_i \ge \frac{\nu_i}{V_i} = \bar{p}_i + z_\alpha \phi \sqrt{R_i(F_i, M_i)\, H(F_i) + \frac{1}{(1^T F_i)^2} \sum_{\{j / j \ne i\}} F_i^T F_{ij} F_j} \tag{6.17}$$

Solving for $H(F_i)$ one obtains

$$H(F_i) \le \left(\frac{\psi_i - \bar{p}_i}{z_\alpha \phi \sigma_i} \right)^2 - \frac{1}{(\sigma_i V_i)^2} \sum_{\{j | j \ne i\}} F_i^T \; C_{ij}\, F_j \tag{6.18}$$

where

$$\sigma_i = \sqrt{\frac{F_i^T M_i F_i}{F_i^T F_i}} = \sqrt{R_i(F_i, M_i)} \tag{6.19}$$

Single obligor limits per segment are obtained by applying Theorem 5.3:

$$\theta_i \le \left(\frac{\psi_i - \bar{p}_i}{z_\alpha \phi \sigma_i} \right)^2 - \frac{1}{(\sigma_i V_i)^2} \sum_{\{j | j \ne i\}} F_i^T \; C_{ij}\, F_j \tag{6.20}$$

It is interesting to note that the bound on concentration now includes a correction for default correlation with the loans in other groups: namely, the second term on the right-hand side of the inequality. This conforms to intuition, since higher correlation of defaults with the loans in the other groups means that less concentration can be tolerated in group i, namely:

$$\frac{1}{(\sigma_i V_i)^2} \sum_{\{j | j \ne i\}} F_i^T \; C_{ij}\, F_j \tag{6.21}$$

6.3.2 Overall capital adequacy in a segmented portfolio

Note that all of the above expressions are obtained from ν_i / V_i, so that the weight of the segments within the portfolio is not accounted for. Therefore, a simple summation of terms can be misleading as to the overall capital adequacy of the segmented portfolio. Letting $\gamma_i = \frac{V_i}{V}$, then if 6.17 is satisfied for all the segments, $\psi = \sum_{i=1}^{h} \gamma_i \psi_i$ ensures capital adequacy for the portfolio.

Example 6.2

Refer to the portfolio of the previous examples. The partition is shown in Table 6.1.[25] The loans vector is partitioned as: $F^T = (F_1 \;\; F_2 \;\; F_3)$,

Table 6.1

Rating	F_1	Rating	F_2	Rating	F_3
A1	$4,728	B1	$5,528	A2	$7,728
C2	$3,204	C1	$3,138	B2	$5,848
C4	$4,912	C3	$4,831	C5	$5,435
D1	$5,320	E2	$5,042	D2	$5,765
D3	$20,239	E3	$15,411	E1	$1,800
F1	$1,933	F3	$2,411	F2	$2,317
F4	$2,598	G1	$358	G3	$2,652
G2	$1,090	G5	$6,467	G4	$4,929
				G6	$6,480
Total	$44,024	Total	$43,186	Total	$42,954

Next, the default probabilities vector and the covariance matrix are partitioned to be consistent with the partition of the loans vector as:

$$\pi^T = (\pi_1 \quad \pi_2 \quad \pi_3) \text{ and } M = \begin{bmatrix} M_1 & C_{1,2} & C_{1,3} \\ C_{2,1} & M_2 & C_{2,3} \\ C_{3,1} & C_{3,2} & M_3 \end{bmatrix} \text{ where:}$$

- M_1, M_2, and M_3 are the idiosyncratic covariance matrices for the three groups respectively.
- $C_{12}^T = C_{21}$ is the covariance matrix between the loans of groups one and two. Likewise, $C_{13}^T = C_{31}$ is the covariance matrix between the loans of the first and third groups and $C_{23}^T = C_{32}$ is the covariance matrix between the loans of the second and third (see Appendix A).

Table 6.2 shows the value of the loans of each segment, the corresponding HHI, and the associated capital allocation γ_i.

Table 6.2

Segment *i*	V_i	$H(F_i)$	γ_i	K_i
1	$44,024	0.2613	0.3382	$20,293
2	$43,186	0.2008	0.3318	$19,907
3	$42,954	0.1293	0.33	$19,800

[25] A1 is the first A-rated loan, C2 is the second C-rated loan and so on.

Refer to Appendix A for the variance covariance matrix used for this example. The S_i matrices for each segment have the form:

$$S_1 = \frac{1}{2}\begin{bmatrix} 2M_1 & C_{12} & C_{13} \\ C_{21} & 0 & 0 \\ C_{31} & 0 & 0 \end{bmatrix},\ S_2 = \frac{1}{2}\begin{bmatrix} 0 & C_{12} & 0 \\ C_{21} & 2M_2 & C_{23} \\ 0 & C_{32} & 0 \end{bmatrix} \text{ and } S_3 = \frac{1}{2}\begin{bmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{31} & C_{32} & 2M_3 \end{bmatrix}.$$

Note that $\psi_i = \frac{K_i}{V_i} = \frac{\gamma_i \times K}{V_i} = \frac{K}{V} = \frac{60{,}000}{130{,}164} = 0.4610$ for all segments, since $\gamma_i = V_i / V$.

From 6.15, parameter ϕ, which allows summation of individual VaRs, is:

$$\phi = \frac{\sqrt{F^T MF}}{\sum_{i=1}^{3}\sqrt{F^T S_i F}} = 0.5783$$

Calculation of ν_i with 6.16, using a 5% confidence limit and assuming normality, yields:

$\nu_1 = \$16{,}255 < K_1 = \$20{,}293$,

$\nu_2 = \$19{,}368 < K_2 = \$19{,}907$,

$\nu_3 = \$20{,}060 > K_3 = \$19{,}800$.

First note that:

$$VaR_\alpha = \sum_{i=1}^{3} \nu_i = \$55{,}683$$

Moreover,

$$\psi = 0.4610 \geq \sum_{i=1}^{3} \frac{\nu_i}{V_i} = \frac{VaR}{V} = \frac{55{,}684}{130{,}164} = 0.4278$$

Thus, the portfolio as a whole exhibits capital adequacy, in spite of the fact that the third segment does not comply with its individual capital requirement. This means that the segment will not satisfy any of the other conditions. Using the data in Tables 6.1 and 6.2, the equivalent correlation for each segment is calculated from equation (6.13) and the risk concentration measure from (6.11). The results are summarised in Table 6.3:

Table 6.3

p	ρ	$H(F)$	H'	$H'/H(F)$	Loss std dev
0.0774	0.1404	0.2613	0.365	1.3969	0.1614
0.1162	0.1746	0.2008	0.3403	1.6947	0.1869
0.1339	0.2792	0.1293	0.3724	2.8801	0.2078

With these values, one can verify all the capital adequacy relations. As was to be expected, the third segment does not comply with the limit on concentration.

$$H(F_3) = 0.1293 > \left(\frac{\psi_i - \bar{p}_i}{z_\alpha \phi \sigma_i}\right)^2 - \frac{1}{(\sigma_i V_i)^2} \sum_{\{j|j \neq i\}} F_i^T C_{ij} F_j = 0.1115$$

Now single obligor limits can be obtained:

$\theta_1 \leq 1.1478 - 0.3895 = 0.7583$; $f_1 \leq 0.7583 \times \$44{,}024 = \$33{,}384$

$\theta_2 \leq 0.5314 - 0.2860 = 0.2454$; $f_2 \leq 0.2454 \times \$43{,}186 = \$10{,}596$

$\theta_3 \leq 0.2492 - 0.1377 = 0.1115$; $f_3 \leq 0.1115 \times \$42{,}954 = \$4{,}790$

In summary, no loan in the first group exceeds its limit, while the $15,411 loan exceeds its limit in the second group. As was to be expected, the third group is the most problematic, since only the three smallest loans in the segment comply with the limit.

Note that although the third segment is the least numerically concentrated as measured by $H(F)$, it has the highest level of risk concentration H'. Although the first segment also exhibits high risk concentration, since it has the lowest average default probability it is the least risky of the three. Note also that the first is the numerically more concentrated segment, but since its equivalent correlation is relatively low, its risk concentration relative to its HHI is the smallest of the three. These numbers also illustrate the interplay between default probabilities and concentration in the loss variance of each segment, pointing to the third segment as the riskiest, because its equivalent correlation, risk concentration and average default probability are the largest of the three, providing the highest standard deviation of losses.

The example evidences the analytical power of the model. If one had restricted the exercise to using the general model without analysing individual segments, the risky third segment would have passed undetected. It is also clear that the results depend on the segmentation criterion used, since one can classify the loans in such a way that all segments comply with the relevant relations, and risky groups of loans will remain undetected. However, the example also indicates how one can obtain insight into the ex ante concentration issue, in the worst case by trial and error.

7. Accounting for recovery rates

It is simple to extend all the relations so far obtained to include loan recovery rates. Doing so leads to less restrictive limits in terms of tolerable concentration along the different dimensions where concentration can occur. Basically, there are two ways to account for recovery rates. The first is to define F directly as the vector of "*loss given default*" *(LGD), as opposed to the outstanding balance*, where it is assumed that nothing is recovered if loans default. This would be very much in line with current practice.[26] Thus if an estimation of the LGD vector is at hand, one can simply use this in the relations derived without any changes, although they should be reinterpreted accordingly.

Alternatively, assuming that the portfolio is segmented such that recovery rates are the same for all loans in the group, let r_i be the recovery rate for defaulted loans in segment i, so that the loss-given-default vector is simply $L_i = (1 - r_i)F_i$. Proceeding in the usual manner for each segment leads to:

$$H(F_i)R_i(M_i,F_i) + \frac{2}{V_i^2(1-r_i)}\sum_{\{j/j\neq i\}}(1-r_j)\,F_i^T\ C_{ij}\ F_j \leq \left(\frac{\psi_i - (1-r_i)\bar{p}_i}{z_\alpha \phi\,(1-r_i)}\right)^2 \tag{7.1}$$

and adding over all segments:

$$\sum_i H\,(F_i)R_i(M_i,F_i) + 2\sum_i \frac{1}{V_i^2(1-r_i)}\sum_{\{j/j\neq i\}}(1-r_j)\,F_i^T\ C_{ij}\ F_j \leq \sum_i \left(\frac{\psi_i - (1-r_i)\bar{p}_i}{z_\alpha \phi\,(1-r_i)}\right)^2 \tag{7.2}$$

The expression shows that any change in recovery rates has a double impact. On the one hand, the importance of each segment's correlation with loans of other segments is increased or decreased, depending on the ratio of loss rates between the loans in the segment with respect to that of the others. Additionally, its contribution to the expected loss also decreases (increases) in the numerator of the right-hand side, increasing (decreasing) the established bound on concentration. It is not difficult to show that the denominator of the right-hand side behaves accordingly, decreasing as the recovery rate increases and vice-versa. So, if recovery rate data is inadequate or non-existent, one can perform exercises using different recovery rates, or using some kind of reference.

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26 See Basel Committee on Banking Supervision (1999).

8. The normality assumption

Up to this point, it has been assumed that the loss distribution is normal. In this section we discuss the approximation of the loss distribution using a gamma distribution, which can also be characterised by its mean and variance and captures the asymmetry typically observed in credit loss distributions. The gamma density function can be written as:[27]

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{x}{\beta}}$$

The mean and the variance are $E(x) = \alpha\beta$ and $VAR(x) = \alpha\beta^2$ respectively, and there is only one solution for any given pair of parameters (α, β).

Several exercises have been performed to date, to compare the results of the model presented here and CreditRisk$^+$, on random portfolios from the SENICREB database of the central bank.[28] Without claiming to have conducted a rigorous and exhaustive study, we can say that the results obtained are encouraging. In the next example the results for the best and worst fits are shown.

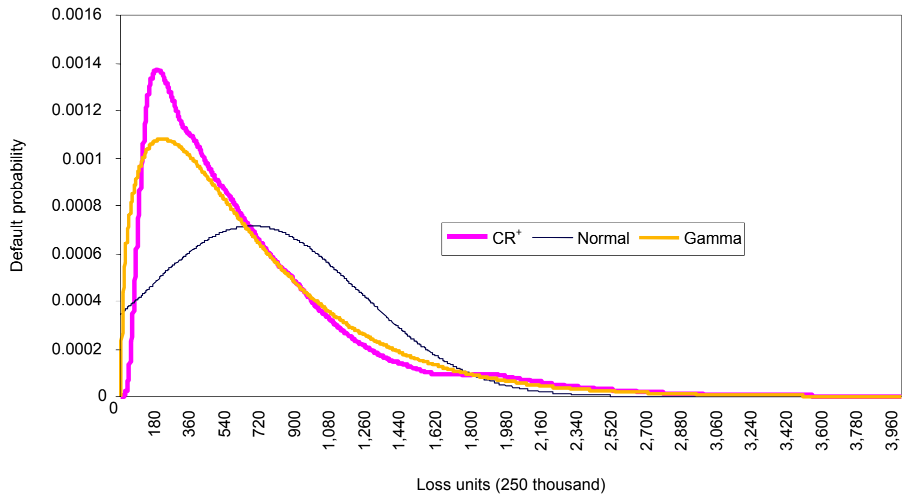
Example 8.1

The results for the first exercise, Figure 8.1, compare the loss distributions obtained for a random portfolio of 3,000 loans in the SENICREB database. Whereas the normal approximation can differ with CreditRisk$^+$ as much as 37.7% in VaR at the 99% confidence level, the difference using the gamma approximation is only 0.45%.

Figure 8.2 shows the results on a random portfolio of 1,320 loans from the same source. The loss distribution obtained using CreditRisk$^+$ has two "humps". This is because this sample contained a very large loan in comparison with the other loans in the portfolio, which, due to the bucketing procedure required by CreditRisk$^+$, creates discontinuities and gaps in the possible losses. As shown in the table, the largest difference of VaR between the two methodologies using the gamma approximation is 12.34% at the 99% confidence level. The figures using the normal approximation are worse.

Figure 8.1

Comparison of loss distributions on a random sample of 3,000 loans



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[27] There are many ways in which the gamma distribution can be written. The one adopted here follows the convention used in CreditRisk$^+$.

[28] SENICREB (Servicio Nacional de Información de Créditos Bancarios), is a loan database of the Mexican banking system, and is managed by the Bank of Mexico.

Table 8.1

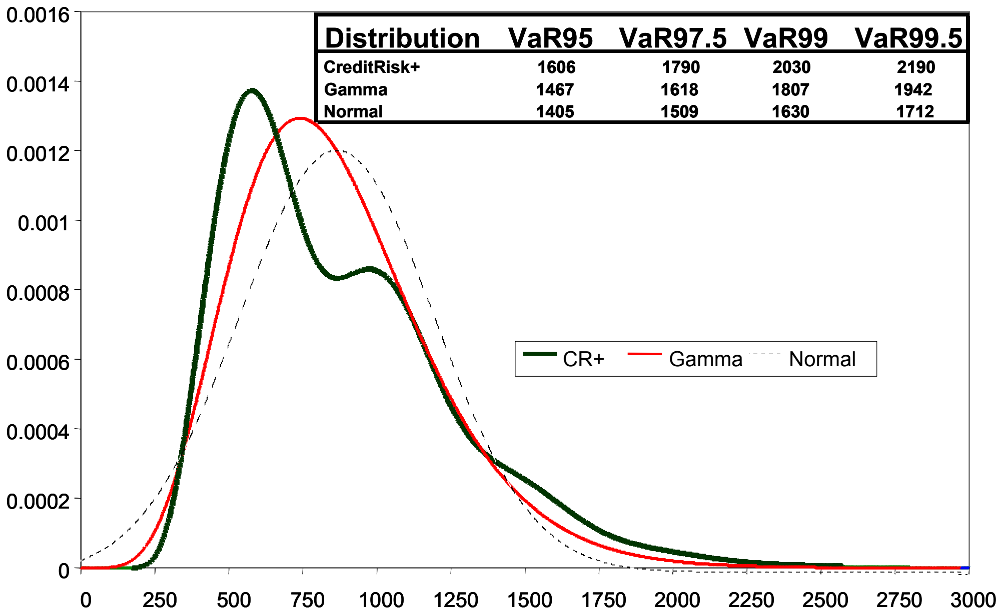
VaR comparative statistics for the sample

VaR confidence level	0.95	0.975	0.99	0.995
CR$^+$	1,878	2,212	2,623	2,932
Normal	1,590	1,765	1,969	2,108
Gamma	1,770	2,120	2,577	2,919

Loss distribution statistics	Mean	Variance	Std dev	alfa	beta
CR$^+$	673	312,277	559	–	–
Normal	674	310,116	557	–	–
Gamma	674	310,116	557	1.46	460.26

Figure 8.2

Comparison of loss distributions on a random sample of 1,320 loans



It should be pointed out that not all of the exercises produced VaR differences where the model underestimated the results of CreditRisk$^+$. Some of the random portfolios provided results where the opposite occurred using the gamma approximation. In all of these cases the differences were small.

It is not always the case that the VaR obtained by CyRCE is less than the corresponding VaR obtained using CreditRisk$^+$. Although the preceding examples are interesting and serve to illustrate the kind of results obtained by both methods, they are far from being a rigorous comparative study. In particular, it is interesting to examine how the two methodologies behave as the number of loans in the portfolio increases. In order to explore this behaviour, a simulation experiment was carried out, taking random samples of portfolios of increasing numbers of loans, and their VaR was calculated by both methods, for different confidence levels.[29] The results of the exercise are summarised in Figure 8.3. The number of loans in the portfolios is shown on the x-axis, while the y-axis shows the average of the following statistic:

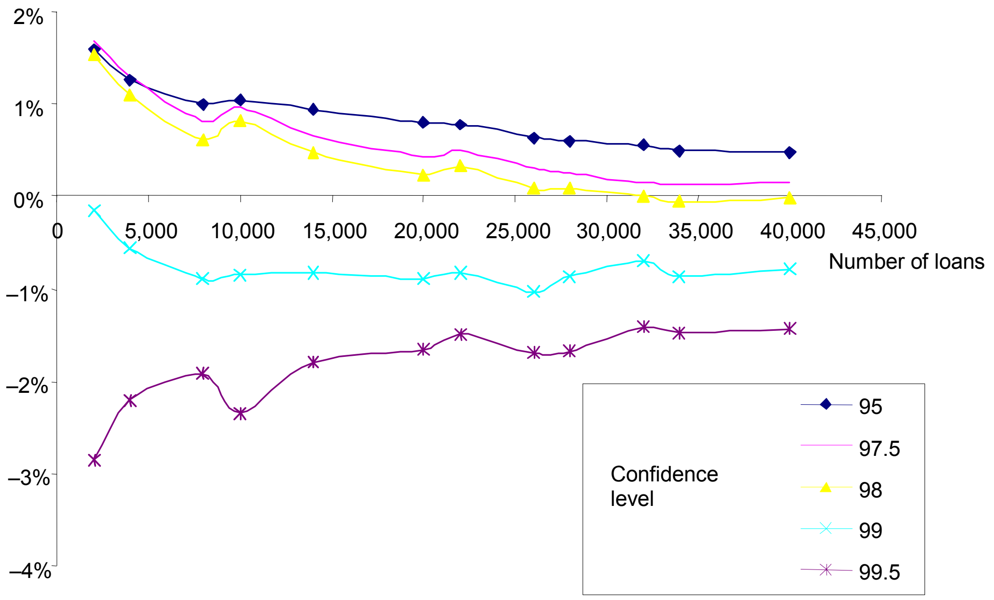
$$\Delta \frac{VaR_{CyRCE} - CaR_{CreditRisk^{+}}}{\text{Total value of loan portfolio}}$$

The curves in the graph represent the average differences in VaR relative to the value of the portfolio, for different confidence levels. The gamma distribution was used for approximating the loss distribution obtained by CyRCE.

Figure 8.3

Comparison between CreditRisk⁺ and CyRCE as the number of loans increases in the portfolio, for different confidence levels

$$\frac{\text{VaR CyRCE} - \text{VaR CreditRisk}^{+}}{\text{Total loan}}$$



First, it is interesting to note that the average difference of VaRs relative to the size of the portfolio decreases as the number of loans increases. This provides some empirical evidence that there is some sort of large numbers effect. Next, the graph shows that, on average, the VaR obtained by CyRCE overestimates that obtained by CreditRisk^{+} for confidence levels below 98%, and underestimates them for higher confidence levels. Undoubtedly, this is due to the heavier tails of the loss distribution generated by CreditRisk^{+}.

9. Application of the model with limited portfolio information

Any credit risk model requires two types of information: a description of the loans in the portfolio, and the default behaviour of the loans it contains (ie default probabilities and correlations). The model

[29] Thus, for each of the 23 sizes of portfolio, between 2,000 and 64,000 loans, 500 simulation runs were performed. Due to the characteristics of the SENICREB database, sampling was done with replacement for the larger sizes.

presented here allows several options for performing calculations with limited information. Regardless of the quality of information available on default rates of loans in a portfolio, it is the author's experience that in the worse case, bankers have some idea of what these are, even if this information is not available in some sort of systematised database. The estimation of default probabilities and correlations from default rates is a topic in itself and will not be dealt with here. On the other hand, the difficulty in obtaining portfolio information is of particular relevance to regulators, and probably constitutes the largest stumbling block for effective credit risk supervision. Banks are reluctant to provide regulators with this information on an ongoing basis simply because of the huge quantities of data involved. Even if the data could be obtained in an appropriate and systematic way, it would be difficult to handle. Private banks with large portfolios would also benefit from reducing information requirements to run their models. We now address this issue.

As seen in the derivation of the model, it is not strictly necessary to know the credit portfolios in detail. Given an adequate segmentation of the portfolio, the only information required by the model is:

a. The total value of the loans in each segment V_i.

b. Enough information about the loan distribution within each segment, which allows an estimate of its HHI.

c. Estimates of p_i, ρ_i and ρ_{ij}.

In what follows, we will discuss how estimates of HHI can be obtained from some very basic statistics. Thus, suppose that the portfolio has been segmented into h segments. If for each segment one knows the value of the segment, V_i, and the value of the largest loan in each segment, f_j^*, then Theorem 5.3 states that:

$$H(F_i) \le \theta_i = \frac{f_i^*}{V_i} \tag{9.1}$$

Therefore, $H(F_i) = \theta_i$, is an estimate of HHI for each segment, although perhaps somewhat crude. In fact, Theorem 5.3 can be used to obtain a slightly tighter bound. To see this, remember that the largest concentration occurs when the portfolio has the following distribution as a proportion of its value V:

$$f_k = \begin{cases} \theta; & k = 1, 2, \ldots, n \\ \varepsilon; & k = n+1 \\ 0; & k = n+2, \ldots, N \end{cases}$$

$$\sum f_k = n\theta + \varepsilon$$

For this distribution,

$$H(F) = n\theta^2 + \varepsilon^2 = n\theta \cdot \theta + \varepsilon^2 = (1 - \varepsilon)\theta + \varepsilon^2.$$

This expression is minimum when $\varepsilon = 0.5\theta$. Since it is virtually impossible to have such a distribution in practice, if only the largest loan in each segment is known, one could argue that a good bound on HHI is:

$$H(F) < \theta(1 - 0.5\theta) \tag{9.2}$$

If the number of loans per segment N_i is known, as well as the average size loan $\overline{f_i}$ and the variance $\overline{\sigma}_i^2$, then HHI can be obtained. To see this, first note that $V_i = N_i \overline{f_i}$ is the value of each segment and $V = \sum_{i=1}^{h} V_i$ is the value of the portfolio. Then, by the definition of variance:

$$\overline{\sigma}_i^2 = \frac{\sum_k \left(f_k^i - \overline{f_i}\right)^2}{N_i - 1} = \frac{\left(N_i \overline{f_i}\right)^2}{(N_i - 1)\left(N_i \overline{f_i}\right)^2} \left\{ \sum_k \left(f_k^i\right)^2 - N_i \left(\overline{f_i}\right)^2 \right\}$$

$$= \frac{(V_i)^2}{(N_i - 1)} \left\{ H(F_i) - \frac{1}{N_i} \right\}$$

Solving for HHI one obtains:

$$H(F_i) = \frac{(N_i - 1)}{N_i^2}\left(\frac{\sigma_i}{\bar{f}_i}\right)^2 + \frac{1}{N_i} \tag{9.3}$$

Now, having estimates of HHI for each segment, one can obtain the HHI of the whole portfolio as follows:

$$H(F) = \sum_{i=1}^{h}\left(\frac{V_i}{V}\right)^2 H(F_i) \tag{9.4}$$

Note that (9.3) and (9.4) are *exact* values for the concentration indices, and that they can be obtained with very limited information.

10. Systemic credit risk analysis of the Mexican banking system

Using the SENICREB database, the model is currently being used to analyse the credit risk profile and capital adequacy of the 20 banks of the Mexican financial system. The results are presented to the board of governors of the central bank on a monthly basis. In the exercise presented here for illustrative purposes, we assume that the underlying loss distribution is normal and VaR computations are for a monthly horizon at the 2.5% confidence level. Due to the centralisation that characterises the Mexican economy, it is difficult to segment by geographical region. As a starting point, the loan portfolio of the system has been segmented by bank and economic activity. Loans rated by one or more of the major rating agencies form a separate segment, because there are relatively few rated loans so the observed history of their default behaviour is insufficient for default probability and correlation estimation. Fortunately, the rating agencies themselves provide good estimates of these parameters.

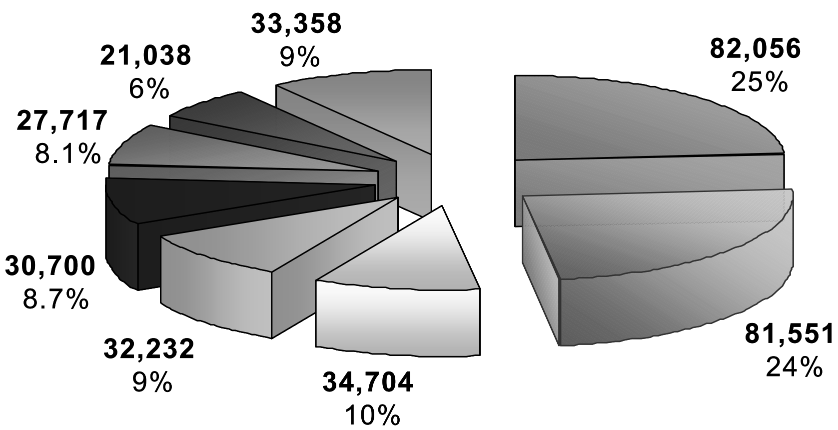
Figure 10.1 shows how the loan market was distributed among the major banks in Mexico at the end of March 2002. Two banks have 48% of the market, and 91% of all loans are held by seven banks.

Figure 10.1

Distribution of the loan portfolio by major banks

March 2002

Loan portfolio of the system = 343,357[1]



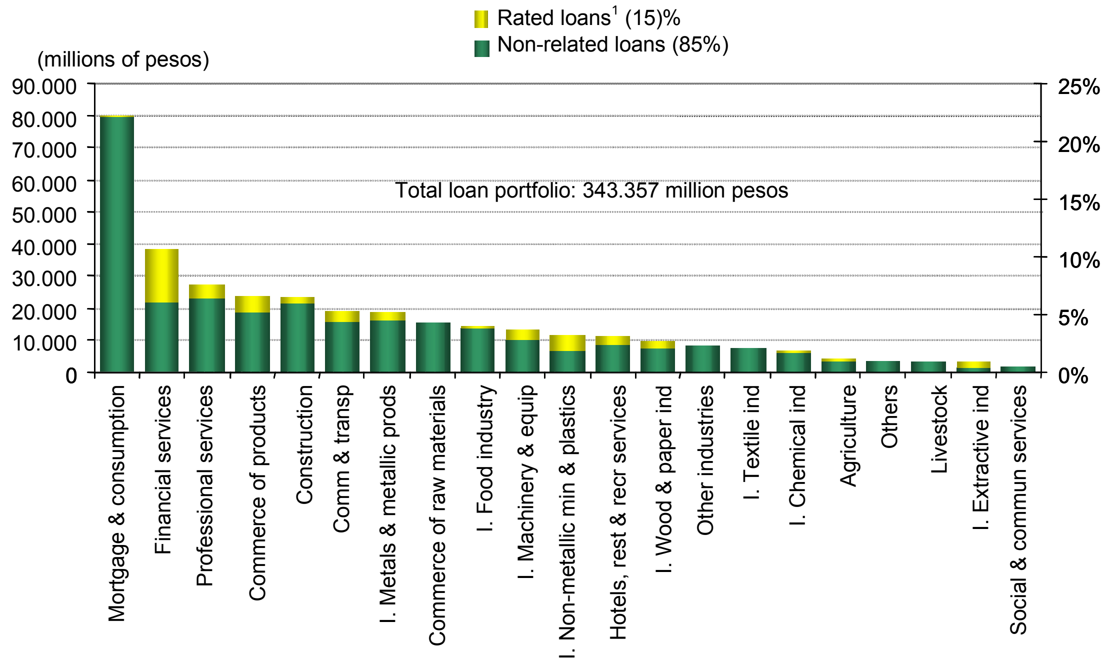
[1] Millions of pesos.

Figure 10.2 shows the distribution of the loan portfolio of the banking system segmented by economic activity. From the bar chart, we see that the largest segment is represented by mortgage and consumer loans, followed by financial services and so on.

Figure 10.2

Distribution of the loan portfolio by economic activity

March 2002

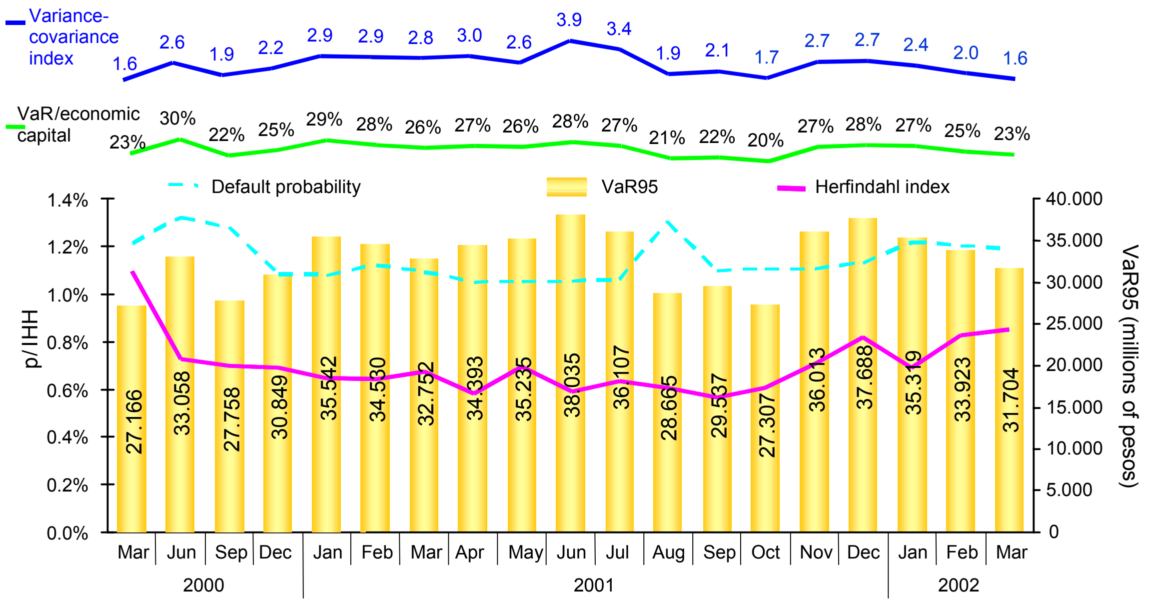


[1] Rated by Standard & Poor's, Moody's and Fitch.

Figure 10.3, shows the variation over time of VaR, default probabilities, HHI concentration index, covariation index and VaR relative to economic capital for the banking system.

Figure 10.3

Evolution of the risk profile of the Mexican banking system



	Mar	Jun	Sep	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar
p	1.2%	1.3%	1.3%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.3%	1.1%	1.1%	1.1%	1.1%	1.2%	1.2%	1.2%
IHH	1.1%	0.7%	0.7%	0.7%	0.6%	0.6%	0.7%	0.6%	0.7%	0.6%	0.6%	0.6%	0.6%	0.6%	0.7%	0.8%	0.7%	0.8%	0.9%

This graph summarises how systemic VaR responds to the main risk drivers, and provides a first indication of how well the banking system is capitalised relative to the level of risk taken.

Figure 10.4 shows the concentration of risk in the system. Banks are sorted by their contribution to the overall VaR of the system, and arranged from largest to smallest. A Lorenz curve is constructed so that the amount of risk concentrated in a specific number of banks can be seen. The bars represent the ratio of each bank's VaR relative to its net capital. The horizontal line is the average VaR/net capital ratio for the system, so that one can see the relative position of each bank with respect to the system. Thus, 80% of the risk is concentrated in five banks and the third of these have a VaR/net capital ratio of 64%, which is relatively high when compared to the 23% average of the system.

Figure 10.4

Individual banks' contribution to systemic risk

March 2002

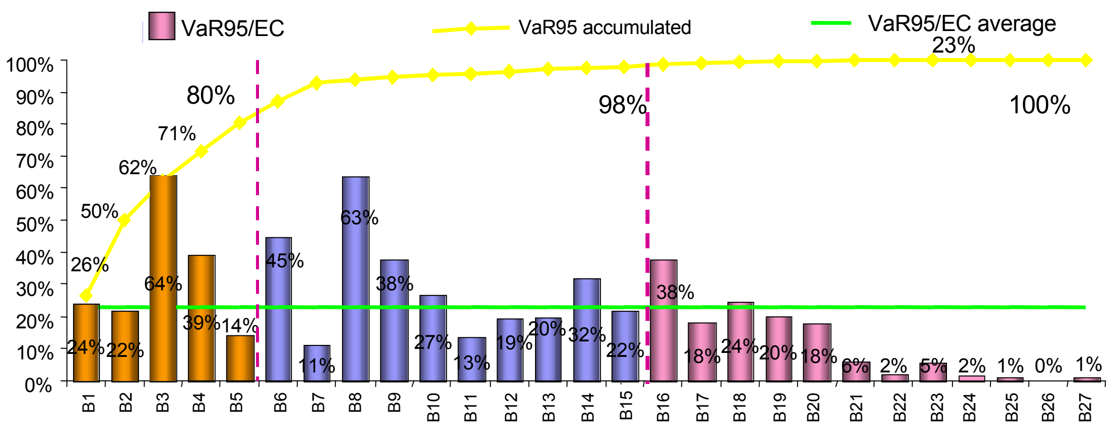


Figure 10.5

Contribution to systemic risk by economic activity

March 2002

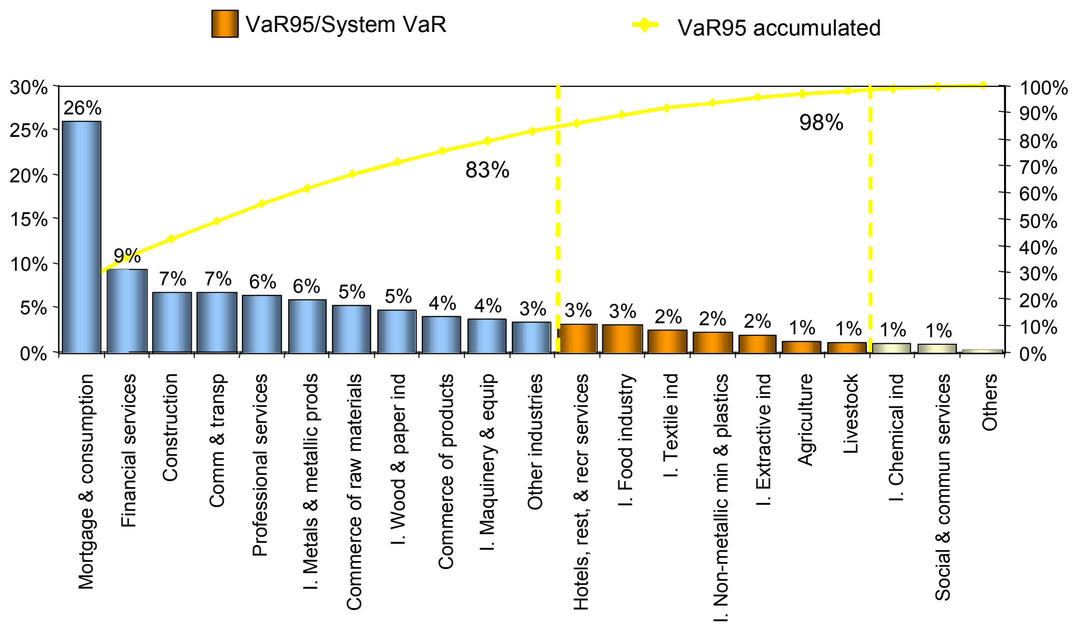


Figure 10.5, shows the contribution of the individual segments of economic activity. Most of the risk is in consumer and mortgage loans, followed by financial services. Note that construction and communications and transportation have moved up to third and fourth place respectively, from the fifth and sixth positions they occupy in terms of loan value. This is due to relatively high concentration and default probabilities within these sectors. Note also how the food industry, which is in the eighth position in terms of loan value, occupies the 13th slot in terms of risk.

Figures 10.6 to 10.8 provide some statistical results on the behaviour of the credit risk driver, ie default probabilities, concentration and loss variation-covariation indices, for all banks and segments considered.

Figure 10.6

Default probability histogram

April 2001-March 2002

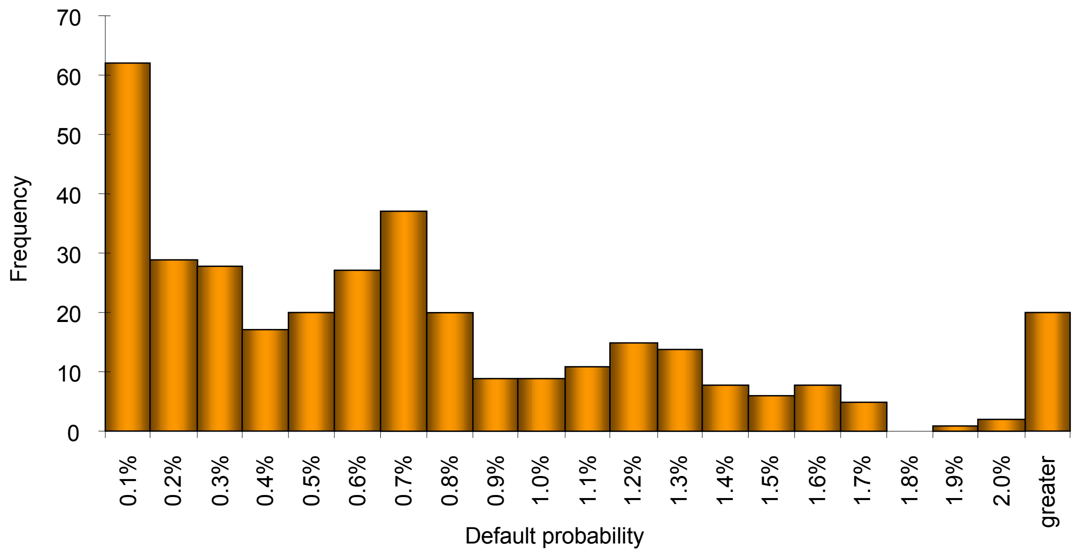


Figure 10.7

Herfindahl-Hirschman index histogram

April 2001-March 2002

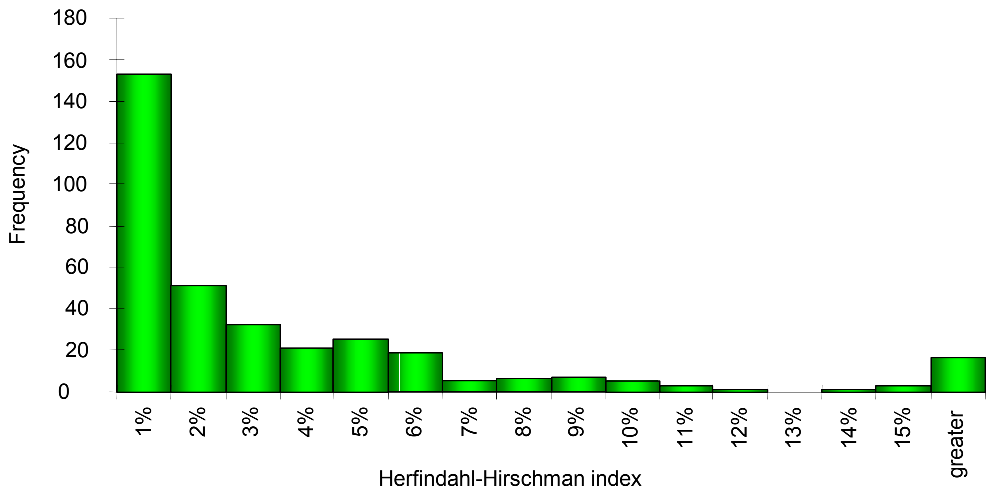


Figure 10.8

Loss variance-covariance index histogram

April 2001-March 2002

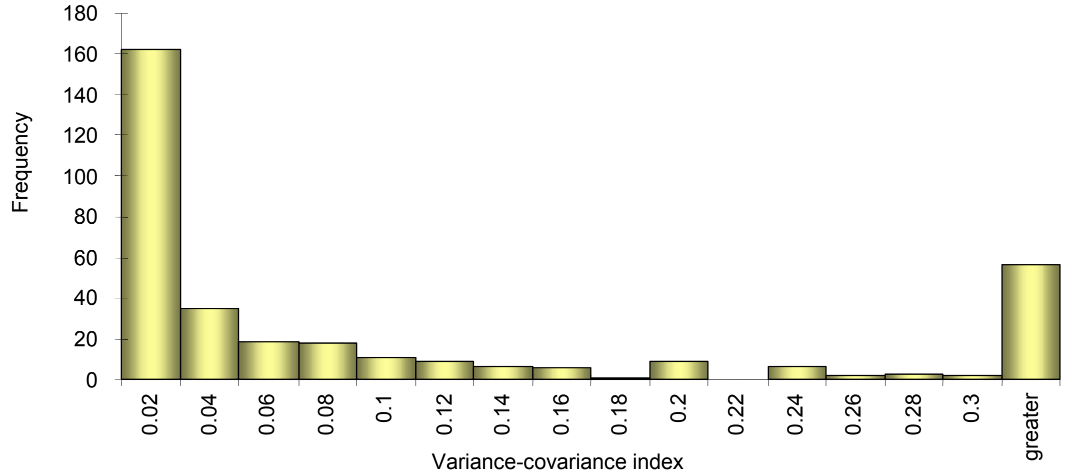


Figure 10.9 is a histogram of how well banks are capitalised as measured by the capital adequacy relationship. On average, the excess of net capital over VaR as a proportion of the value of the portfolios is around 25%, and in the past two years no negative quantities have been observed.

Figure 10.9

Capital adequacy histogram (EC – VaR95)/loan portfolio value > 0

April 2001-March 2002

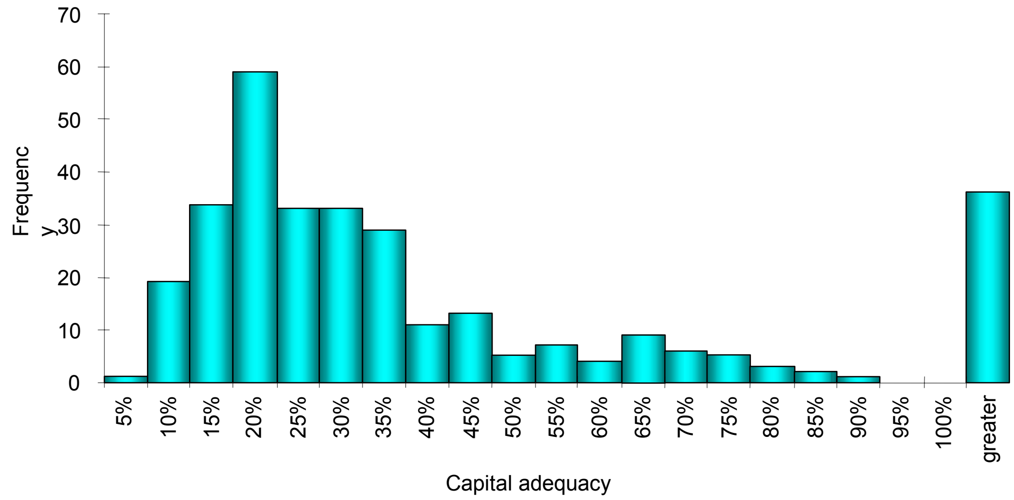
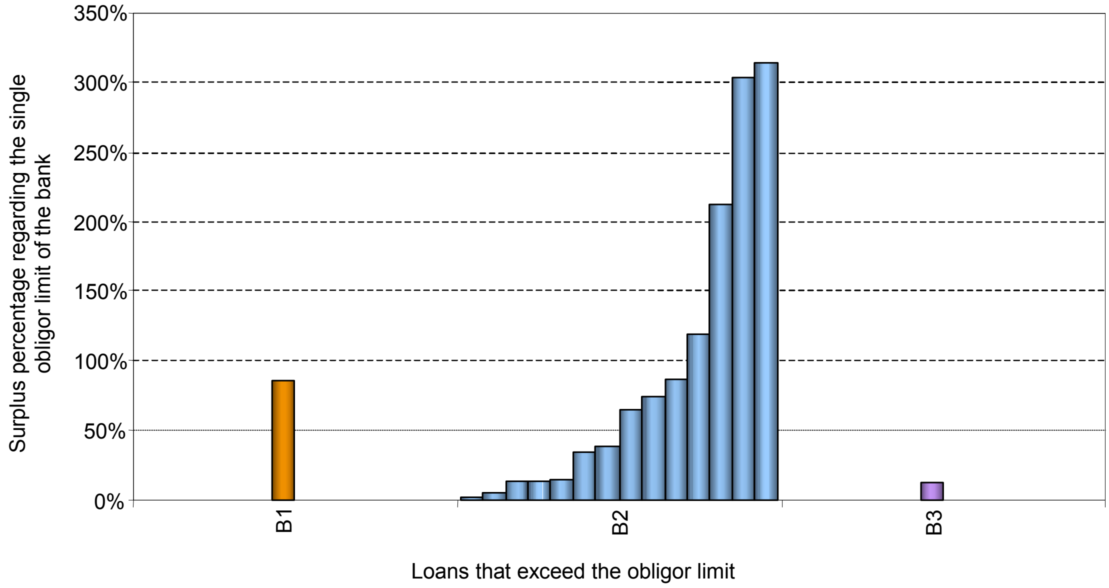


Figure 10.10 shows the analysis of compliance with the theoretical single obligor limits. Among the five banks that account for 80% of the risk of the system, only the three shown in the graph have loans that exceed the limit. Bank number two of this graph, which is the same as bank number three of Figure 10.5, has many such loans. Analysis of the bank's risk revealed that the large VaR/EC ratio is due to some extent to concentration. However, it should be emphasised that the limit is only a sufficient condition for capital adequacy but not a necessary one. So even if it helps to know how many and which loans are in violation of the condition, further analysis is needed in order to assess the gravity of the situation.

Figure 10.10

Loans that exceed single obligor limits in five major banks

March 2002



These graphs give a good idea of the type of analysis that is possible using the model. The same amount of detail is obtained for every bank in the system, permitting a more in-depth analysis of their situation.

11. Concluding remarks

The results obtained are very appealing for managing credit risk, since they provide explicit formulae to measure risk and permit a precise quantification of the policy actions that should be adopted in order to maintain capital adequacy. If default or recovery rates change, or concentration along a particular segment is excessive, the relations can be used to determine the adjustments to the capitalisation ratio and/or concentration composition of the portfolio that would re-establish capital adequacy. If banks have control over default or recovery rates to some degree, these can be part of the management instruments that can be used to maintain capital adequacy.

Since single obligor limits and the HHI are related to concentration, and since the measures are subject to the same bound, either one can be used as a policy instrument. In fact, both measures can be used in conjunction. Whereas the single obligor limit is easy to implement and supervise, it may lead to overly constrained loan distributions. For example, if a greedy bank manager decides to grant a loan exceeding his limit, the gravity of the transgression may be assessed using the HHI. It may be that, apart from misbehaviour, the infraction is not serious in terms of risk.

It is clear that default probability distributions as well as recovery rates exhibit random behaviours through time, depending on economic and financial factors. In contrast to market risk, where risk factors can be modelled using continuous processes, because loan defaults are discrete events in time, default behaviour in a certain group can also change in pronounced discrete jumps. This is one reason why it is difficult to establish ex ante concentration. Under different economic conditions, default probabilities and correlations can increase for a certain group of debtors, which otherwise appeared to be unrelated (eg mortgage loans to employees of a large company that goes bankrupt). Since the model can handle arbitrary segmentations of the portfolio, and it is relatively simple to stress particular segments and analyse the consequences, it provides a means for detecting ex ante concentration.

There is no reason why default probabilities cannot be made to depend on risk factors and credit drivers through logit models as in Credit Portfolio View, or as linear combinations of these factors.

Since these are determined exogenously, the results can be embedded in simulation models which generate scenarios of trajectories of these variables through time that exhibit the discontinuities typical of default-related events. From these, one can obtain stressed loss distributions and all the related statistics for each segment and the portfolio. This type of experiment may be what is needed to set an appropriate policy on the capitalisation ratio and single obligor limits. Note that since the simulation process in this case is for only a few variables, it can be done very efficiently.

Whatever the dynamics, it is always possible to make the necessary adjustments through time by monitoring only a few variables.

Appendix A:
Variance-covariance matrix

Example 6.1

Table A.1

M_1

	1	2	3	4	5	6	7	8
1	0.0162	0.0050	0.0050	0.0060	0.0060	0.0082	0.0082	0.0105
2	0.0050	0.0475	0.0086	0.0103	0.0103	0.0140	0.0140	0.0180
3	0.0050	0.0086	0.0475	0.0103	0.0103	0.0140	0.0140	0.0180
4	0.0060	0.0103	0.0103	0.0694	0.0125	0.0169	0.0169	0.0217
5	0.0060	0.0103	0.0103	0.0125	0.0694	0.0169	0.0169	0.0217
6	0.0082	0.0140	0.0140	0.0169	0.0169	0.1275	0.0230	0.0295
7	0.0082	0.0140	0.0140	0.0169	0.0169	0.0230	0.1275	0.0295
8	0.0105	0.0180	0.0180	0.0217	0.0217	0.0295	0.0295	0.2100

Table A.2

$C_{12}^T = C_{21}$

	1	2	3	4	5	6	7	8
9	0.0063	0.0108	0.0108	0.0130	0.0130	0.0177	0.0177	0.0227
10	0.0081	0.0138	0.0138	0.0166	0.0166	0.0226	0.0226	0.0290
11	0.0081	0.0138	0.0138	0.0166	0.0166	0.0226	0.0226	0.0290
12	0.0111	0.0190	0.0190	0.0229	0.0229	0.0311	0.0311	0.0399
13	0.0111	0.0190	0.0190	0.0229	0.0229	0.0311	0.0311	0.0399
14	0.0132	0.0226	0.0226	0.0273	0.0273	0.0370	0.0370	0.0475
15	0.0169	0.0290	0.0290	0.0350	0.0350	0.0475	0.0475	0.0609
16	0.0169	0.0290	0.0290	0.0350	0.0350	0.0475	0.0475	0.0609

Table A.3

$C_{13}^T = C_{31}$

	1	2	3	4	5	6	7	8
17	0.0039	0.0067	0.0067	0.0081	0.0081	0.0109	0.0109	0.0140
18	0.0052	0.0089	0.0089	0.0108	0.0108	0.0146	0.0146	0.0188
19	0.0067	0.0114	0.0114	0.0138	0.0138	0.0187	0.0187	0.0240
20	0.0081	0.0138	0.0138	0.0167	0.0167	0.0226	0.0226	0.0290
21	0.0092	0.0157	0.0157	0.0190	0.0190	0.0257	0.0257	0.0330
22	0.0109	0.0187	0.0187	0.0226	0.0226	0.0306	0.0306	0.0393
23	0.0140	0.0240	0.0240	0.0290	0.0290	0.0393	0.0393	0.0504
24	0.0140	0.0240	0.0240	0.0290	0.0290	0.0393	0.0393	0.0504
25	0.0140	0.0240	0.0240	0.0290	0.0290	0.0393	0.0393	0.0504

Table A.4

M_2

	9	10	11	12	13	14	15	16
9	0.0291	0.0086	0.0086	0.0118	0.0118	0.0140	0.0180	0.0180
10	0.0086	0.0475	0.0109	0.0150	0.0150	0.0179	0.0230	0.0230
11	0.0086	0.0109	0.0475	0.0150	0.0150	0.0179	0.0230	0.0230
12	0.0118	0.0150	0.0150	0.0900	0.0207	0.0246	0.0316	0.0316
13	0.0118	0.0150	0.0150	0.0207	0.0900	0.0246	0.0316	0.0316
14	0.0140	0.0179	0.0179	0.0246	0.0246	0.1275	0.0376	0.0376
15	0.0180	0.0230	0.0230	0.0316	0.0316	0.0376	0.2100	0.0483
16	0.0180	0.0230	0.0230	0.0316	0.0316	0.0376	0.0483	0.2100

Table A.5

$C_{32}^{T} = C_{23}$

	17	18	19	20	21	22	23	24	25
9	0.0070	0.0093	0.0119	0.0144	0.0164	0.0195	0.0250	0.0250	0.0250
10	0.0089	0.0119	0.0152	0.0184	0.0209	0.0249	0.0320	0.0320	0.0320
11	0.0089	0.0119	0.0152	0.0184	0.0209	0.0249	0.0320	0.0320	0.0320
12	0.0122	0.0164	0.0209	0.0253	0.0288	0.0343	0.0440	0.0440	0.0440
13	0.0122	0.0164	0.0209	0.0253	0.0288	0.0343	0.0440	0.0440	0.0440
14	0.0146	0.0195	0.0249	0.0301	0.0343	0.0408	0.0524	0.0524	0.0524
15	0.0187	0.0250	0.0320	0.0386	0.0440	0.0524	0.0672	0.0672	0.0672
16	0.0187	0.0250	0.0320	0.0386	0.0440	0.0524	0.0672	0.0672	0.0672

Table A.6

M_3

	17	18	19	20	21	22	23	24	25
17	0.0162	0.0093	0.0119	0.0144	0.0164	0.0196	0.0251	0.0251	0.0251
18	0.0093	0.0291	0.0160	0.0193	0.0220	0.0262	0.0336	0.0336	0.0336
19	0.0119	0.0160	0.0475	0.0247	0.0281	0.0335	0.0429	0.0429	0.0429
20	0.0144	0.0193	0.0247	0.0694	0.0340	0.0404	0.0519	0.0519	0.0519
21	0.0164	0.0220	0.0281	0.0340	0.0900	0.0461	0.0591	0.0591	0.0591
22	0.0196	0.0262	0.0335	0.0404	0.0461	0.1275	0.0704	0.0704	0.0704
23	0.0251	0.0336	0.0429	0.0519	0.0591	0.0704	0.2100	0.0903	0.0903
24	0.0251	0.0336	0.0429	0.0519	0.0591	0.0704	0.0903	0.2100	0.0903
25	0.0251	0.0336	0.0429	0.0519	0.0591	0.0704	0.0903	0.0903	0.2100

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