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The Information Content of Interest Rate Futures Options*

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Abstract

Option prices are being increasingly employed to extract market expectations and views about monetary policy. In this paper, Eurodollar options are monitored to examine the evolution of market sentiment over the possible future values of Eurodollar rates. Risk-neutral probability functions are employed to synopsize the information contained in the prices of Eurodollar futures options. Several common methods of estimating risk-neutral probability density functions are examined. A method based on a mixture of lognormals density is found to rank first and a method based on a Hermite polynomial approximation is found to rank second. Several standard summary statistics are also examined, namely volatility, skewness and kurtosis. The volatility measure is fairly robust across methods, while the skewness and kurtosis measure are model-sensitive. As a concrete example, the days surrounding the September 1998 Federal Market Open Committee are examined.

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Introduction and overview

Timely information is crucial to central banks for formulating and implementing monetary policy. There are of course many sources of information. Macroeconomic data releases, regional industry visits and surveys, and financial market data are all examples of sources that central Banks use. This paper focusses on the latter source—in particular, the derivative markets sector of financial markets, which has gained prominence as a source of information.

Derivative markets have the desirable property of being forward-looking in nature and thus are a useful source of information for gauging market sentiment about future values of financial assets. Indeed, several studies have used option prices to extract market expectations and views about monetary policy [Bahra (1996), Söderlind and Svensson (1997), Söderlind (1997), Butler and Davies (1998), and Levin, Mc Manus, and Watt (1998)]. In particular, Bahra noted that option prices may prove to be useful to monetary authorities as valuable sources to (i) assess monetary conditions, (ii) assess monetary credibility, (iii) assess the timing and effectiveness of monetary operations, and (iv) identify market anomalies.

In this paper, eurodollar futures options are monitored to examine the evolution of market sentiment over the possible future values of eurodollar rates. The key tool used to synopsize the information contained in the prices of eurodollar futures options is the risk-neutral probability density function (PDF). Risk-neutral PDFs provide the probabilities attached by a risk-neutral agent to particular outcomes for future values of eurodollar rates. In addition, changes in the shape and location of the risk-neutral PDF can point to changes in the tone of the market.

Many methods exist to extract risk-neutral PDFs from option prices. This paper compares several common methods of estimating risk-neutral PDFs with the aim of determining which method most accurately prices observed market options. Encouragingly, the mixture of lognormals method ranked first—this method is now used at the Bank for examining the information content of foreign exchange futures options.[1] However, the mixture of lognormal method can occasionally run into problems. When it does, an alternative method called the Hermite polynomial method is more appropriate. The Hermite method ranked second and yielded similar results to the mixture of lognormal method.

Several standard summary statistics can be derived from the risk-neutral PDFs, namely volatility, skewness, and kurtosis. Invariably, these statistics are always quoted in conjunction with the risk-neutral PDF estimates.

A second objective of the paper is to ascertain the robustness and usefulness of these statistics. The volatility measure was found to be fairly robust across the different risk-neutral PDFs. However, the estimates of skewness and kurtosis were found to be model-dependent. The skewness measure for the exchange rate is

1. Foreign exchange futures options are examined to monitor the evolution of the markets' sentiment over future Canadian dollar exchange rates.

now quoted weekly at the Bank. The results of this paper show that further research needs to be conducted on an appropriate measure of market sentiment asymmetry.

As a concrete example, the days surrounding the September 1998 Federal Open Market Committee (FOMC) meeting are examined using the risk-neutral PDF methodology. Risk-neutral PDFs are used to monitor the response of market sentiment over the future levels of the eurodollar rates to the 29 September FOMC statement. The risk-neutral PDFs indicated an increase in market uncertainty prior to the 29 September meeting date, a lessening of uncertainty on the meeting date, and a renewed increase in uncertainty the day after the meeting. The risk-neutral PDFs clearly suggest a bearish market sentiment for the eurodollar rate, both prior to and after the FOMC meeting. Thus, some market participants expected the Fed easing and also anticipated further rate cuts would follow before mid-December 1998.

This paper is organized as follows: Section 1 reviews exchange-traded interest rate futures and interest rate futures options. Section 2 presents the general theory behind the pricing of interest rate futures options. Section 3 gives an overview of several of the common methods that are used to extract risk-neutral PDFs. (Those readers not interested in the technical details of the various option-pricing models may wish to skip section 3.) Section 4 describes the data. Section 5 compares the risk-neutral PDFs from the various estimation methods. Section 6 presents a study of the September 1998 FOMC meeting, focusing on the response of the risk-neutral PDF to the meeting. Section 7 concludes the paper and discusses possible further work.

The work in the present paper closely follows the work and methodologies of Jondeau and Rockinger (1998), and Coutant, Jondeau, and Rockinger (1998).

1. The instruments

The primary focus of this paper is exchange-traded interest rate futures and interest rate futures options. In the United States and Canada, the main exchanges for interest rate products are the Chicago Merchantile Exchange (CME) and the Montreal Exchange (ME). The CME lists a host of contracts on short-term U.S. and foreign securities. For example, both futures and futures options are listed for 3-month eurodollars, 1-month LIBOR, 13-week Treasury bills, euroyen and eurocanada. On the other hand, the ME lists relatively few interest rate futures, namely, 1-month Canadian bankers' acceptance futures (BAR), 3-month Canadian bankers' acceptance futures (BAX), 5-year Government of Canada bond futures (CGF), and 10-year Government of Canada bond futures (CGB). Futures options are listed for the 3-month Canadian bankers' acceptance futures (OBX) and the 10-year Government of Canada bond futures (OGB). Options are also listed for a small selection of Government of Canada bonds.

According to the CME, the eurodollar futures (ED) are "the most liquid exchange-traded contracts in the world when measured in terms of open interest" (Chicago Mercantile Exchange 1999). For example, a snapshot of the futures market on 15 January 1999 reveals that the March 99 ED contract had a trading volume of 76,109 and an open interest of 465,398. The eurodollar futures options (ZE) on this contract, March 99 ZE, had a combined trading volume of 27,939 and a combined open interest of 748,664. The numbers for the eurocanada futures contract pale in comparison; on 14 January 1999 the March 99 futures contract had zero trading volume and an open interest of only 190.

Statistics from the ME reveal that the BAX contract is the most actively traded contract at that exchange. The average daily volume and open interest for all BAX contracts for 1998 was 27,104 and 171,354, respectively. In comparison, the OBX futures options had an average daily volume and open interest of 840 and 15,505, respectively. The OBX volume and open interest are minuscule compared with the figures for the ZE contracts, especially considering the fact that the OBX data is aggregated across all maturity dates trading while the ZE data refers to a single maturity date. Thus, for the remainder of the paper, only CME futures and futures option data will be used.

ED contracts are listed for the quarterly cycle of March, June, September, and December, and also for the two nearest serial (non-quarterly) months. ED futures contracts are traded using a price index. The futures interest rate is calculated by subtracting the futures price from 100. For example, a ED price of 95.80 corresponds to a futures interest rate of 4.20 per cent. Thus if investors expect short-term interest rates to decline (increase), they would go long (short) the futures contract. ED contracts have a contract size of U.S.$1 million. They also feature a minimum allowable price move or tick size of 0.01, with the single exception of when a futures contract is in its expiration month, in which case the minimum tick size is reduced to 0.005. A tick value of 0.01 corresponds to a value of U.S.$25 (Contract size ¥ Tick Value ¥ Maturity of the underlying futures contract = 1,000,000 ¥ 0.01/100 ¥ 3/12). Futures contracts cease trading at 11:00 am London time on the second London business day prior to the third Wednesday of the contract month.

The ZE contract cycle, maturity date, and minimum tick size are the same as those of the underlying ED contract. The ZE contract size is simply one futures contract. Eurodollar futures options consist of American-style[2] call and put[3] options written on the underlying ED futures contract. A 3-month ED futures call option gives the holder the right but not the obligation to buy a 3-month ED futures contract. Now, investors who expect U.S. short-term interest rates to decline would also be expecting the price of the futures contract to increase. Thus, they might be inclined to purchase a 3-month ED futures call option to speculate on their belief. Hence, an exchange-listed interest rate futures call option is equivalent to a put option on the futures interest rate because of the inverse relationship between prices and interest rates, and the fact that exchange-listed interest rate futures options are quoted in units of price rather than percentage interest rates.

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2. An American option allows the holder to exercise the option on any date up to and including the maturity date—the maturity date is also referred to as the expiration date or the exercise date. European options only allow exercise on the expiration date. American options are always more expensive than European options with the same characteristics because of the added feature of early exercise. In general, the early exercise feature of American options makes these options more difficult to price than European options.
3. A call option gives the holder the right but not the obligation to buy the underlying asset at a predetermined strike price. A put option gives the holder the right but not the obligation to sell the asset at the strike price.

For notational convenience, exchange-listed call (put) options that are quoted in units of price are converted to put (call) options that have units of interest rate, that is, to percentage interest rates.

2. General theory

The valuation of interest rate futures options is best illustrated by first considering the pricing of European-style options. Let $\tilde{r}(t)$ denote the futures interest rate at time *t*—recall $\tilde{r}(t) = 100 - \tilde{p}(t)$, where $\tilde{p}(t)$ is the listed futures price at time *t*. Let *X* and *T* denote the strike price and the time to maturity of the option, respectively. Note that the strike price of a call option on the futures interest rate is equal to 100 minus the listed strike price of an interest rate futures put option. First, note that on their maturity dates the price of a call and put option will be

$$\begin{aligned} \tilde{C}(T, X) &= \max\{0, \tilde{r}(t) - X\} \equiv (\tilde{r}(t) - X)^{+} \\ \tilde{P}(T, X) &= \max\{0, X - \tilde{r}(t)\} \equiv (X - \tilde{r}(t))^{+} \end{aligned} . \tag{1}$$

Prior to maturity, European options are priced by taking the expectation of the discounted future cash flows. In this case, the future cash flows are the possible payouts of the options at maturity; see equation (1). The cash flows are discounted using the future values of the instantaneous risk-free rate. Thus, the value of European call and put options prior to maturity are given by the following formulae, respectively:

$$\begin{aligned} C(0, X) &= E_0\left[\exp\left\{-\int_0^T \tilde{r}_i(\tau)\ d\tau\right\}\ \tilde{C}(T, X)\right] \\ P(0, X) &= E_0\left[\exp\left\{-\int_0^T \tilde{r}_i(\tau)\ d\tau\right\}\ \tilde{P}(T, X)\right] \end{aligned} , \tag{2}$$

where E_0 represents the risk-neutral expectation, as opposed to the true or actual expectation, and $\tilde{r}_i(\tau)$ refers to the continuously compounded instantaneous interest rate. To simplify matters, the instantaneous rate is taken to be a fixed risk-free interest rate r_f. Strictly speaking, this assumption is incorrect, however it is common practice among market participants and academics alike.

Thus, the value of the European call and put options can then be expressed as:

$$\begin{aligned} C(0, X) &= \exp\{-r_f T\}\ E_0[(\tilde{r}(T) - X)^{+}] \\ P(0, X) &= \exp\{-r_f T\}\ E_0[(X - \tilde{r}(T))^{+}] \end{aligned} . \tag{3}$$

2.1 American-style interest rate futures options

Exchange- traded interest rate futures options are typically American-style options. Thus, the above pricing formulae for European-style options needs to be adjusted to account for the possibility of early exercise. Explicit formulae for American-style options are generally not available. However, Melick and Thomas (1997), Leahy and Thomas (1996), and Söderlind (1997) have shown that the following bounds can be placed on the prices of American-style currency futures options:

$$\begin{aligned} \bar{C}_A(0, X) &= E_0[\max\{0, \tilde{r}(T) - X\}] \\ \underline{C}_A(0, X) &= \max\{E_0[\tilde{r}(T)] - X, \exp(-r_f T)E_0[\max\{0, \tilde{r}(T) - X\}]\} \\ \bar{P}_A(0, X) &= E_0[\max\{0, X - \tilde{r}(T)\}] \\ \underline{P}_A(0, X) &= \max\{X - E_0[\tilde{r}(T)], \exp(-r_f T)E_0[\max\{0, X - \tilde{r}(T)\}]\} \end{aligned} . \tag{4}$$

American-style options can then be priced as a weighted average of the upper and lower bounds, namely:

$$\begin{aligned} C_\theta(0, X) &= \omega_i \bar{C}_A(0, X) + (1 - \omega_i) \underline{C}_A(0, X) \\ P_\theta(0, X) &= \omega_i \bar{P}_A(0, X) + (1 - \omega_i) \underline{P}_A(0, X) \end{aligned} \quad \text{where } i = 1,2 \text{ and } 0 \le \omega_i \le 1 . \tag{5}$$

Following Melick and Thomas (1997), the weights applied will depend on whether the particular option is in-the-money[4] or out-of-the-money. That is, by convention, i = 1 for in-the-money call or put options, and i = 2 for out-of-the-money call or put options.

2.2 General methodology

The formulae for the prices of European options, (3), can be written explicitly in terms of the risk-neutral PDF, $q[\tilde{r}(T)]$, as follows:

$$\begin{aligned} C(0, X) &= \exp\{-r_f T\} \int_X^\infty \{\tilde{r}(T) - X\}\ q[\tilde{r}(T)]\ d\tilde{r}(T) \\ P(0, X) &= \exp\{-r_f T\} \int_0^X \{X - \tilde{r}(T)\}\ q[\tilde{r}(T)]\ d\tilde{r}(T) \end{aligned} . \tag{6}$$

The risk-neutral PDF for the interest rate, $q[\tilde{r}(T)]$, provides the probabilities attached by a risk-neutral agent today (that is, time $t = 0$) to particular outcomes for future interest rates[5] that could prevail on the maturity date of the option contract.

Various methodologies have been proposed to obtain the risk-neutral PDF from observed futures option prices.[6] The techniques used in this paper—a full discussion follows later—all allow the risk-neutral

4. A European interest rate call (put) option is in-the-money if the futures interest rate is above (below) the strike interest rate, out-of-the-money if the futures interest rate is below (above) the strike interest rate, and at-the-money if the futures interest rate equals the strike interest rate.
5. In the context of this paper, the future interest rate refers to the 3-month eurodollar rate.

PDF to be expressed in a parametric form. Thus, it is helpful to introduce the following notation: let θ denote the parametric vector for the risk-neutral PDF—of course the makeup of this vector will vary depending on the technique being used. Now, let $C_\theta(0, X)$, and $P_\theta(0, X)$ be the theoretical call and put futures option prices with exercise price X [the theoretical prices are calculated from equation (5) with the aid of equations (4) and (6)]. Also, let $C(X)$ and $P(X)$ be the observed call and put futures option prices with exercise price X. Finally, let the theoretical interest rate futures price derived from the option-pricing model under risk-neutral density, $q[\tilde{r}(T)]$, be given by $F_\theta(0, T)$ (= $E_0[\tilde{r}(T)]$), and let the observed interest rate futures price be given by $F(0, T)$.

The parameters of the risk-neutral PDFs, θ, are estimated by minimizing the squared pricing errors associated with the call futures option prices, the put futures options prices, and the interest rate futures price. The minimization problem is:

$$\min_{\theta}\left[\sum_{i=1}^{n}[C(X_i) - C_\theta(0, X_i)]^2 + \sum_{j=1}^{m}[P(X_j) - P_\theta(0, X_j)]^2 + [F(0, T) - F_\theta(0, T)]^2\right] \quad (7)$$

where the number of call and put options are allowed to differ.

3. Overview of some specific techniques

As mentioned earlier, many techniques exist to extract risk-neutral PDFs from option prices. In this section, the theory behind some of the more common techniques is reviewed. In general, the techniques considered in this paper fall, with one exception, into two broad categories: a stochastic process for the evolution of the short-term interest rate is specified, or a parametric form for the risk-neutral PDF over the interest rate on the maturity date of the option is specified. The former category contains Black's model and a jump-diffusion model. The latter category contains methods based on a mixture of lognormal density functions and a Hermite polynomial expansion. The single exception is the method of maximum entropy.

3.1 Black's model

Black's model (1976) is the baseline model for pricing futures options. The model is very similar to the Black–Scholes model (1973). The futures interest rate, $\tilde{r}(t)$, is assumed to follow a lognormal process

$$d\tilde{r}(t) = \sigma\,\tilde{r}(t)\,dW(t) \quad (8)$$

6. There are four main methods of extracting risk-neutral PDFs from option prices: (i) specify a generalized stochastic process for the price of the underlying asset, (ii) specify a parametric form for the risk-neutral PDF, (iii) smooth the implied volatility function, and (iv) use non-parametric techniques. For a broad review of these techniques see Levin, Mc Manus, and Watt (1998).

where σ is the volatility of the futures interest rate, and dW is a Wiener process, that is *W(t)* is a geometric Brownian motion process in a risk-neutral world. For such a process, the risk-neutral PDF is a lognormal density:

$$q[\tilde{r}(T)] = \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}\ \ \tilde{r}(T)} \exp\left\{-\frac{1}{2}\left(\frac{\log(F(0,T)/\tilde{r}(T)) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)^2\right\} \quad , \tag{9}$$

where $F(0,T)$ is the interest rate futures rate. Furthermore, in Black's model the theoretical prices of European call and put futures options are given by

$$C_\theta(0, X) = \exp\{-r_f T\}\,[F(0,T)N(d_1) - XN(d_2)] \tag{10}$$

$$P_\theta(0, X) = \exp\{-r_f T\}\,[XN(-d_2) - F(0,T)N(-d_1)]\,, \tag{11}$$

where

$$d_1 = \frac{\log\{F(0,T)/X\}}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T} \tag{12}$$

and $N(x)$ represents the standardized cumulative normal probability distribution function evaluated at x.

At this point, it is worthwhile giving an example of how interest rate futures options are priced using Black's model. Consider the March 1999 ED futures and futures options listed on the CME on January 29, 1999. The March 1999 three-month ED futures contract had a listed settlement price of 95.04. The ZE call contract with strike price 95.00 had a settlement price of 0.060 and the ZE put contract with the same strike price had a settlement price of 0.020. The other inputs required for Black's model are the time-to-maturity of the contracts, the risk-free rate, and the instantaneous volatility. There are 45 days until the expiration of the contracts on March 15. Thus, the time to maturity is T = 0.125 (= 45/360). The risk-free rate is 4.97 per cent, which was calculated as weighted average of 30-day and 60-day eurodollar spot rates. The volatility is 6.02 per cent. First, convert the futures price and the strike price to interest rates. Thus $F(0,T)$ = 4.96 per cent (=100 – 95.04) and X = 5.00 per cent (= 100 – 95.00). Recall that a price call is equivalent to an interest rate put. Hence, the listed call can be priced by using equation (11) to yield a theoretical price of 0.065. The listed put can be priced using equation (10) to yield a theoretical price of 0.025. The theoretical prices are fairly close to the listed prices. Note that the discrepancies in the theoretical and listed price increase as the strike price moves away from the futures price. Table 1 compares the listed and theoretical option prices for a few different strike prices.

Table 1: Listed and theoretical prices of eurodollar futures options

Strike	CME call price	CME put price	Theoretical call price	Theoretical put price
94.875	0.170	0.005	0.167	0.003
95.000	0.060	0.020	0.065	0.025
95.125	0.020	0.105	0.012	0.097

The option contracts refer to March 1999 3-month eurodollar futures options. The CME prices are settle prices for these options for 29 January 1999. The settlement price for 3-month eurodollar futures contract on that date is 95.04. The theoretical prices are calculated using Black's model with a risk-free interest rate of 4.97 per cent and a volatility of 6.2 per cent.

3.2 Mixture of lognormals

A popular choice for the risk-neutral PDF is that of a weighted sum of independent lognormal density functions, which is referred to as a mixture of lognormals. Levin, Mc Manus, and Watt (1998) used this technique to extract the Canada–U.S. exchange rate from Canadian dollar futures options listed on the CME. The mixture of lognormal distributions is a flexible way to deal with departures from the assumptions underlying Black's model without having to specify a stochastic process for the evolution of the futures rate. As well, the mixture of lognormals has the advantage of retaining Black's model as a special subcase. The number of lognormals is usually dictated by the data constraints. Two lognormals are chosen for the present study.

The risk-neutral PDF with a weighted mixture of two lognormal distributions is given by

$$q[\tilde{r}(T)] = \phi_1 q_1[r(T)] + (1-\phi_1) q_2[\tilde{r}(T)], \tag{13}$$

where $0 < \phi_1 \le 1$ and

$$q_i[\tilde{r}(T)] = \frac{1}{\sqrt{2\pi}\sigma_i \ \tilde{r}(T)} \exp\left\{-\frac{1}{2}\left(\frac{\log(\tilde{r}(T)) - \mu_i}{\sigma_i}\right)^2\right\}, \text{ for } i = 1,2.$$

Black's model is given by the special case $\phi_1 = 1$, $\mu_1 = \log F(0, T) - \frac{1}{2}\sigma^2 T$ and $\sigma_1 = \sigma\sqrt{T}$.

The theoretical European call and put prices for the mixture of lognormals are

$$C_\theta(0, X) = \phi_1\left[\exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)N(d_1) - XN(d_2)\right]$$
$$+(1-\phi_1)\left[\exp\left(\mu_2 + \frac{1}{2}\sigma_2^2\right)N(d_3) - XN(d_4)\right]$$
$$P_\theta(0, X) = \phi_1\left[-\exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)N(-d_1) + XN(-d_2)\right] \tag{14}$$
$$+(1-\phi_1)\left[-\exp\left(\mu_2 + \frac{1}{2}\sigma_2^2\right)N(-d_3) + XN(-d_4)\right]$$

where

$$d_1 = \frac{1}{\sigma_1}[\mu_1 + \sigma_1^2 - \log(X)]\ ,\ d_2 = d_1 - \sigma_1$$
$$\mathrm{d}_3 = \frac{1}{\sigma_2}[\mu_2 + \sigma_2^2 - \log(X)]\ ,\ d_4 = d_3 - \sigma_2 \tag{15}$$

The theoretical futures price is given by

$$F_\theta(0, T) = \phi_1\exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right) + (1-\phi_1)\exp\left(\mu_2 + \frac{1}{2}\sigma_2^2\right)\ . \tag{16}$$

3.3 Jump diffusion

Black's model can be extended to account for asymmetries by adding a jump-diffusion process to Black's basic model. Thus, $\tilde{r}(T)$ is assumed to follow a lognormal jump-diffusion process. The evolution is characterized by two components, a lognormal process and a Poisson jump process,

$$d\tilde{r}(t) = (\mu - \lambda E[k])\,\tilde{r}(t)\,dt + \sigma_\omega\,\tilde{r}(t)\,dW(t) + k\tilde{r}(t)dq_{0,t}, \tag{17}$$

where $dq_{0,t}$ is a Poisson counter on the time interval (0,*t*), λ is the average rate of occurrence of the jumps, and *k* is the jump size. In other words, the probability that one jump occurs within the time interval *dt* is $\mathrm{Prob}[dq_{0,dt} = 1] = \lambda\ dt$ and the probability that no jumps occur is $\mathrm{Prob}[dq_{0,dt} = 0] = 1 - \lambda\ dt$. For simplicity, *k* is assumed to be constant. In general *k* is stochastic.

Bates (1991) showed that a European call could be priced as

$$C(0, X) = \exp\{-r_f T\}\sum_{n=0}^{\infty}\mathrm{Prob}\begin{bmatrix} n \text{ jumps} \\ \text{occur}\end{bmatrix}\ E_0\left[(\tilde{r}(T) - X)^+\middle|\ \begin{matrix} n \text{ jumps} \\ \text{occur}\end{matrix}\right], \tag{18}$$

where

$$\mathrm{Prob}\begin{bmatrix} n \text{ jumps} \\ \text{occur}\end{bmatrix} = \frac{(\lambda T)^n}{n!}\,e^{-\lambda T}\ .$$

A similar formula exists for European puts. For simplicity, assume that at most one jump can occur over the lifetime of the option [see Malz (1996, 1997)]. Ball and Torous (1983, 1985) call this the Bernoulli version of the model. The price of a European call then becomes

$$\begin{aligned} C_\theta(0, X) &= \exp\{-r_f T\} \ \text{Prob}\begin{bmatrix}\text{no jumps}\\ \text{occur}\end{bmatrix} \ E_0\left[(\tilde{r}(T) - X)^+ \middle| \begin{matrix}\text{no jumps}\\ \text{occur}\end{matrix} \right] \\ &\quad + \exp\{-r_f T\} \ \text{Prob}\begin{bmatrix}\text{1 jump}\\ \text{occurs}\end{bmatrix} \ E_0\left[(\tilde{r}(T) - X)^+ \middle| \begin{matrix}\text{1 jump}\\ \text{occurs}\end{matrix} \right] \\ &= (1 - \lambda T) \ \exp\{-r_f T\}\left[\frac{F(0, T)}{1 + \lambda k T} N(d_1) - X N(d_2)\right] \\ &\quad + \lambda T \ \exp\{-r_f T\}\left[\frac{F(0, T)}{1 + \lambda k T}(1 + k) N(d_3) - X N(d_4)\right] \end{aligned} \tag{19}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma_\omega \sqrt{T}}\left[\log\left(\frac{F(0, T)}{1 + \lambda k T}\right) - \log(X) + \frac{1}{2}\sigma_\omega^2 T\right], \quad d_2 = d_1 - \sigma_\omega \sqrt{T} \\ d_3 &= \frac{1}{\sigma_\omega \sqrt{T}}\left[\log\left(\frac{F(0, T)}{1 + \lambda k T}(1 + k)\right) - \log(X) + \frac{1}{2}\sigma_\omega^2 T\right], \quad d_4 = d_3 - \sigma_\omega \sqrt{T} \end{aligned} \tag{20}$$

The price of a European call then becomes

$$\begin{aligned} P_\theta(0, X) &= (1 - \lambda T) \ \exp\{-r_f T\}\left[-\frac{F(0, T)}{1 + \lambda k T} N(-d_1) + X N(-d_2)\right] \\ &\quad + \lambda T \ \exp\{-r_f T\}\left[-\frac{F(0, T)}{1 + \lambda k T}(1 + k) N(-d_3) + X N(-d_4)\right] \end{aligned} \tag{21}$$

Furthermore, the theoretical futures price is $F_\theta(0, T) = F(0, T)$. Note that the future interest rates conditional on no jump occurring and one jump occurring are

$$\begin{aligned} E_0\left[\tilde{r}(T) \middle| \begin{matrix}\text{no jumps}\\ \text{occur}\end{matrix}\right] &= \frac{F(0, T)}{1 + \lambda k T} \\ E_0\left[\tilde{r}(T) \middle| \begin{matrix}\text{1 jump}\\ \text{occurs}\end{matrix}\right] &= \frac{F(0, T)}{1 + \lambda k T}(1 + k) \end{aligned} \tag{22}$$

Thus, the option-pricing formulae consist of a weighted sum of Black's option-pricing formulae where the weights are given by the probability of no jumps occurring and one jump occurring over the lifetime of the option. The option-pricing formulae are very similar to the formula for the mixture of lognormals. Indeed, the jump diffusion is a subcase of the mixture of lognormals PDF. The jump-diffusion PDF is given by equation (13) with $\phi_1 = 1 - \lambda T$, $\mu_1 = \log F(0, T) - \log(1 + \lambda\kappa T) - \frac{1}{2}\sigma_\omega^2 T$, $\mu_2 = \mu_1 + \log(1 + \kappa)$, and $\sigma_1 = \sigma_\omega \sqrt{T} = \sigma_2$.

3.4 Hermite polynomial approximation

Asymmetries in the option data can also be modelled by adding perturbations to Black's baseline model. The Hermite polynomial approximation is a scheme to add perturbations such that successive perturbations are orthogonal. A Hermite polynomial expansion around the baseline lognormal solution is analogous to performing a Fourier expansion. Each additional term in the Hermite polynomial expansion is related to higher moments of the distribution. The general idea is that the Hermite polynomials act as a basis for the set of risk-neutral PDFs. In other words, the risk-neutral PDF can be approximated by a linear summation of Hermite polynomials—the more polynomials the better the approximation; in theory, an infinite series of polynomials gives an almost perfect fit. The technique was developed by Madan and Milne (1994) and later employed to price eurodollar futures options by Abken, Madan, and Ramamurtie (1996).

As a starting point, consider the following lognormal diffusion process:

$$d\tilde{r}(t) = \mu\,\tilde{r}(t)\,dt + \sigma\,\tilde{r}(t)\,dW(t)\,, \tag{23}$$

which can be solved to yield

$$\tilde{r}(t) = F(0,T)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z\right], \tag{24}$$

where z is distributed as standard normal, that is $z \sim \mathrm{N}(0,1)$. The Hermite polynomial adjustments are constructed with respect to the normalized variable

$$z = \frac{\log[\tilde{r}(T)/\mathrm{F}(0,T)] - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \tag{25}$$

The risk-neutral PDF for z is denoted by $Q(z)$ and can be written as

$$Q(z) = \lambda(z)n(z)\,, \tag{26}$$

where $n(z)$ is the reference PDF and $\lambda(z)$ captures departures from the reference PDF. The reference PDF is taken as the standardized unit normal PDF, $n(z) = \exp[-z^2/2]/\sqrt{2\pi}$. The departures from normality are captured by an infinite summation of Hermite polynomials, that is:

$$\lambda(z) = \sum_{k=0}^{\infty} b_k\phi_k(z)\,, \tag{27}$$

where b_k are constants and

$$\phi_k(z) = \frac{(-1)^k}{\sqrt{k!}}\,\frac{1}{n(z)}\,\frac{d^k n(z)}{dz^k} = \frac{-1}{\sqrt{k}}\,\frac{d\phi_{k-1}(z)}{dz} + \frac{1}{\sqrt{k}}\,z\,\phi_{k-1}(z) \tag{28}$$

are an orthogonal system of standardized Hermite polynomials.[7]

The price of any contingent claim payoff $g(z)$ is given by

$$\begin{aligned} V[g(z)] &= \exp\{-r_f T\}\ E_0[g(z)] = \exp\{-r_f T\}\int g(z)\sum_{k=0}^{\infty} b_k\phi_k(z)n(z)dz \\ &= \exp\{-r_f T\}\sum_{k=0}^{\infty} g_k\, b_k \end{aligned}, \tag{29}$$

where $g_k = \int g(z)\phi_k(z)n(z)\,dz$. Now, European call and put options have the contingent claim payoffs, respectively

$$\begin{aligned} g(z;\text{call}) &= \left(F(0,T)\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z\right]-X\right)^+ \\ g(z;\text{put}) &= \left(X-F(0,T)\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z\right]\right)^+ \end{aligned}. \tag{30}$$

Thus, European call and put prices can be written as

$$\begin{aligned} C_\theta(0,X) &= \exp\{-r_f T\}\sum_{k=0}^{\infty}\alpha_k\, b_k \\ P_\theta(0,X) &= \exp\{-r_f T\}\sum_{k=0}^{\infty}\beta_k\, b_k \end{aligned}, \tag{31}$$

where $\alpha_k = \int g(z;\text{call})\phi_k(z)n(z)\,dz$ and $\beta_k = \int g(z;\text{put})\phi_k(z)n(z)\,dz$. Madan and Milne (1994) show that

$$\alpha_k = \frac{1}{\sqrt{k!}}\left.\frac{\partial^k\Phi(u)}{\partial u^k}\right|_{u=0}, \tag{32}$$

where the generating function $\Phi(u)$ is given by

$$\begin{aligned} \Phi(u) &= F(0,T)\ \exp\{\mu T+\sigma\sqrt{T}u\}\ N[d_1(u)]-XN[d_2(u)] \\ d_1(u) &= \frac{\log\{F(0,T)/X\}}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}+u \\ d_2(u) &= d_1(u)-\sigma\sqrt{T} \end{aligned}, \tag{33}$$

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7. The first four standardized Hermite polynomials are $\phi_0(z) = 1$, $\phi_1(z) = z$, $\phi_2(z) = (z^2-1)/\sqrt{2}$, $\phi_3(z) = (z^3-3z)/\sqrt{6}$ and $\phi_4(z) = (z^4-6z^2+3)/\sqrt{24}$. Higher-order Hermite polynomials can be easily calculated using the recurrence relationship $\phi_k(z) = \frac{z}{\sqrt{k}}\phi_{k-1}(z)-\sqrt{\frac{k-1}{k}}\phi_{k-2}(z)$. The polynomials are orthogonal because $\int_{-\infty}^{\infty}\phi_k(z)\phi_j(z)n(z)\ dz$ equals one if $k = j$ and zero otherwise.

and that

$$\beta_k = \begin{cases} \alpha_0 + X - F(0,T)\exp\{\mu T\} & \text{if } k = 0 \\ \alpha_k - \dfrac{\sigma\sqrt{T}}{\sqrt{k!}}\, F(0,T)\exp\{\mu T\} & \text{if } k > 0 \end{cases} . \tag{34}$$

For empirical work, the Hermite polynomial expansion must be truncated at a finite order in *z*. Two approximations are considered in this paper, a fourth-order and a sixth-order approximation. First consider the sixth-order approximation. The risk-neutral PDF for the sixth-order Hermite approximation is given by:

$$Q(z) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]\left[\left(b_0 - \frac{b_2}{\sqrt{2}} + \frac{3b_4}{\sqrt{24}} - \frac{15b_6}{\sqrt{720}}\right) + \left(b_1 - \frac{3b_3}{\sqrt{6}} + \frac{15b_5}{\sqrt{120}}\right)z + \left(\frac{b_2}{\sqrt{2}} - \frac{6b_4}{\sqrt{24}} + \frac{45b_6}{\sqrt{720}}\right)z^2 + \left(\frac{b_3}{\sqrt{6}} - \frac{10b_5}{\sqrt{120}}\right)z^3 + \left(\frac{b_4}{\sqrt{24}} - \frac{15b_6}{\sqrt{720}}\right)z^4 + \frac{b_5}{\sqrt{120}}z^5 + \frac{b_6}{\sqrt{720}}z^6\right] . \tag{35}$$

Under the reference measure, *z* is normally distributed with a mean of 0 and a variance of 1. Under the measure *Q(z)*, *z* has mean $E_Q[z] = b_1$ and variance $E_Q[(z - E_Q[z])^2] = b_0 + \sqrt{2}b_2 - b_1^2$. Furthermore, $\int Q(z)dz = b_0$. Thus, the restriction $b_0 = 1$ must be imposed to insure that the PDF Q integrates to unity. The following restrictions on b_1 and b_2, $b_1 = 0$ and $b_2 = 1$ are imposed to insure that *z* to have mean zero and unit variance with respect to the probability density *Q(z)*. Hence, under the above restrictions the risk-neutral PDF for *z* is

$$Q(z) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]\left[\left(1 + \frac{3b_4}{\sqrt{24}} - \frac{15b_6}{\sqrt{720}}\right) + \left(-\frac{3b_3}{\sqrt{6}} + \frac{15b_5}{\sqrt{120}}\right)z + \left(-\frac{6b_4}{\sqrt{24}} + \frac{45b_6}{\sqrt{720}}\right)z^2 + \left(\frac{b_3}{\sqrt{6}} - \frac{10b_5}{\sqrt{120}}\right)z^3 + \left(\frac{b_4}{\sqrt{24}} - \frac{15b_6}{\sqrt{720}}\right)z^4 + \frac{b_5}{\sqrt{120}}z^5 + \frac{b_6}{\sqrt{720}}z^6\right] \tag{36}$$

and the risk-neutral PDF for $\tilde{r}(T)$ is

$$q[\tilde{r}(T)] = \frac{1}{\sigma\sqrt{T}}\,\frac{1}{\tilde{r}(T)}Q\left[\frac{\log[\tilde{r}(T)/F(0,T)] - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right]. \tag{37}$$

Finally, the futures price is given by

$$F_\theta(0,T) = F(0,T)\exp\{\mu T\}\left[\sum_{k=0}^{6}\frac{b_k}{\sqrt{k!}}(\sigma\sqrt{T})^k\right] . \tag{38}$$

The fourth-order approximation is simply given by setting $b_5 = 0$ and $b_6 = 0$ in the above equations.

3.5 Method of maximum entropy

The concept of entropy originated in the world of classical thermodynamics as a measure of the state of disorder of a system. Shannon (1948) later introduced the idea to information theory, where entropy was taken as a measure of missing information. Jaynes (1957, 1982) extended the idea to the field of statistical interference using the principle of maximum entropy (PME). Buchen and Kelly (1996) applied the PME to estimating risk-neutral PDFs from option prices. This estimate "will be the least prejudiced estimate, compatible with the given price information in the sense that it will be maximally noncommittal with respect to missing or unknown information."

The PME is a Bayesian method of statistical inference that only uses the price information given and makes no parametric assumptions about the form of the risk-neutral PDF. The method starts with a definition of the entropy of a distribution q:

$$S(q) = -\int_0^{\infty} q(x) \log[q(x)]\, dx, \tag{39}$$

which is maximized subject to the constraints

$$\begin{aligned} 1 &= \int_0^{\infty} q(x)\ dx \\ C_i &= \exp\{-r_f T\} \int_0^{\infty} q(x)\ c_i(x)\ dx \qquad \text{where } i = 1 \ldots m, \\ F(0, T) &= \int_0^{\infty} x\, q(x)\ dx \end{aligned} \tag{40}$$

where C_i is the market price of the contingent claim whose payoff at time T is given by $c_i(x)$. The risk-neutral PDF is then given by:

$$\begin{aligned} q(x) &= \frac{1}{\mu} \exp\left\{\lambda_0 x + \sum_{i=1}^{m} \lambda_i c_i(x)\right\} \\ \mu &= \int_0^{\infty} \exp\left\{\lambda_0 x + \sum_{i=1}^{m} \lambda_i c_i(x)\right\} dx \end{aligned} . \tag{41}$$

Now suppose that the contingent claims consist of European call and put options. Estimating the parameters $\{\lambda_i\}_{i=0}^{m}$ is simplified if only one type of contingent claim is used. Thus, convert the put options to call options using put–call parity. Hence, a futures put option with strike price X and observed price P is converted to a futures call option with the same strike price and observed price

$C = P - \exp\{-r_f T\} \ [X - F(0, \mathrm{T})]$. For notational convenience, order the resulting set of call options in terms of increasing strike prices, that is, $X_1 < X_2 < \ldots < X_m$.

The futures contract is also considered to be a call option with strike price $X_0 = 0$ and an observed price of $C_0 = \exp\{-r_f T\} \ F(0, T)$. Thus, the constraints for the futures contract and the futures call options can be written as

$$C_i = \exp\{-r_f T\} \int_0^{\infty} q(x)\,(x - X_i)^+ \, dx \qquad \text{where } i = 0 \ldots m \quad . \tag{42}$$

Coutent, Jondeau, and Rockinger (1998) show that the risk-neutral PDF can be written as:

$$q(x) = \begin{cases} \dfrac{1}{\mu} \exp[a_i x + b_i] & \text{for } X_i \le x < X_{i+1} \text{ where } i = 0 \ldots m-1 \\ \dfrac{1}{\mu} \exp[a_m x + b_m] & \text{for } X_m \le x \end{cases} \tag{43}$$

where $a_i = a_{i-1} + \lambda_i$ for $i \ge 1$ with $a_0 = \lambda_0$ and $b_i = b_{i-1} - (a_i - a_{i-1}) X_i$ for $i \ge 1$ with $b_0 = 0$. The normalization constant is given by

$$\mu = -\frac{1}{a_m} \exp[a_m X_m + b_m] + \sum_{i=0}^{m-1} \frac{1}{a_i} \{\exp[a_i X_{i+1} + b_i] - \exp[a_i X_i + b_i]\}. \tag{44}$$

Furthermore, the theoretical European call price for strike price X_i is given by C_i where

$$\exp\{r_f T\} \ C_i = \left\{ \begin{array}{l} -\left(\dfrac{X_m - X_i}{a_m} - \dfrac{1}{a_m^2}\right) \exp[a_m x + b_m] \\ + \displaystyle\sum_{k=i}^{m-1} \left\{ \left(\dfrac{X_{k+1} - X_i}{a_k} - \dfrac{1}{a_k^2}\right) \exp[a_k X_{k+1} + b_k] - \left(\dfrac{X_k - X_i}{a_k} - \dfrac{1}{a_k^2}\right) \exp[a_k X_k + b_k] \right\} \end{array} \right. \tag{45}$$

[see Coutent, Jondeau, and Rockinger (1998) for details].

The risk-neutral PDF is characterized by the parameters $\{a_i\}_{i=0}^{m}$, which are estimated by minimizing the squared call pricing errors; see equation (7). The convergence of the estimation process is enhanced by picking initial values for the parameters that are reasonable. Coutent, Jondeau, and Rockinger (1998) suggest choosing parameters so that the risk-neutral PDF, (43), is approximately equal to Black's risk-neutral PDF, (9).[8]

4. Data

The data consist of end-of-the-day settlement prices for American-style eurodollar futures options and eurodollar futures that are traded on the CME, and covers 23 September 1998 to 30 September 1998, inclusively. The dates were chosen to include the Federal Open Market Committee (FOMC) meeting on 29 September 1998. The data also consists of 60- and 90-day spot eurodollar rates. The risk-free rate was constructed by linearly interpolating between these rates and then converting the result to a continuously compounded rate.

The average daily trading volume of the ED futures and the Dec98 ED futures options over the period 23–30 September 1998 was 110,217 and 74,030 contracts, respectively. The average number of Dec98 ED futures options traded was 16. These contracts had a wide range of strike prices, typically from 4.0 per cent to 6.5 per cent (see Table 2 for details).

5. Comparing the models

The various methods outlined in section 3 are compared in this section. First, the models are compared according to their pricing errors—the pricing error is the difference between the theoretical option price and the observed option price. Second, the models are compared using several summary statistics, notably the mean, annualized volatility, skewness, and kurtosis (see the Appendix for further discussion on these quantities). Third, the models are compared by examining the risk-neutral PDFs. This comparison is both graphical and analytic—the analytic analysis consists of comparing the cumulative distribution functions for the various PDFs.

5.1 Metrics for comparison

As mentioned above, the models are compared by examining the pricing errors associated with each model. The pricing error, which is the basic building block, is the difference between the theoretical option price and the observed option price. Thus, the pricing errors for call and put futures options are $C_\theta(0, X_i) - C(X_i)$ and $P_\theta(0, X_j) - P(X_j)$, respectively. These raw pricing errors are illustrated in Figures 1 through 6 [hollow bullets (o) indicate pricing errors for call options and asterisks (*) indicate pricing errors for put options]. Strike prices are marked along the horizontal axis. Black's model clearly gives the highest pricing errors. The method of maximum entropy appears to give the lowest pricing errors. The mixture of lognormals and the Hermite polynomial approximations yield similar pricing errors. Not surprisingly, the mixture of lognormals

8. The initial values of the parameters s $\{a_i\}_{i=0}^{m}$ can be estimated as follows. First, generate a data set of interest rates, $\{x\}$, and the corresponding Black's risk-neutral PDF, $\{q_B(x)\}$. Next, estimate the parameters $\{\lambda_i\}_{i=0}^{m}$ for the regression $\log[q_B(x)] = -\log\mu + \lambda_0 x + \sum_{i=1}^{m}\lambda_i (x - X_i)^+ + \varepsilon$. The initial values are then according to the algorithm used for equation (43).

method has smaller pricing errors than the jump-diffusion method, and the sixth-order Hermite polynomial approximation has smaller pricing errors than the fourth-order Hermite approximation. The mixture of lognormals and both the Hermite methods tend to have similar pricing errors.

An alternative to looking at the raw pricing errors is to combine the pricing errors into a single quantity that measures the accuracy of fit. Several measures of accuracy of fit exist in the literature. However, only two measures will be considered in this paper: the mean squared error (MSE) and the mean squared percentage pricing error (MSPE). The choice of measures is motivated by the fact that the loss function (7) is quadratic in the pricing errors. The MSE and the MSPE are calculated as follows:

$$\begin{aligned} \text{MSE} &= \frac{1}{n+m-k}\sum_{i=1}^{n}[C(X_i)-C_\theta(0,X_i)]^2+\frac{1}{n+m-k}\sum_{j=1}^{m}[P(X_j)-P_\theta(0,X_j)]^2 \\ \text{MSPE} &= \frac{1}{n+m-k}\sum_{i=1}^{n}\left[\frac{C(X_i)-C_\theta(0,X_i)}{C(X_i)}\right]^2+\frac{1}{n+m-k}\sum_{j=1}^{m}\left[\frac{P(X_j)-P_\theta(0,X_j)}{P(X_j)}\right]^2 \end{aligned} \tag{46}$$

where n and m are the number of observed call and put prices, and k is the number of independent parameters for the risk-neutral PDF being used, $k = \#\{\theta\}$. The MSE places more weight on larger errors than smaller errors. The MSPE is dimensionless, and thus facilitates comparison across both different methods and different data sets.

Neither the MSE nor the MSPE measures point to a single method that always ranks first. However, averaging the measures over the sample period yields a clear ranking. Both the MSE and the MSPE measures rank the mixture of lognormal method first, the sixth-order Hermite polynomial approximation a close second, and the fourth-order Hermite approximation third (see Table 3). The results may of course be dependent on the ranking scheme employed. However, the other ranking schemes that were considered ranked the mixture of lognormals method first and either one of the Hermite approximations or the method of maximum entropy second. Finally, the results may be dependent on the sample. Only further testing with more diverse data sets will resolve this issue.

5.2 Summary statistics

The models can also be compared according to summary statistics that are calculated with respect to the logarithm of the futures rate. The standard statistics examined are the mean, annualized volatility, skewness, and kurtosis (see the Appendix for a more in-depth explanation). For any given day, the means calculated from each model are practically identical. This result is not too surprising, given that the PDFs are risk-neutral.

The evolution of volatility over the event period follows a fairly consistent pattern. All methods have volatility increasing from 23 September to 24 September, decreasing from 28 September to 29 September, and increasing again from 29 September to 30 September (see Figure 7 and Tables 4a to 9a). The level of volatility from 24 September to 28 September varies across models. On average, the mixture of lognormals yields the highest estimates of volatility and Black's model yields the lowest estimates. Also, the jump model tends to yield higher volatilities than the sixth-order Hermite approximation, the sixth-order Hermite approximation tends to yield higher volatilities than the fourth-order Hermite approximation, and the fourth-order Hermite approximation tends to yield higher volatilities than the method of maximum entropy.

The skewness estimates vary widely across the models (see Figure 7), although all models have negative skewness for each day of the study period. However, no consistent pattern exists for the day-to-day evolution of skewness across methods. For example, from 25 September to 28 September, the mixture of lognormals method measure of skewness becomes more negative while both the Hermite approximations becomes less negative. Likewise, the kurtosis estimates vary dramatically across models. All the models do, however, yield kurtosis numbers greater than 3, indicating fat-tailed (leptokurtotic) distributions.

In summary, the lower moments of the distribution, namely the mean and the volatility, tend to be consistent across models. But the discrepancies between the distributions tend to be exaggerated when higher moments are considered. The skewness and kurtosis measures appear to be very model-dependent, and thus are probably not reliable as indicators of market sentiment.

5.3 The shape of things to come

The risk-neutral PDFs implied by the various models for 23 September to 30 September are illustrated in Figures 1 through 6. The PDFs for the mixture of lognormals method, the jump-diffusion method, and the Hermite polynomial-approximation methods are invariably bimodal. The higher peak is situated almost directly above the futures rate, and in most cases a much lower second peak is situated above a eurodollar rate that is roughly 100 basis points lower than the futures rate (see Figures 1through 6). However, most of the mixture of lognormal risk-neutral PDFs have no lower peak. Instead, they have heavy left tails, indicating negative skewness. The Black risk-neutral PDF is always unimodal. The method of maximum entropy risk-neutral PDF is extremely spiky for all the dates considered. The method of maximum entropy estimates one parameter for every strike price, and thus tends to overfit when there is a large number of strike prices, which is the case in this study. Furthermore, the method of maximum entropy PDFs appear choppy because the first derivative of the PDF is discontinuous at the strike prices.

The cumulative distribution functions (CDFs) are helpful in comparing models. The CDFs are more easily interpreted than the PDFs, since they give the probabilities that the futures rate will be less than a given rate on the maturity date of the futures contract. (Analytic expressions for the CDFs for the various models are in the Appendix.) A selection of the probabilities can be found in Tables 4b through 9b. The CDFs are plotted in Figures 8a through 8d. Black's model consistently underestimates the probabilities in the left tail of the distribution compared with the other models. Not surprisingly, the method of maximum entropy CDF is very different from the other CDFs. The CDFs for the mixture of lognormals method, and the fourth- and sixth-order Hermite polynomial approximation are very close to each other, as can seen both from Tables 4b, 5b, 6b, 7b, 8b, and 9b and from Figure 8d. (For clarity, the aforementioned CDFs are only plotted in Figure 8d).

5.4 General comments on estimation procedures

The method of maximum entropy tends to overfit. This is directly related to the small number of degrees of freedom. Furthermore, the estimation procedure was the slowest to converge. The mixture of lognormals method can also be slow to converge, especially if the true risk-neutral PDF is close to being lognormal. The problem is that there is not a unique set of parameter values that gives a lognormal distribution. Likewise, the jump-diffusion method is plagued by the same problem. The jump-diffusion method works well when there is a reasonable likelihood of a jump occurring. However, as with the mixture of lognormals method, the jump-diffusion method has degenerate parameterizations for lognormal distributions. The Hermite polynomial-approximation methods are quick to converge and do not admit degenerate parameterizations. The Hermite method always converges; the fourth-order approximation converges faster than the sixth-order approximation. The only drawback with the Hermite polynomial-approximation methods is that the estimation of the risk-neutral PDF can occasionally yield negative probability values. These negative probability values can occur because the Hermite method employed is an approximation method that involves truncating an infinite series.

Overall, the mixture of lognormals method and the sixth-order Hermite polynomial-approximation method are probably the best methods to use for extracting risk-neutral PDFs from interest rate option prices. Coutant, Jondeau, and Rockinger (1998) favoured the fourth-order Hermite polynomial-approximation method in their comparison of various methods using French data.

Finally, given the variability of the skewness estimates across methods and the relative consistency of the CDFs, a more accurate measure of skewness could probably be constructed by comparing the tails of the PDFs as opposed to using the third central moment of the distribution. Such a measure exists in the literature: relative intensity [see Campa, Chang, and Reider (1997)] compares the likelihood of large upward movements in the eurodollar rate to large downward movements.

6. The event

As mentioned earlier, the dates of the study were chosen to coincide with the FOMC meeting on 29 September 1998. The FOMC is a 12-member committee, consisting of the seven members of the Board of Governors of the Federal Reserve System, the president of the Federal Reserve Bank of New York, and four of the presidents of the other 11 Reserve Banks; the latter positions rotate yearly.

The FOMC meets eight times a year and has primary responsibility for conducting monetary policy. The committee decides on the desired level of the federal funds rate. Press releases are often posted immediately after meetings, especially if the Fed’s stance on monetary policy has changed. For example, the press release following the 29 September 1998 meeting started: “The Federal Open Market Committee decided today to ease the stance of monetary policy slightly, expecting the federal funds rate to decline 1/4 percentage point to around 5 1/4 per cent.” This reduction was the first of a series of reductions in the Fed fund target rate in 1998. Two later reductions of 25 basis points each occurred on 15 October 1998 and 17 November 1998.

The annualized volatility numbers generally increased over the first half of the period—based on the results of the previous section, the analysis of the present section uses the risk-neutral PDF from the mixture of lognormals method—starting off at 17.82 per cent on 23 September, rising to a high of 19.20 per cent on 28 September, falling to a low of 15.24 per cent on 29 September, and finally starting upwards again on 30 September to 16.92 per cent. Thus, uncertainty, as measured by annualized volatility, initially increased, and peaked the day prior to the FOMC meeting. Uncertainty reduced on the day of the meeting but started to increase again the following day.

The probability of the ED futures rate being below 5.00 per cent on 14 December 1998 rose from 33 per cent to 38 per cent over the period. In addition, the probability of the ED futures rate being below 5.25 per cent rose from 63 per cent to 75 per cent. Furthermore, the skewness numbers remained negative over the entire period, indicating a bearish market tone. Interestingly, skewness became even more negative the day after the Fed easing, indicating that a further Fed easing was expected by some market participants. These findings are consistent with the general market views of the time. Anecdotal evidence suggests that, while market participants anticipated an easing at the 29 September FOMC meeting, some were disappointed by the size of the move (25 basis points) and immediately priced in a further rate reduction by the November meeting.

7. Conclusion

The information content of exchange-traded eurodollar futures options were examined in this paper. Several techniques for extracting risk-neutral PDFs from ED futures option prices were compared. The mixture of lognormals method ranked first, with both the lowest MSE and MSPE. However, this method is occasionally slow to converge due to degeneracies in the parameter space. Typically, the lack of convergence occurs when the risk-neutral PDF appears to be close to a single lognormal distribution. In this case, the alternative sixth-order Hermite polynomial-approximation method yields better results. The Hermite method is quick to converge and gives comparable results to the mixture of lognormals method. However, the method occasionally yields PDFs that have negative probabilities—these negative probabilities are an artifact of the approximation method and are not too worrisome, since they tend to occur near the tails of the distribution.

The higher central moments of the risk-neutral PDFs, namely skewness and kurtosis, are unstable across estimation techniques and thus are probably not overly informative as measures of asymmetry in market sentiment. In contrast, the CDF was found to be stable across the three methods that yielded the lowest MSPEs, namely the mixture of lognormals and the two Hermite polynomial-approximation methods. Thus, measures of skewness based on the CDF are probably more appropriate. One candidate is relative intensity, which compares the likelihood of large upward movements in the ED rate to the likelihood of large downward movements.

Risk-neutral PDFs are useful tools for monitoring market sentiment, as was indicated by the analysis of the 29 September 1998 FOMC meeting. Various methods were used to extract risk-neutral PDFs from ED futures options over the period around the FOMC meeting in order to examine the evolution of market sentiment over the future values of ED rates. Uncertainty grew in the market prior to the meeting and abated on the day of the meeting, only to increase again the following day. Market participants remained bearish on future ED rates both prior to and after the Fed easing, indicating that some of them expected further rate cuts.

Information extracted from option prices can be used to monitor market sentiment. However, the best way to present this information is still up for debate. In particular, work needs to be done on appropriate measures of asymmetry and the predictive power of these measures.

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Table 2: Federal Open Market Committee meeting, September 1998

September 1998	60-day euro-dollar rate	90-day euro-dollar rate	Risk-free rate	Euro-dollar futures rate	Trading volume of euro-dollar futures	Number of different option contracts	Trading volume of euro-dollar futures options
Wednesday 23	5.5313	5.5000	5.3620	5.115	101,026	16	79,626
Thursday 24	5.5000	5.4688	5.3333	5.035	121,205	18	74,215
Friday 25	5.3907	5.3594	5.2306	5.040	124,453	15	96,714
Monday 28	5.3594	5.3282	5.2039	5.060	78,949	15	84,918
Tuesday 29	5.3438	5.3750	5.2217	5.110	142,304	14	50,615
Wednesday 30	5.3594	5.4063	5.2430	5.050	93,363	18	58,089

Note: The day of Federal Open Market Committee meeting is highlighted.

Table 3: Eurodollar futures options:
Pricing errors for call and put futures options, September 1998

Measure	Model	23 Sept.	24 Sept.	25 Sept.	28 Sept.	29 Sept.	30 Sept.	Average	Ranking
Mean squared error ($\times 10^{-5}$)	Black	10.610	8.066	9.771	8.626	7.000	9.230	8.884	6
	MLN[a]	0.777	0.558	0.987	0.863	0.796	1.108	0.848	1
	Jump	1.482	0.604	0.930	0.928	1.983	1.735	1.277	4
	Hermite (4)	0.857	0.591	1.193	1.046	0.945	1.099	0.955	3
	Hermite (6)	0.779	0.608	1.093	0.892	0.667	1.181	0.870	2
	Maximum entropy	2.763	2.974	0.945	0.720	0.609	2.588	1.767	5
Mean squared per-centage pricing error ($\times 10^{-2}$)	Black	7.458	12.754	16.310	18.210	9.302	18.798	13.805	5
	MLN	0.292	0.151	0.201	3.401	0.179	2.735	1.160	1
	Jump	7.071	0.121	0.209	6.914	1.357	5.054	3.454	4
	Hermite (4)	0.249	0.378	2.221	4.808	0.098	2.731	1.748	3
	Hermite (6)	0.896	0.563	0.538	1.887	0.114	3.506	1.251	2
	Maximum entropy	60.805	40.399	4.005	3.146	0.560	38.907	24.637	6
See Section 5.1 for further details.									

a. Mixture of lognormals

Table 4a: Eurodollar futures options, 23 September 1998

23 September	Mean	Volatility	Skewness	Kurtosis
Black	1.629	15.85	0	3
MLN[a]	1.629	17.82	–0.956	5.877
Jump	1.629	17.52	–1.133	5.254
Hermite (4)	1.629	17.66	–0.866	5.438
Hermite (6)	1.629	17.26	–0.719	3.774
Maximum entropy	1.627	16.55	–1.170	5.341

a. Mixture of lognormals

Table 4b: Eurodollar futures options, 23 September 1998
Probabilities for the eurodollar rate on 14 December 1998

23 September	$\mathrm{Prob}[\tilde{r}(T) \leq R]$					
	4.50	**4.75**	**5.00**	**5.25**	**5.50**	**5.75**
Black	0.05	0.17	0.40	0.65	0.84	0.94
MLN[a]	0.08	0.14	0.32	0.63	0.87	0.96
Jump	0.07	0.14	0.33	0.62	0.85	0.96
Hermite (4)	0.08	0.13	0.33	0.63	0.86	0.96
Hermite (6)	0.10	0.14	0.32	0.63	0.87	0.96
Maximum entropy	0.10	0.14	0.30	0.70	0.83	0.99

The probabilities are the risk-neutral probabilities that the market assigns on the given date for the eurodollar rate on 14 December 1998 to be less than the stated *R* value. See the Appendix for details.

a. Mixture of lognormals

Table 5a: Eurodollar futures options, 24 September 1998

24 September 1998	Mean	Volatility	Skewness	Kurtosis
Black	1.613	16.97	0	3
MLN[a]	1.613	18.35	–0.808	3.987
Jump	1.613	18.47	–0.978	4.801
Hermite (4)	1.613	18.37	–0.806	4.524
Hermite (6)	1.613	18.48	–0.998	4.592
Maximum entropy	1.611	17.56	–1.023	4.622

a. Mixture of lognormals

Table 5b: Eurodollar futures options, 24 September 1998
Probabilities for the eurodollar rate on 14 December 1998

24 September 1998	$\text{Prob}[\tilde{r}(T) \leq R]$					
	4.50	**4.75**	**5.00**	**5.25**	**5.50**	**5.75**
Black	0.09	0.25	0.48	0.71	0.87	0.95
MLN[a]	0.09	0.20	0.43	0.70	0.88	0.97
Jump	0.09	0.20	0.43	0.70	0.89	0.97
Hermite (4)	0.10	0.20	0.43	0.70	0.89	0.97
Hermite (6)	0.09	0.20	0.43	0.70	0.88	0.97
Maximum entropy	0.12	0.18	0.47	0.68	0.89	0.99

The probabilities are the risk-neutral probabilities that the market assigns on the given date for the eurodollar rate on 14 December 1998 to be less than the stated *R* value. See the Appendix for details.

a. Mixture of lognormals

Table 6a: Eurodollar futures options, 25 September 1998

25 September 1998	Mean	Volatility	Skewness	Kurtosis
Black	1.614	16.75	0	3
MLN[a]	1.614	19.03	–1.208	5.646
Jump	1.613	19.19	–1.280	6.305
Hermite (4)	1.614	18.26	–0.813	4.735
Hermite (6)	1.614	18.96	–1.227	5.971
Maximum entropy	1.614	18.26	–0.697	3.680

a. Mixture of lognormals

Table 6b: Eurodollar futures options, 25 September 1998
Probabilities for the eurodollar rate on 14 December 1998

25 September 1998	$\text{Prob}[\tilde{r}(T) \leq R]$					
	4.50	**4.75**	**5.00**	**5.25**	**5.50**	**5.75**
Black	0.08	0.24	0.48	0.71	0.87	0.96
MLN[a]	0.08	0.19	0.42	0.69	0.88	0.97
Jump	0.08	0.19	0.43	0.69	0.88	0.97
Hermite (4)	0.09	0.19	0.42	0.70	0.89	0.97
Hermite (6)	0.07	0.19	0.43	0.70	0.88	0.97
Maximum entropy	0.13	0.15	0.49	0.67	0.90	0.96

The probabilities are the risk-neutral probabilities that the market assigns on the given date for the eurodollar rate on 14 December 1998 to be less than the stated *R* value. See the Appendix for details.

a. Mixture of lognormals

Table 7a: Eurodollar futures options, 28 September 1998

28 September 1998	Mean	Volatility	Skewness	Kurtosis
Black	1.619	16.21	0	3
MLN[a]	1.618	19.20	–1.712	10.699
Jump	1.618	18.66	–1.563	7.842
Hermite (4)	1.619	17.55	–0.749	5.017
Hermite (6)	1.618	18.51	–1.168	6.897
Maximum entropy	1.617	17.43	–0.596	4.681

a. Mixture of lognormals

Table 7b: Eurodollar futures options, 28 September 1998
Probabilities for the eurodollar rate on 14 December 1998

28 September 1998	$\text{Prob}[\tilde{r}(T) \leq R]$					
	4.50	**4.75**	**5.00**	**5.25**	**5.50**	**5.75**
Black	0.06	0.21	0.45	0.70	0.87	0.96
MLN[a]	0.06	0.16	0.40	0.69	0.89	0.97
Jump	0.06	0.16	0.40	0.69	0.89	0.97
Hermite (4)	0.08	0.16	0.39	0.69	0.90	0.97
Hermite (6)	0.05	0.16	0.41	0.69	0.89	0.98
Maximum entropy	0.11	0.14	0.45	0.67	0.89	0.98

The probabilities are the risk-neutral probabilities that the market assigns on the given date for the eurodollar rate on 14 December 1998 to be less than the stated *R* value. See the Appendix for details.

a. Mixture of lognormals

Table 8a: Eurodollar futures options, 29 September 1998

29 September 1998	Mean	Volatility	Skewness	Kurtosis
Black	1.629	13.74	0	3
MLN[a]	1.629	15.24	–0.711	6.681
Jump	1.629	15.46	–1.754	9.955
Hermite (4)	1.629	15.07	–0.608	6.026
Hermite (6)	1.629	14.45	–0.952	3.206
Maximum entropy	1.629	14.86	–0.718	5.984

a. Mixture of lognormals

Table 8b: Eurodollar futures options, 29 September 1998
Probabilities for the eurodollar rate on 14 December 1998

29 September 1998	$\mathrm{Prob}[\tilde{r}(T) \le R]$					
	4.50	**4.75**	**5.00**	**5.25**	**5.50**	**5.75**
Black	0.02	0.12	0.37	0.68	0.89	0.97
MLN[a]	0.05	0.11	0.30	0.70	0.92	0.97
Jump	0.03	0.10	0.33	0.67	0.90	0.98
Hermite (4)	0.06	0.09	0.31	0.69	0.92	0.98
Hermite (6)	0.08	0.09	0.30	0.71	0.90	0.95
Maximum entropy	0.06	0.10	0.30	0.71	0.92	0.96

The probabilities are the risk-neutral probabilities that the market assigns on the given date for the eurodollar rate on 14 December to be less than the stated *R* value. See the Appendix for details.

a. Mixture of lognormals

Table 9a: Eurodollar futures options, 30 September 1998

30 September 1998	Mean	Volatility	Skewness	Kurtosis
Black	1.617	14.19	0	3
MLN[a]	1.617	16.92	–1.434	8.794
Jump	1.617	16.88	–1.755	8.810
Hermite (4)	1.618	15.58	–0.848	5.623
Hermite (6)	1.618	15.39	–0.700	5.294
Maximum entropy	1.615	15.09	–1.216	5.216

a. Mixture of lognormals

Table 9b: Eurodollar futures options, 30 September 1998
Probabilities for the eurodollar rate on 14 December 1998

30 September 1998	$\text{Prob}[\tilde{r}(T) \leq R]$					
	4.50	**4.75**	**5.00**	**5.25**	**5.50**	**5.75**
Black	0.04	0.18	0.45	0.74	0.91	0.98
MLN[a]	0.07	0.13	0.37	0.75	0.93	0.98
Jump	0.05	0.14	0.40	0.73	0.92	0.99
Hermite (4)	0.07	0.13	0.38	0.74	0.94	0.99
Hermite (6)	0.08	0.14	0.38	0.74	0.94	0.99
Maximum entropy	0.07	0.14	0.40	0.76	0.95	1.00

The probabilities are the risk-neutral probabilities that the market assigns on the given date for the eurodollar rate on 14 December 1998 to be less than the stated *R* value. See the Appendix for details.

a. Mixture of lognormals

Figure 1: Eurodollar futures options, 23 September 1998

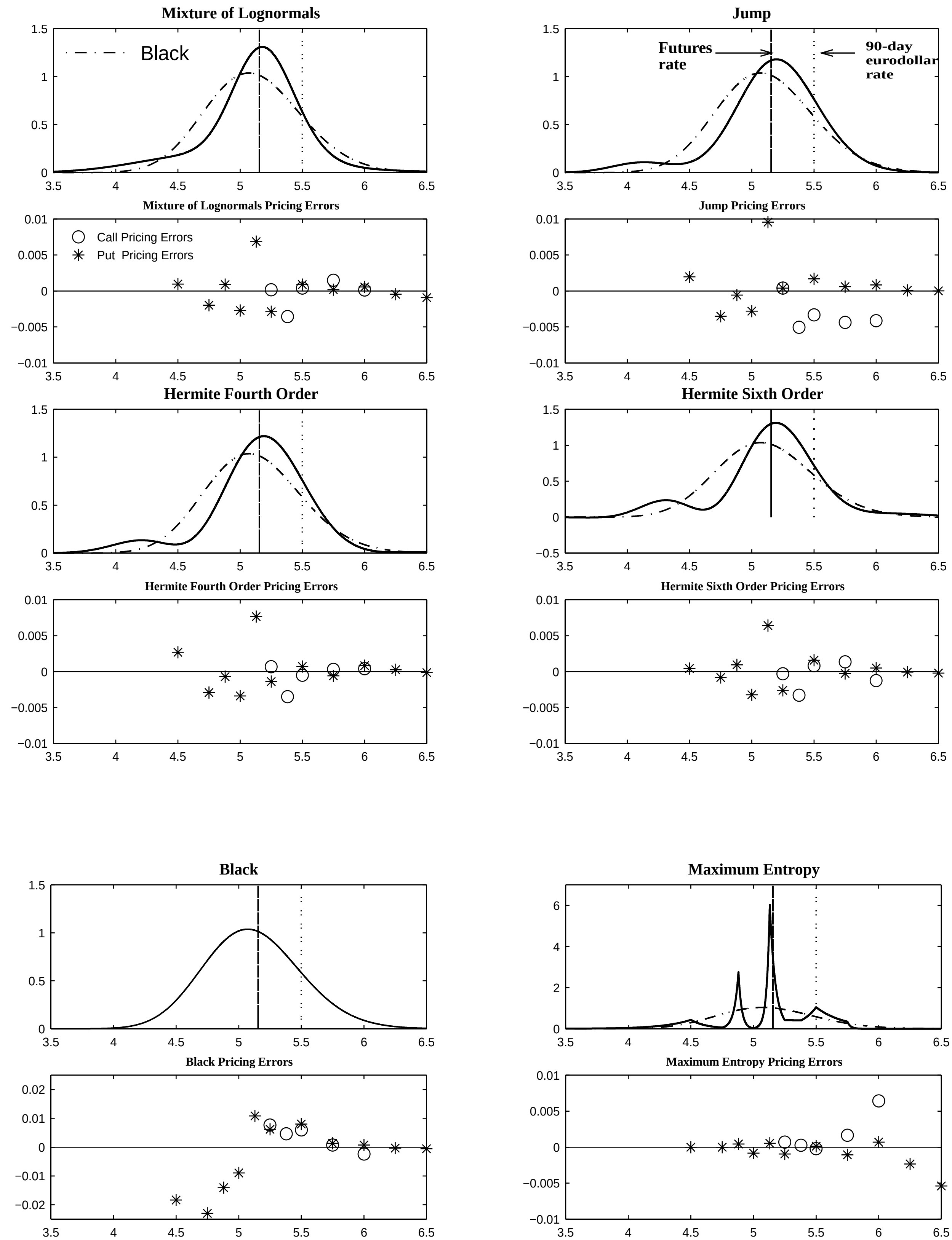


Figure 2: Eurodollar futures options, 24 September 1998

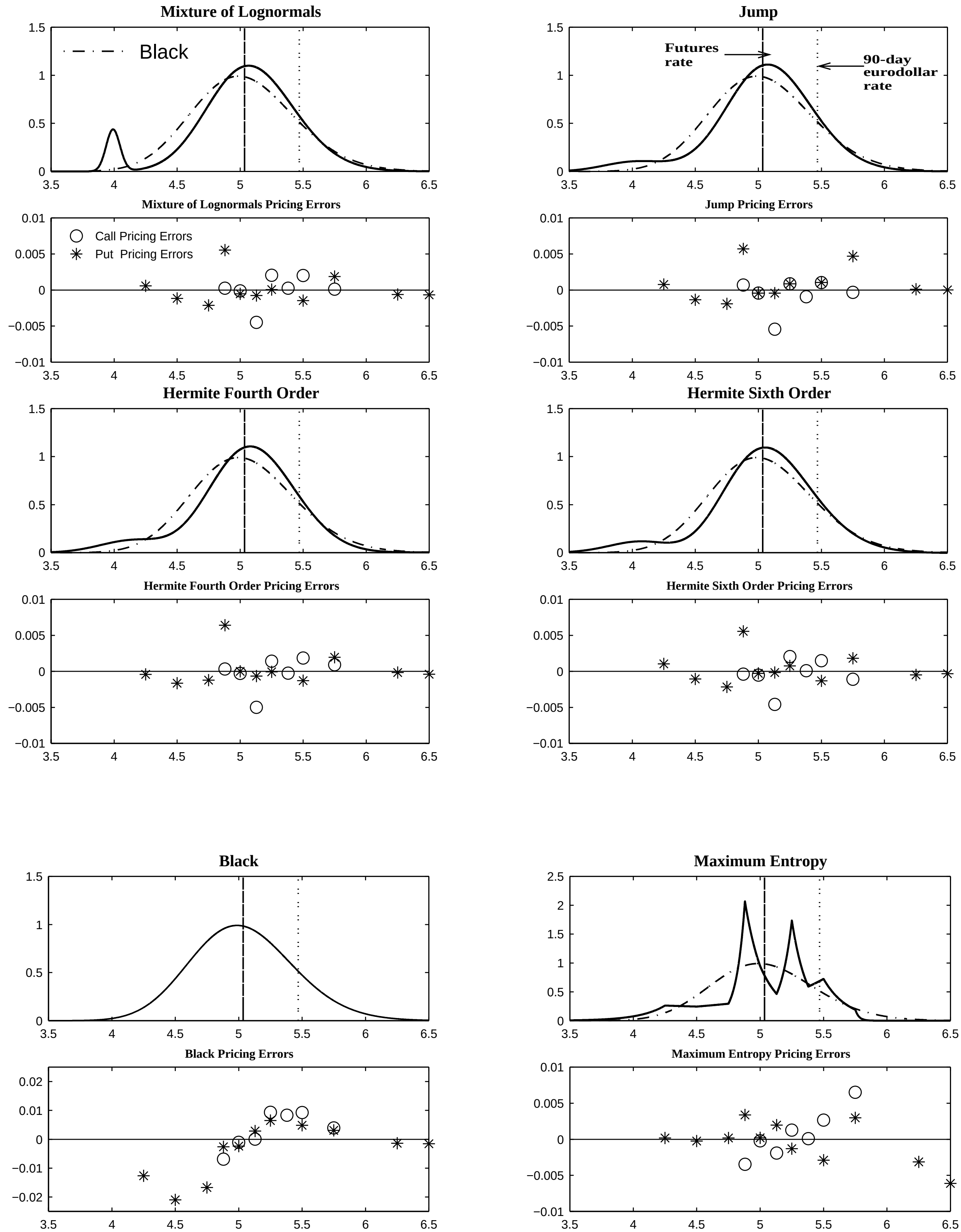


Figure 3: Eurodollar futures options, 25 September 1998

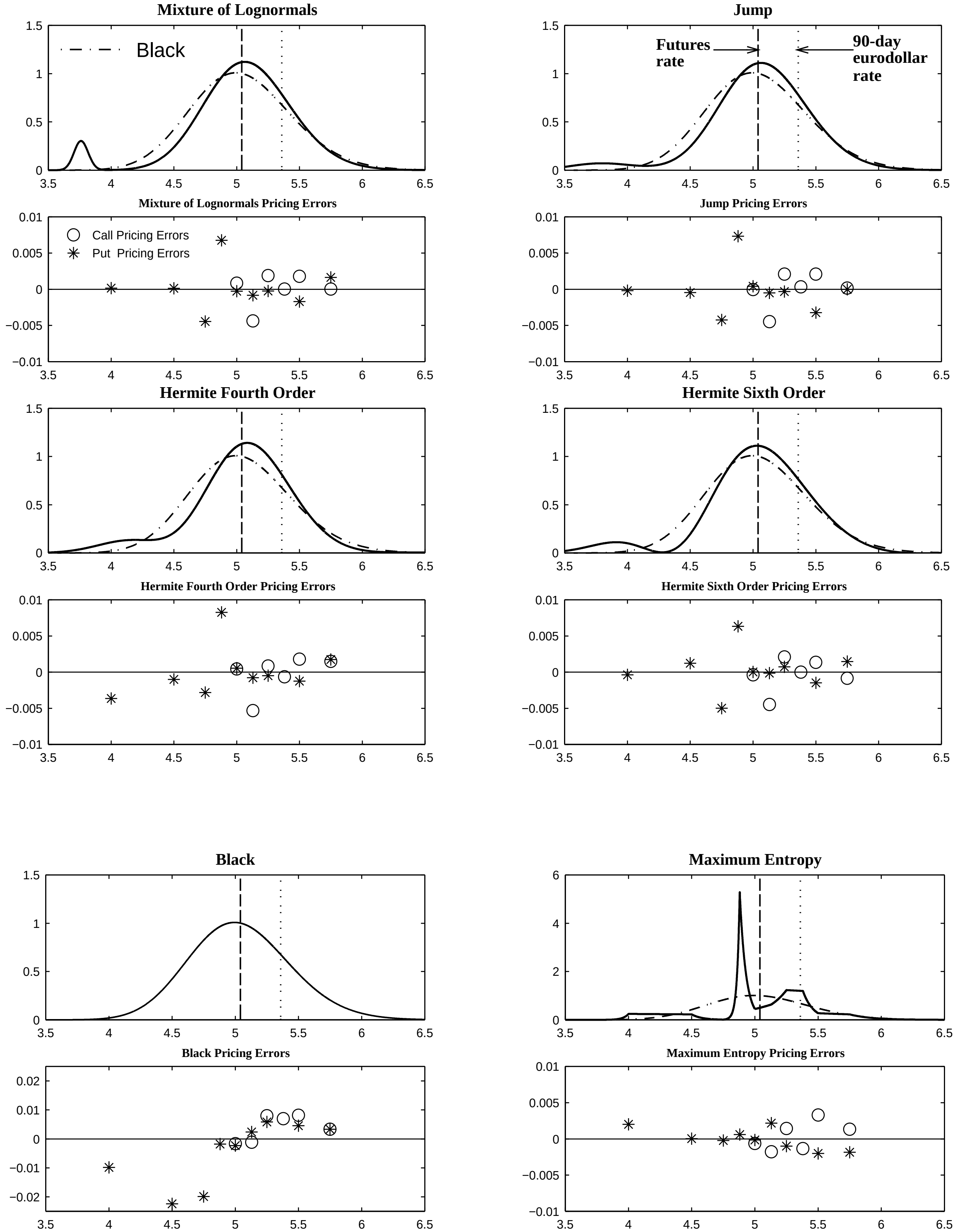


Figure 4: Eurodollar futures options, 28 September 1998

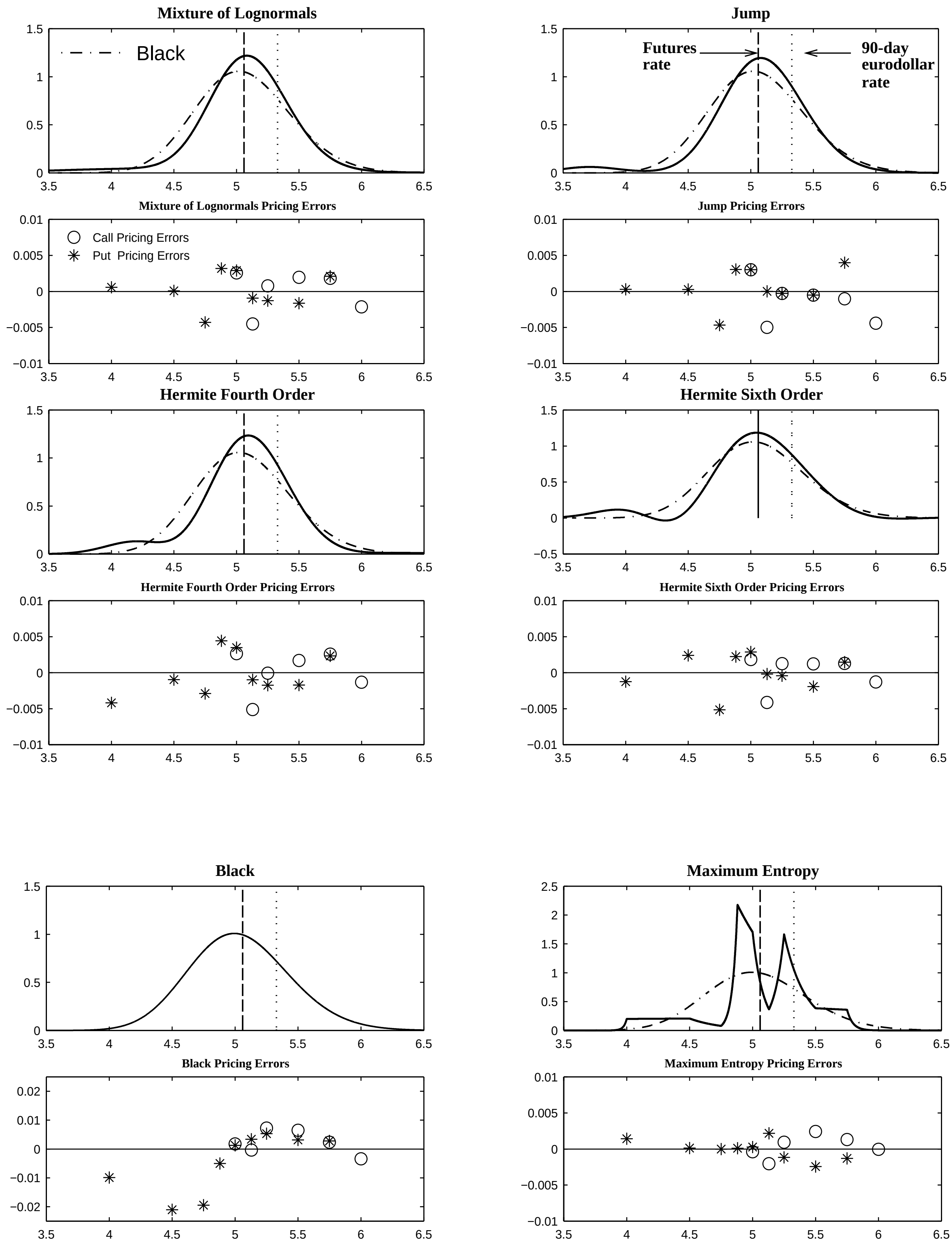


Figure 5: Eurodollar futures options, 29 September 1998

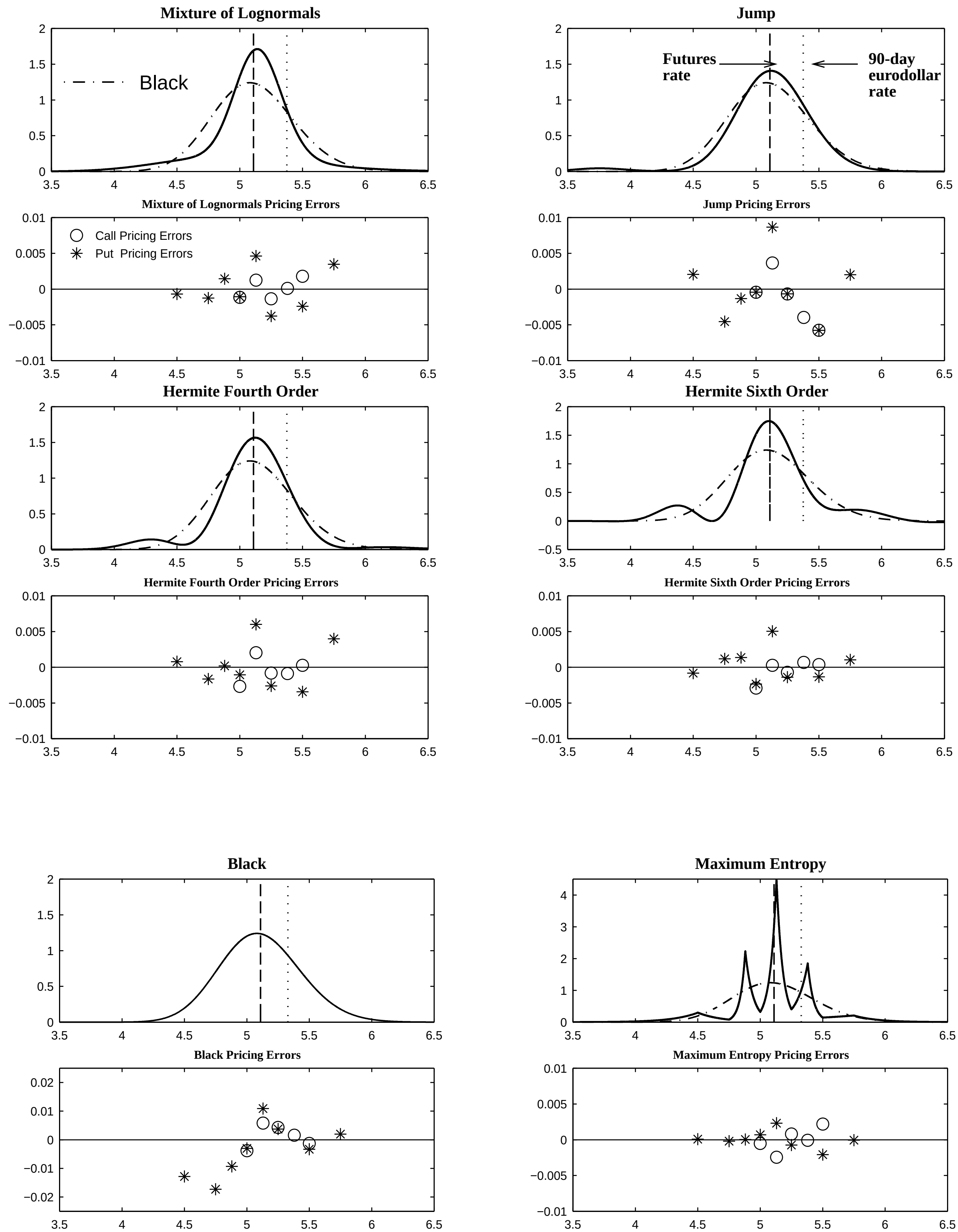


Figure 6: Eurodollar futures options, 30 September 1998

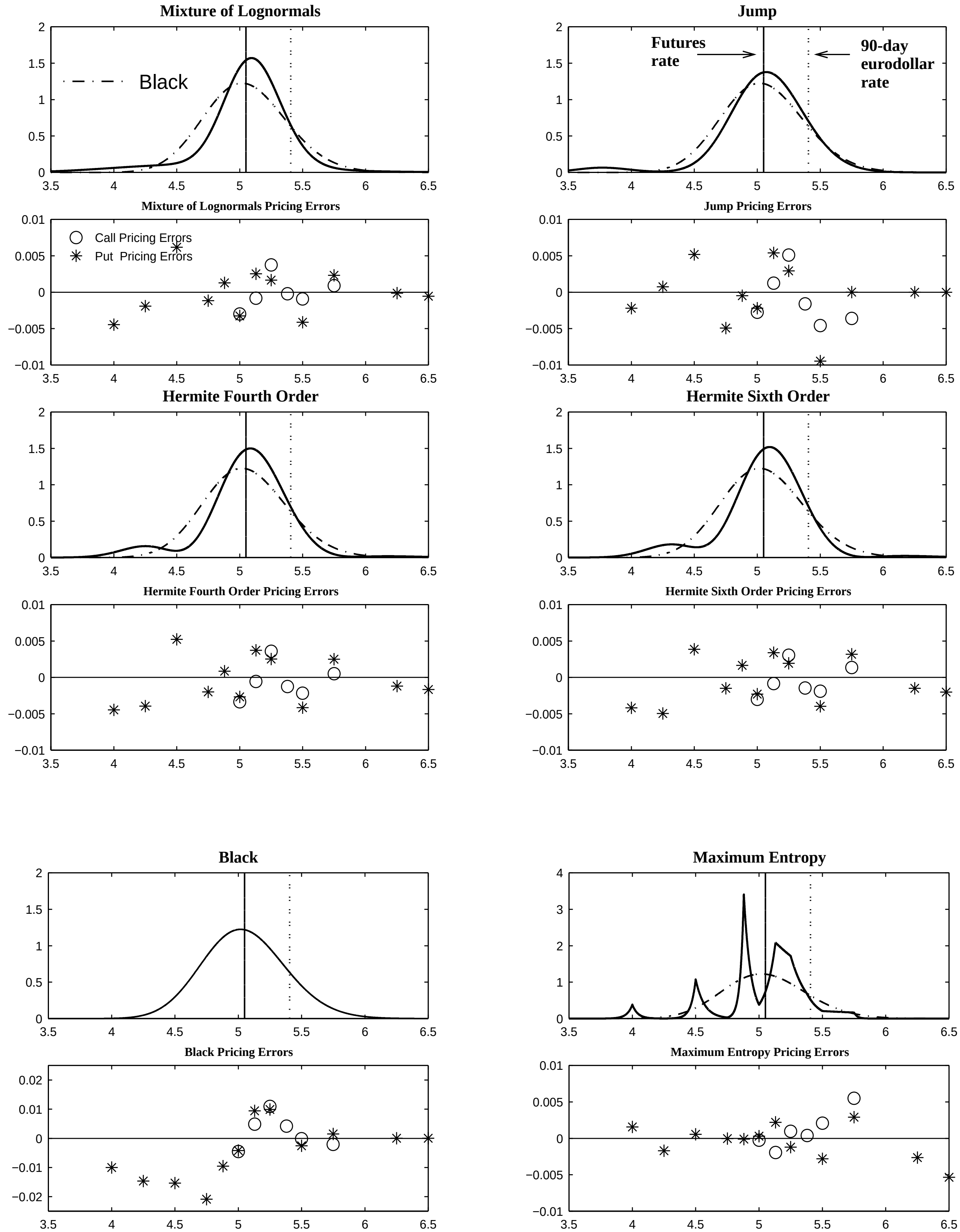


Figure 7: Eurodollar futures options, moments

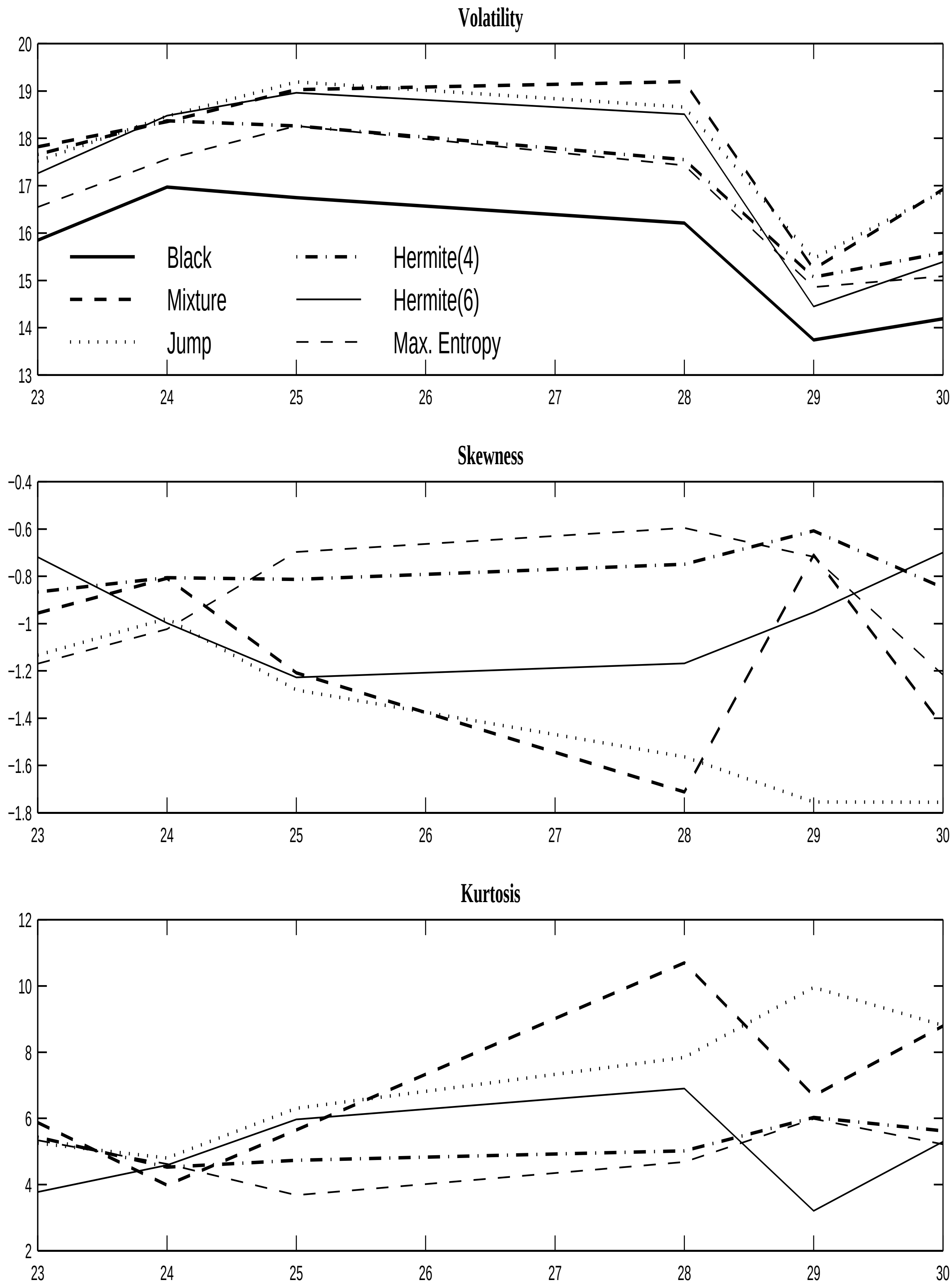


Figure 8a: Cumulative distributions

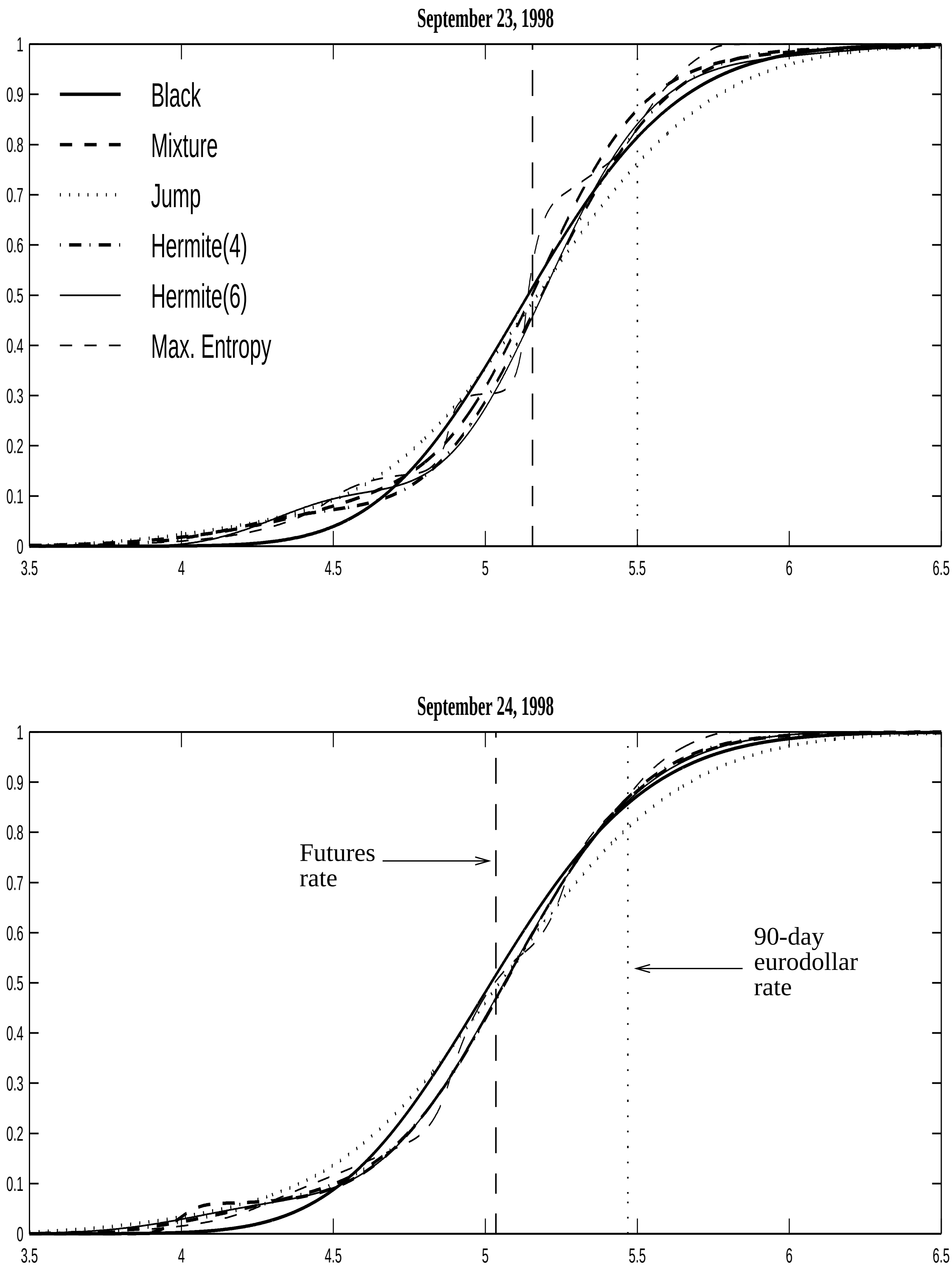
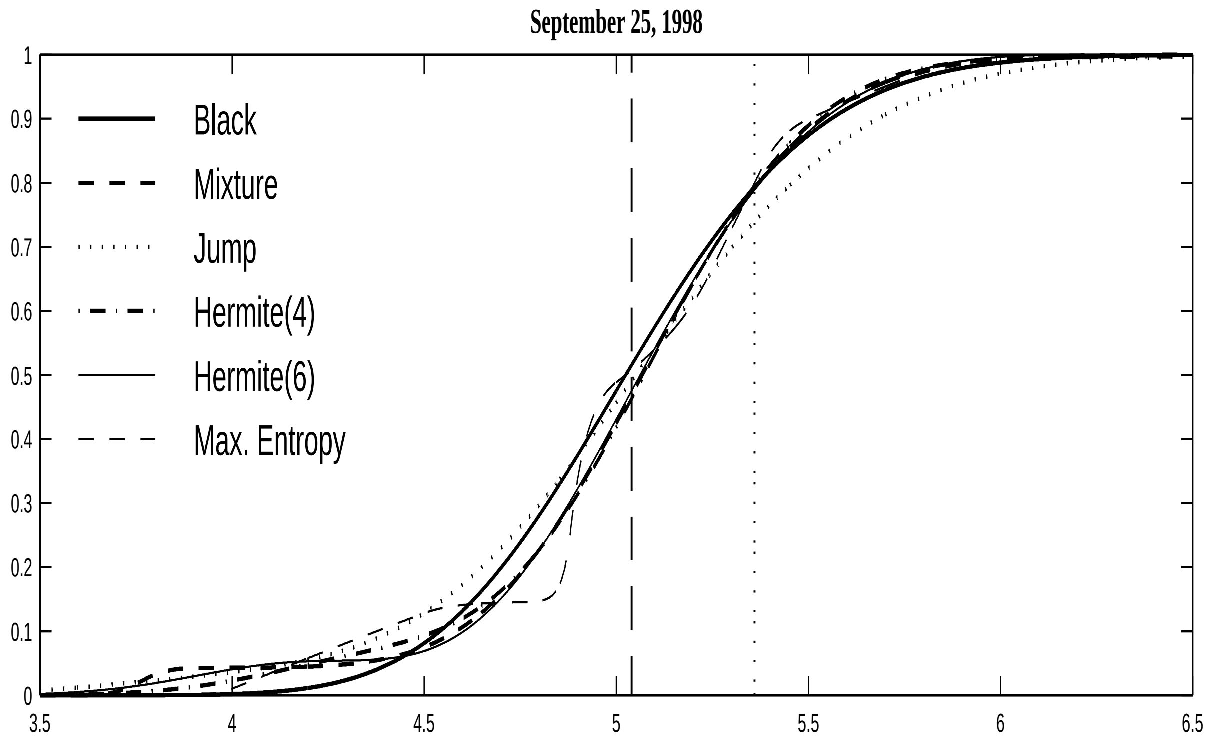


Figure 8b: Cumulative distributions



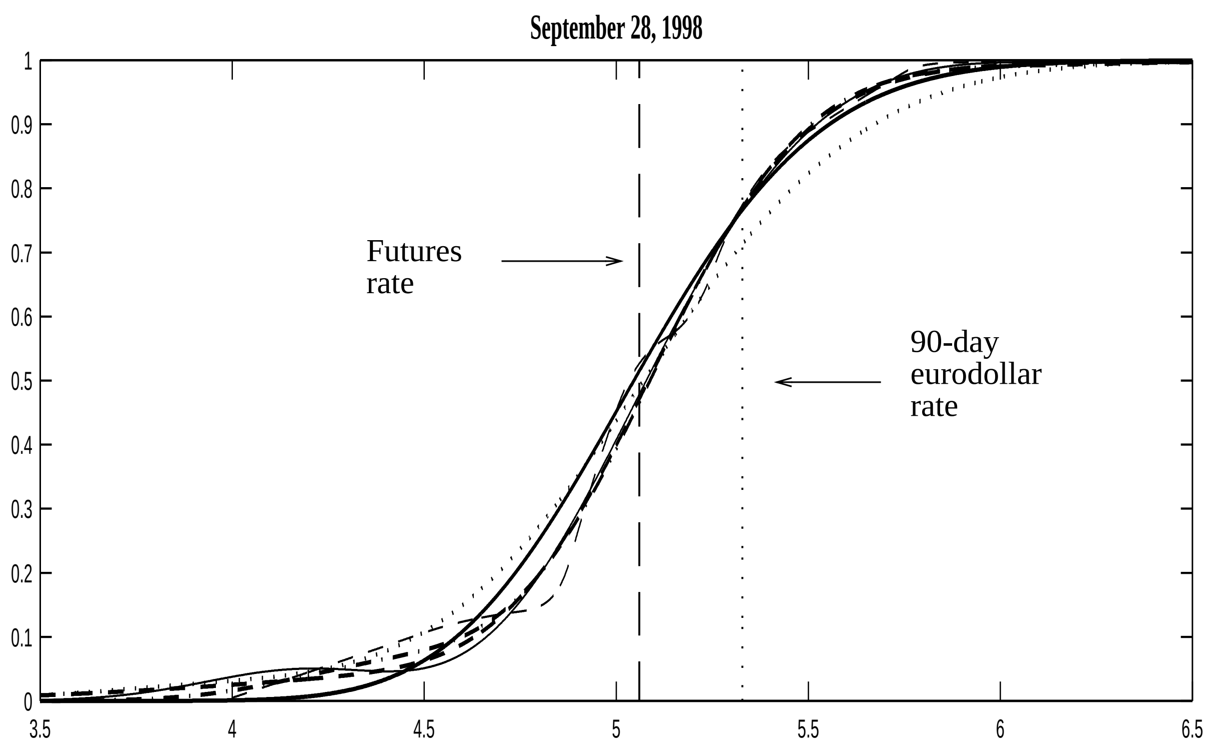
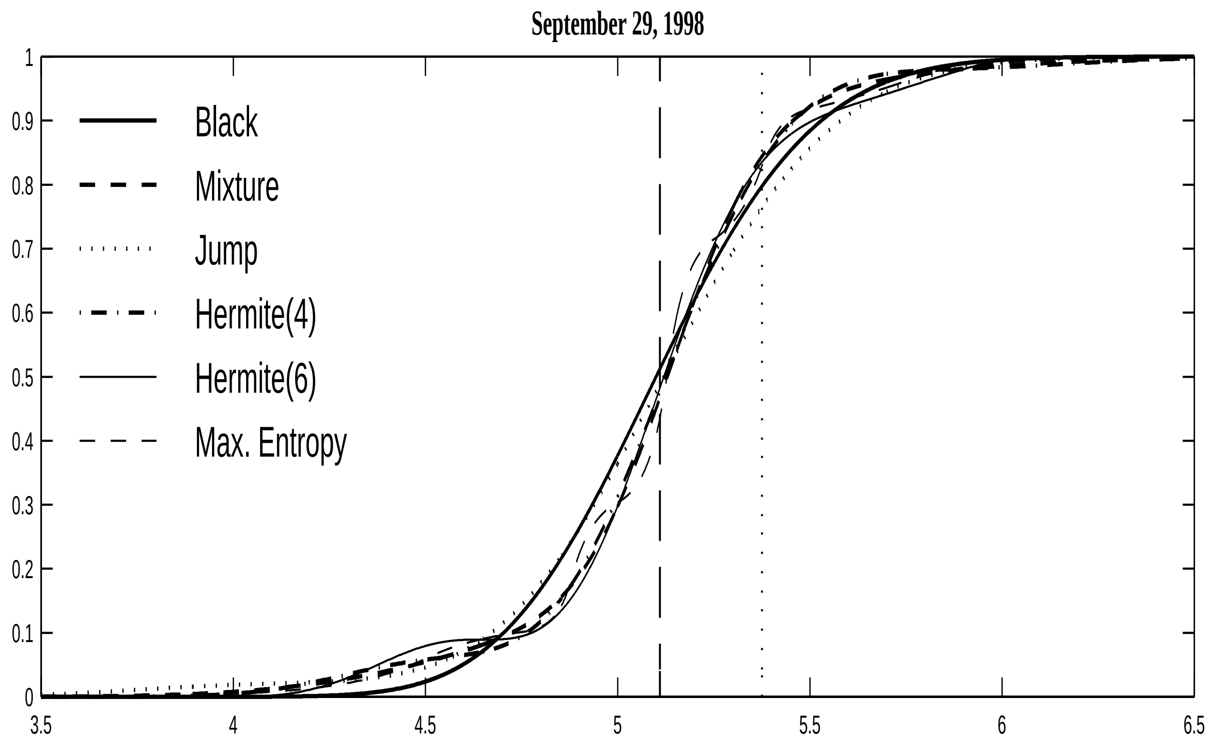


Figure 8c: Cumulative distributions



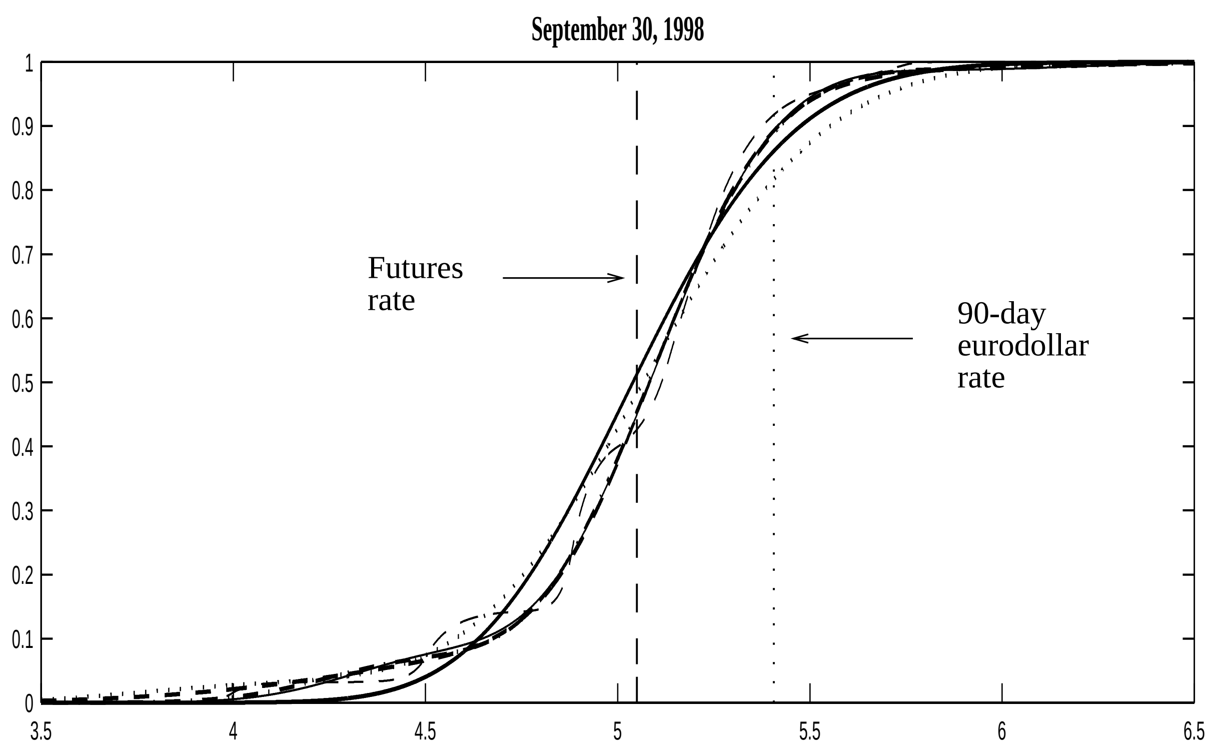
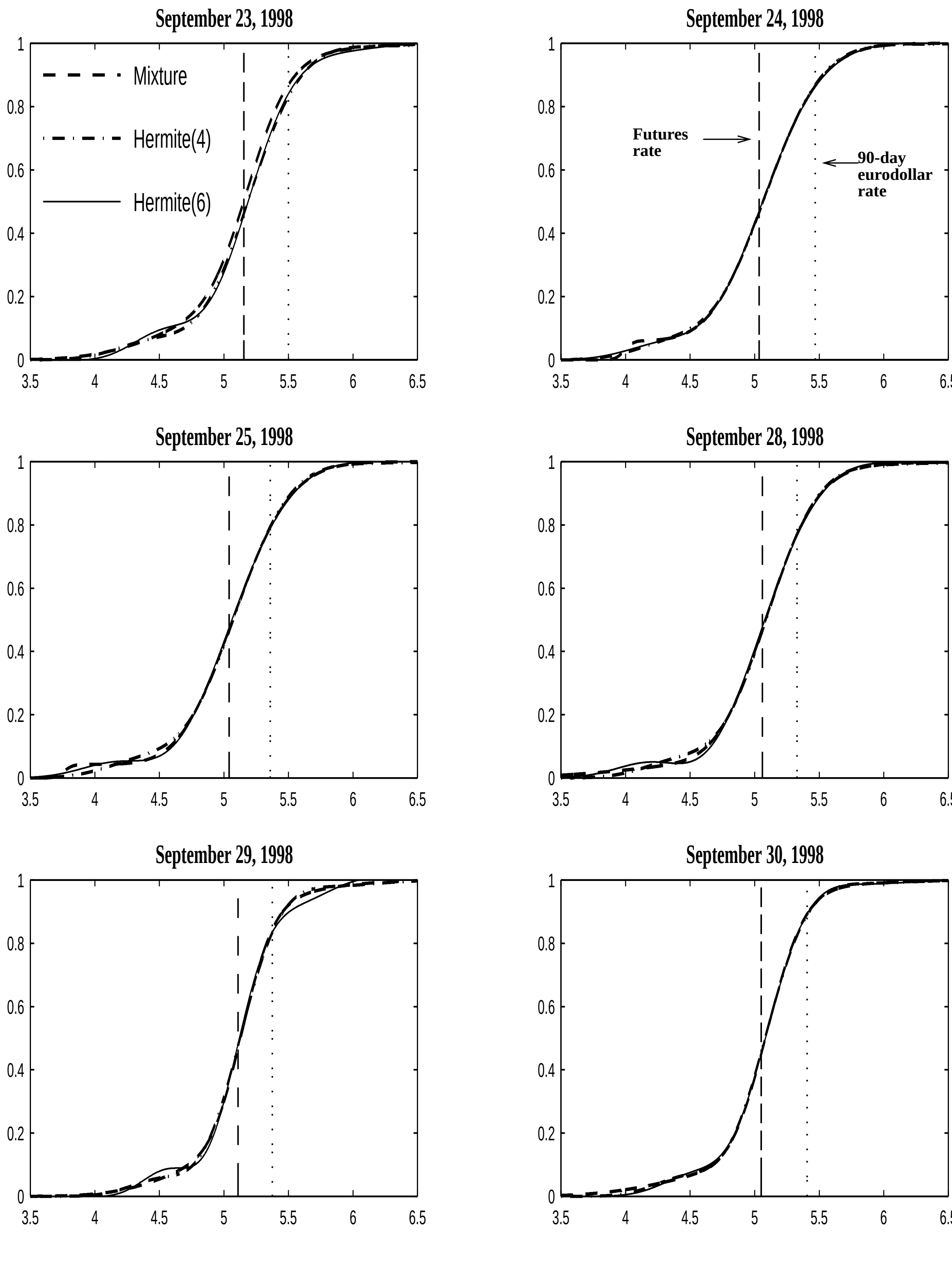


Figure 8d: Cumulative distribution



Appendix: PDF summary statistics

The risk-neutral PDF, $q[\tilde{r}(T)]$, synopsizes the information contained in the price of interest rate futures options. Thus, a graphical depiction of the risk-neutral PDF yields market perceptions over the future value of interest rates. In addition, several numerical statistics also yield helpful insights. In particular, the probability that the futures rate will be less than a given rate, R, on the maturity date of the futures contract is insightful, namely

$$\text{Prob}[\tilde{r}(T) \le R] = \int_0^R q[\tilde{r}(T)]\, d\tilde{r}(T)\,. \tag{47}$$

In addition, several summary statistics calculated with respect to the logarithm of the futures rate are useful, such as the mean, annualized volatility, skewness, and kurtosis. The annualized volatility provides an indication of the dispersion of opinion in the market surrounding the future interest rate. The skewness compares the probability of a large upward movement in the futures rate to the probability of a large downward movement. Risk-neutral PDFs are either symmetric, skewed left or skewed right. A skewed left distribution places greater weight on the likelihood the future interest rate will be far below, as opposed to far above, the current futures price on the maturity date of the option. Finally, kurtosis indicates the possibility of large changes in interest rates prior to the maturity of the futures option.

Note that is the futures rate $\tilde{r}(T)$ has a PDF $q[\tilde{r}(T)]$ then the logarithm of the futures rate, $\log[\tilde{r}(T)]$, has a PDF $Q(\log[\tilde{r}(T)]) = \tilde{r}(T)\,\mathrm{q}[\tilde{r}(T)]$. Thus, the mean, variance, skewness, and kurtosis with respect to the logarithm of the futures rate are given, respectively, by

$$\begin{aligned} \mu &= E_Q[\log\tilde{r}(T)] \\ \text{Var} &= E_Q[(\log\tilde{r}(T)-\mu)^2] \\ \text{Skew} &= \left(E_Q[(\log\tilde{r}(T)-\mu)^3]\right)/\text{Var}^{3/2} \\ \text{Kurt} &= \left(E_Q[(\log\tilde{r}(T)-\mu)^4]\right)/\text{Var}^2 \end{aligned} \tag{48}$$

where E_Q represents expectations with respect to the PDF Q. The annualized volatility is given by $\sigma = \sqrt{\text{Var}/T}$.

The PDF summary statistics for the models outlined in section 3 are as follows.

A.1 Black's model

The cumulative distribution function is

$$\text{Prob}[\tilde{r}(\text{T}) \le R] = N\left(\frac{\log\{R/F(0,T)\}}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) \tag{49}$$

The variance, skewness, and kurtosis are

$$\text{Var} = \sigma^2 T\text{, Skew} = 0\text{, and Kurt} = 3. \tag{50}$$

A.2 Mixture of lognormals

The cumulative distribution function is

$$\text{Prob}[\tilde{r}(T) \le R] = \phi_1 N\left(\frac{\log R - \mu_1}{\sigma_1}\right) + (1-\phi_1)N\left(\frac{\log R - \mu_2}{\sigma_2}\right) \tag{51}$$

The variance, skewness, and kurtosis are

$$\begin{aligned} \text{Var} &= \phi_1\sigma_1^{\;2} + \phi_2\sigma_2^{\;2} + \phi_1\phi_2(\mu_1-\mu_2)^2 \\ \text{Skew} &= \left\{\phi_1\phi_2(\mu_1-\mu_2)\left[3\left(\sigma_1^{\;2}-\sigma_2^{\;2}\right) + (\phi_2-\phi_1)(\mu_1-\mu_2)^2\right]\right\}/\text{Var}^{3/2} \\ \text{Kurt} &= \left\{3\left(\phi_1\sigma_1^{\;4} + \phi_2\sigma_2^{\;4}\right) + 6\phi_1\phi_2(\mu_1-\mu_2)^2\left[\phi_2\sigma_1^{\;2} + \phi_1\sigma_2^{\;2}\right] + \phi_1\phi_2(\mu_1-\mu_2)^4\left(\phi_1^3+\phi_2^3\right)\right\}/\text{Var}^2 \end{aligned} \tag{52}$$

A.3 Jump diffusion

The cumulative distribution and PDF summary statistics are given by equations (51) and (52) above with $\phi_1 = 1 - \lambda T$, $\mu_1 = \log F(0,T) - \log(1+\lambda\kappa T) - \frac{1}{2}\sigma_\omega^2 T$, $\mu_2 = \mu_1 + \log(1+\kappa)$, and $\sigma_1 = \sigma_\omega\sqrt{T} = \sigma_2$.

A.4 Hermite polynomial approximation

The cumulative distribution is

$$\begin{aligned} \text{Prob}[\tilde{r}(T) \le R] = {} & N(Z) + \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{Z^2}{2}\right]\left[\frac{b_3}{\sqrt{6}}(1-Z^2) + \frac{b_4}{\sqrt{24}}Z(3-Z^2)\right. \\ & \left. + \frac{b_5}{\sqrt{120}}(-3+6Z^2-Z^4) + \frac{b_6}{\sqrt{720}}Z(-15+10Z^2-Z^4)\right] \end{aligned} \tag{53}$$

where $Z = \dfrac{\log[R/F(0,T)] - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and the restrictions $b_0 = 1$, $b_1 = 0$ and $b_2 = 1$ have been imposed. (The fourth-order approximation is given by setting $b_5 = 0$ and $b_6 = 0$.) Under the same restrictions, the variance, skewness, and kurtosis are

$$\begin{aligned} \text{Var} &= \sigma^2 T \\ \text{Skew} &= \sqrt{6}\, b_3 \\ \text{Kurt} &= 3 + \sqrt{24}\, b_4 \end{aligned} \quad . \tag{54}$$

A.5 Method of maximum entropy

The cumulative distribution is given by equation (55) if $X_i \le R < X_{i+1}$

$$\text{Prob}[\tilde{r}(T) \le R] = \left(\begin{array}{l} \dfrac{1}{\mu\, a_i} \{\exp[a_i R + b_i] - \exp[a_i X_i + b_i]\} \\ + \displaystyle\sum_{k=0}^{i-1} \dfrac{1}{\mu\, a_k} \{\exp[a_k X_{k+1} + b_k] - \exp[a_k X_k + b_k]\} \end{array} \right. \tag{55}$$

and by equation (56) if $X_m \le R$:

$$\text{Prob}[\tilde{r}(T) \le R] = \left(\begin{array}{l} \dfrac{1}{\mu\, a_m} \{\exp[a_m R + b_m] - \exp[a_m X_m + b_m]\} \\ + \displaystyle\sum_{k=0}^{m-1} \dfrac{1}{\mu\, a_k} \{\exp[a_k X_{k+1} + b_k] - \exp[a_k X_k + b_k]\} \end{array} \right. . \tag{56}$$

(57)

(58)

No closed-form solutions exist for the mean, variance, skewness, and kurtosis. These statistics are calculated by numerical integration. (59)

Discussion of

"The Information Content of Interest Rate Futures Options"

by Des McManus, Bank of Canada

Discussant: James M. Mahoney

This paper attempts to determine which method of estimating the Eurodollar futures' probability density function (PDF), of six methods under consideration, most accurately fits a cross-section of observed Eurodollar futures options prices. The paper uses various metrics to determine which method is flexible enough to price the array of available options prices found in the market, e.g., with the implicit smiles found in the implied volatility – strike price graphs. The paper is well done and advances our understanding of the relative effectiveness in each of the methodologies. Most of my comments are directed at extending the already-impressive amount of work evident in the paper.

Data issues: Observation weights

In coming up with the parameters of the risk-neutral PDF, the estimation procedure is set up to minimize the equally weighted sum of squared deviations between the modeled prices and the market observed prices. There may be some good reasons to weight these observations differently in the minimization problem. First, weighting the observations by trading volume may make sense if there are differences in bid-ask spreads or non-synchronous trading, which translates in differences in measurement errors, across the observations. Second, there may be reasons to underweight deep out-of-the-money options, as little or no information may be available in options whose market prices are near the exchange-mandated minimum tick size (0.01 for Eurodollar futures and options) and where small measurement errors may significantly alter the results of the minimization problem. And third, although not an issue in the empirical section of this paper due to the small sample window, one needs to decide how close to expiration are the prices of option useful in deriving risk-neutral PDFs.

Metrics for comparison

Two criteria are used as metrics of comparison among the six models. First, the mean squared error (MSE) and the mean squared percent error (MSPE) are used. Again, the issue of how these observations should be weighted arises. Equally weighting the observations may not provide reliable results, as, e.g., a small absolute error may yield large percent errors, which would dominate the MSPE metric.

The second metric of comparison in the paper is based on a set of summary statistics: volatility, skewness, and kurtosis. Volatility measures were consistent across methodologies, while the measures of skewness and kurtosis were erratic. A concern here is the lack of robustness in these summary statistics. For example, the volatility, skewness and kurtosis are very different across six methodologies in Table 7a, but the PDFs across the six methodologies are remarkably similar in Table 7b. In addition to robustness, an additional concern is the difficulty in interpreting the summary statistics. For example, again in Table 7, the volatility and kurtosis of the mixture of log-normal (MLN) methodology are greater than the volatility and kurtosis of the Hermite polynomial of degree 4 (H(4)), which might suggest that large movements in interest rates are more likely for the MLN methodology. Yet, the probability of a large move in rates (with rates rising above 5.75% or falling below 4.5%) is actually greater for the H(4) methodology. This counter-intuitive result (driven by difference in skewness in the PDFs) suggests that the summary statistics may not be the most useful gauges of PDFs, as these summary statistics have non-natural units of measurement that are hard to interpret. Perhaps more useful would be a more extensive evaluation of the PDFs from the perspective of various quintiles (i.e., drop Table 7a and extend Table 7b).

Additional metrics for comparison could be used. For example, the cross-sectional PDF could be estimated using a sub-sample of available options prices, and the out-of-sample cross-sectional fit could be tested. This methodology could also be used to test the robustness of the methodology, in the sense that eliminating one observation should not drastically alter the results of the PDF estimation. For an additional metric of comparison, one could use the time series of subsequent observed Eurodollar interest rates to see how well the models predict out of sample.

All in all, the paper provides useful information to the practitioner and researcher alike on the various strengths and weaknesses in six possible ways to fit the cross-section of observed options prices. This paper also helps lay the groundwork for future research on whether any of these methodologies are useful in forecasting future changes in asset prices.

Discussion of
"The Information Content of Interest Rate Futures Options"
by Des McManus, Bank of Canada
Discussant: Roberto Violi, Banca d'Italia, Research Department

The paper applies and compares several common methods to arrive at estimates of the (risk-neutral) probability distribution of future values of Eurodollar rates; its main aim is to determine which method most accurately prices observed market options. As a concrete example of application, the days surrounding some recent FOMC meetings are examined and estimated PDFs are used as indicator of market sentiment to gauge the uncertainty over the future levels of the Eurodollar rates.

While the results of the paper indicate that further research needs to be conducted, several broad conclusions are drawn from the estimates:

- The mixture of lognormals estimation method ranked first in pricing accurately observed market options;
- The Hermite polynomial estimation method ranked second, yielding similar results to the mixture of lognormals; when the latter method fails, the Hermite method constitutes an appropriate alternative.
- Estimates of skewness and kurtosis, unlike the volatility measure, are found to be unstable across estimation techniques, hence model dependent.

The author provides a very articulated review of the estimation techniques and illustrates several summary statistics, suggesting both graphical and analytical methods, in comparing various models.

Two main issues are left open for future research:

- the appropriate measure of asymmetry for estimated PDFs (over and above those which can be constructed out of the more stable cumulative distribution function: like inter-quartile differences);
- evaluation of the predictive power of the measures of asymmetry.

My comments will be mainly concentrated on the issues left open for future research. The basic motivation behind the PDFs' identification and estimation techniques can be found in the weaknesses of Black-Scholes' (BS) theory of option pricing, namely in the divergence between observed options market price and BS-based theoretical options price. The most common bias is the familiar "smile effect" and other "moneyness" biases are fairly well-known. These biases are understood to be related to measures of asymmetry, for example excess kurtosis. Perhaps less understood biases, I believe, relate to the maturity of the option (see Backus, Foresi, Li and Wu, 1997): the upward slope (on average) of the term structure of implied volatility; the excess kurtosis tendency to decline (on average) with maturity. Both biases have been found in foreign exchange currency options, but clear evidence of a term structure of volatility and kurtosis has also been detected for interest rates. It can be shown, in a suitably defined theoretical environment, that both biases eventually decline with maturity. Hence, for long options, the BS formula can be a good approximation of observed option market price in the foreign exchange.

The tendency for kurtosis to decline with maturity in many models is a consequence of a stronger result: the central limit theorem; this statement clearly doesn't apply to all theoretical environment (the unit-root volatility model is an obvious counterexample). Similarly, a mean-reverting process of the underlying rate can be a source of changing volatility at different time horizons (as, for example, in Vasicek, 1977). Stochastic volatility models can also be a way to capture leptokurtosis, as in Heston (1993), or simpler time-varying volatility models, such as ARCH process; according to Das and Sundaram (1997) the degree of conditional leptokurtosis for interest rates varies with the time interval between data observation.

To summarise, jump-diffusion models with mean-reverting short rate seem to allow for parameter choices which match conditional skewness and kurtosis at varying maturities of the term structure of interest rates. When jumps are introduced into a pure diffusion model of interest rate, (Gaussian) volatility drops sharply with respect to its prior level and coefficient of mean reversion for the short rate decline substantially (see Das, 1998). This may imply that jumps account for a substantial component of volatility and provide a source of mean reversion.

Testing these propositions for the option markets requires enough depth across the moneyness spectrum and across maturity, perhaps not readily found for exchange-traded options[1]. The Bates (1991) option pricing model[2], which add a jump-diffusion process to the standard Black's basic model, tested by McManus did not fare particularly well in the cross-sectional comparison[3]. Generally, it appears that the more structural model, including the semi-parametric Hermite as well as the semi-non parametric Gram-Charlier (Edgeworth) polynomial expansion technique, have difficulties in coping with the degree of asymmetry found in the data; non-structural approaches, like a mixture of lognormals[4], seem to warrant a higher degree of flexibility in matching data; yet it is less clear how we should interpret such advantage when confronting the observed option market price. One possible interpretation for the mixture of lognormals is that the return on an asset at any given time can be drawn from one, out of several (oftentimes two), normal distribution; each possible draw has a given probability to occur. The benefit of such specification is that it allows for the possibility that

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1 An attempt along these lines, e.g. incorporating the information embedded in the term structure of implied volatility, can be found in Fornari and Violi (1998).

2 See also Malz (1996) for an application to the estimation of exchange rate realignments probability in the EMS.

3 It is perhaps interesting that Jondeau and Rockinger (1998) estimated a jump-diffusion model of foreign exchange currency option which outperformed all other models at longer maturities; they were able to provide an interpretation for it which should also apply to models of the term structure of interest rates (see also Backus, Foresi and Wu, 1997).

4 This definition is borrowed from Jondeau and Rockinger (1998).

occasionally the return is generated from a distribution with a higher variance, while maintaining the structure of normal densities, conditional on the realisation of a particular draw; this is a simple and effective way to capture fat tails in the return-generating process. Basically, the model allows a jump from one distribution to another, which is similar to the traditional jump-diffusion model allowing the possibility of a jump between an infinite number of normal distributions[5]. In the empirical application of these latter models a simple Bernoulli version is actually used: over the horizon of the option there will be at most one jump of constant size.

Models are estimated at various dates and maturities, yielding a different set of estimates for each date and maturity. This is *prima facie* in sharp contrast with the assumption of constant parameters in the underlying process; the time series of parameters so obtained may also correspond to a process of the underlying asset which might have little to do with historically observed processes. Perhaps more worryingly, estimating parameters tend to display more often than not great instability across dates and maturities. In the literature, and McManus' paper is no exception, the estimates are interpreted as being those perceived to be valid at each point in time by market participants till the expiration of the option. This is clearly inconsistent with the no-arbitrage pricing principle, upon which models are assumed to be based, since parameter changes over time are not treated accordingly within the option pricing model. It also casts serious doubts on the validity of time series information inferred from the volatility and asymmetry measures obtained from the estimated models.

I believe that progress could be made by improving the option price modelling in allowing changing moments over time. As an example of this claim, I have estimated volatility, skewness and excess kurtosis, based on the methodology presented in Fornari and Violi (1998), combining the Soederlind and Svensson (1997) mixture of lognormals model with the Jamshidian (1989) closed form solution for options on discount bonds. As an alternative to Black's model, this model assumes that a single factor model for the term structure of interest rates holds: the overnight rate evolves according to a mean-reverting Gaussian diffusion process, as in Vasicek (1977), with distinct parameters in each of the two regimes; unlike the Black's model, interest rates display a term structure of volatility, though a deterministic one. Only the mixing probabilities of regime switch are allowed to change over time, whereas the parameters of the term structure within each regime are kept constant[6]. No rationale is provided for the changes in the mixing probability; this is clearly a limitation which calls for improvement in future research. Data used are the same as in McManus' paper, comprising CME Eurodollar futures and options contracts with maturity December 1998 for the trading days from September 1, 1998 through November 30, 1998[7].

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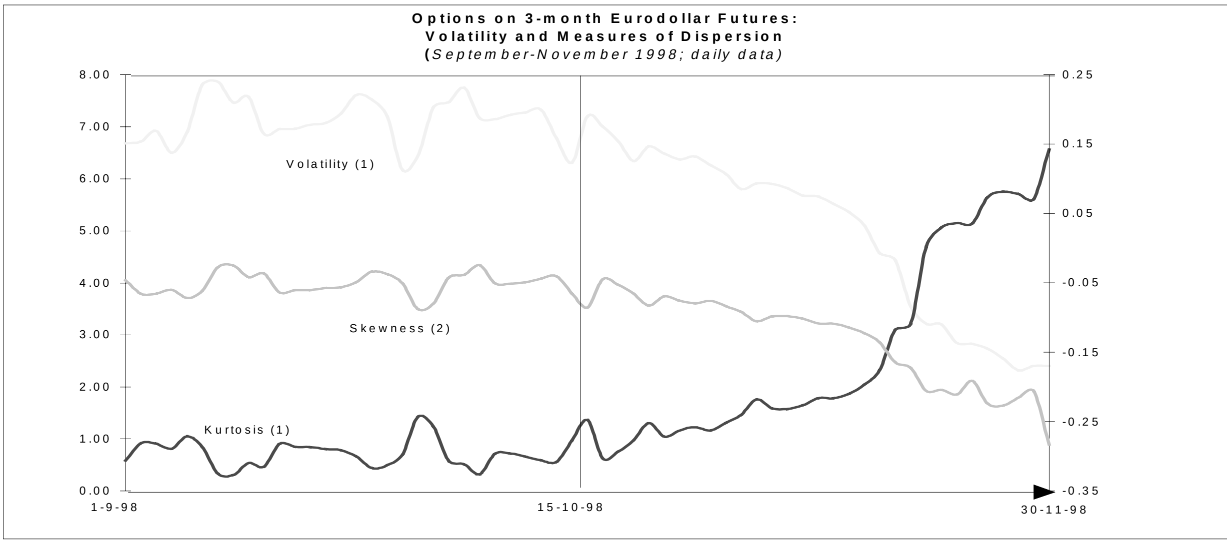
5 See Kon (1984) for details.

6 The TSP 4.4 command LSQ (see the TSP Reference Manual for details) is used to estimate the parameters of the option pricing model; the squared deviation between actual and theoretical option prices is the minimised objective function. No specific handling for the American early exercise premium is adopted.

7 Kindly provided by Gabriele Galati for the BIS workshop exercise on PDF estimation.

Chart 1.

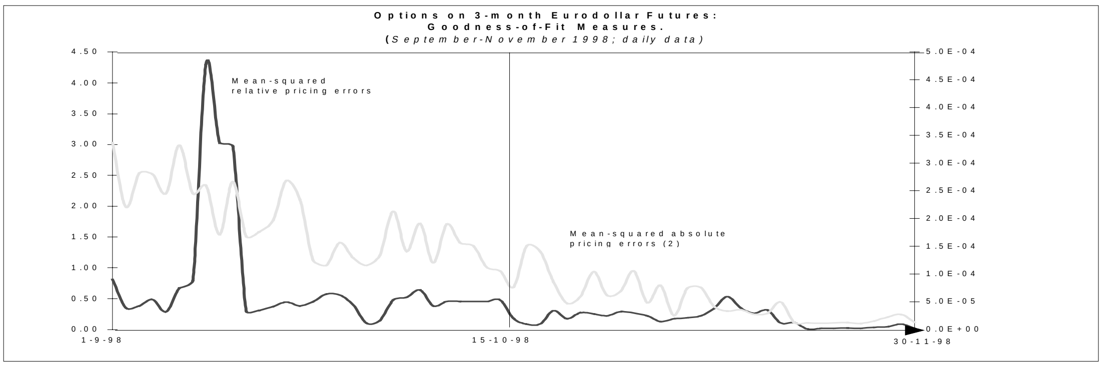


(1) Left-hand scale.
(2) Right-hand scale

Chart 1 corroborates some of the findings reported in the literature; volatility tend to decline and excess kurtosis to rise as maturity approaches; skewness changing of sign relates to the growing perception of FED monetary policy easing bias developing at the US FED in those days.

Pricing errors implied by the estimated model, though larger than the ones reckoned in McManus paper, are still within a reasonable range, as displayed in

Chart 2.



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