Implied Risk Aversion in Options Prices Using Hermite Polynomials

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Abstract

The aim of this paper is to construct a time-varying estimator of the investors' risk aversion function. Jackwerth (1996) and Aït-Sahalia and Lo (1998) show that there exists a theoretical relationship between the Risk Neutral Density (RND), the Subjective Density (SD), and the Risk Aversion Function. The RND is estimated from options prices and the SD is estimated from underlying asset time series. Both densities are estimated on daily French data using Hermite polynomials' expansions as suggested first by Madan and Milne (1994). We then deduce an estimator of the Risk Aversion Function and show that it is time varying.

Résumé

Nous construisons dans ce papier un estimateur variant avec le temps de la fonction d'aversion au risque d'un investisseur. Jackwerth (1996) et Aït-Sahalia et Lo (1998) montrent qu'il existe une relation théorique entre la densité neutre au risque, la densité subjective et la fonction d'aversion au risque. On estime la densité neutre au risque à partir des prix d'options et la densité subjective à partir d'une série chronologique du sous-jacent. Chaque densité est estimée en données quotidiennes sur le marché français, en utilisant à la suite de Madan et Milne (1994) des expansions en polynômes d'Hermite; on en déduit alors un estimateur de la fonction d'aversion au risque pouvant varier dans le temps.

Keywords: Risk aversion function, Index option's pricing, Risk neutral density, Statistical density, Hermite polynomials.

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1 Introduction

An important area of recent research in finance is devoted to the information content in options prices that can be obtained in estimating implied Risk Neutral Densities (RND). Whereas this density gives information about market-makers expectations concerning the future behaviour of the underlying asset, it does not allow to infer anything related to investors' risk aversion. In return, there exists a relationship between the risk neutral density, the subjective density (SD) and the risk aversion function.

Although this theoretical relationship is well known, few works have been interested in the topic in an empirical framework. To our knowledge, the two major studies which deal with are those from Jackwerth (1996) and Aït-Sahalia and Lo (1998). On the one hand, they estimate the RND from options prices and on the other hand they estimate the SD from time series of the underlying asset. By comparing both densities, they conclude that risk aversion is time varying.

Following Jackwerth, and Aït-Sahalia and Lo, we extract both densities (RND and SD) and show that investors' risk aversion function is time varying. The contribution of this study is twofold: first, we investigate French dataset, and second we estimate the model at a daily frequency.

With regard to the RND, in addition to seminal work on options pricing by Black and Scholes (1973) and Merton (1973), we may cite Breeden and Litzenberger (1978) who first found a relationship between options prices and the risk neutral density. Nevertheless their method requires a big range of strike prices; over the past few years, a whole literature has looked into the problem of estimating the RND of the option's underlying asset. We may mention stochastic volatility models such as Hull and White (1987), Chesney and Scott (1989) or Heston (1993); to the latter Bates (1991 and 1996) adds a jump process in the asset return diffusion. Madan and Milne (1994) and Jarrow and Rudd (1982) respectively approximate the RND by Hermite and Edgeworth expansions. Rubinstein (1994), Dupire (1994) and Derman et Kani (1994) suggest to use implied binomial trees. Bahra (1996), and Melick and Thomas (1997) assume lognormal mixture for the RND. Aït-Sahalia (1998) uses kernels estimators of the RND. Lastly we refer to Campa, Chang and Reider (1997), Jondeau and Rockinger (1998) or Coutant, Jondeau and Rockinger (1998) for a comparison of several methods of extracting the RND from options prices on a particular event.

Section 2 first presents a brief review of the investment's theoretical foundations in an economy with a single consumption good, second it describes the traditional Black and Scholes

model and explains why this model is too far from reality. Section 3 describes the model that used: Hermite polynomials approximations and shows how we estimate the risk neutral density using options and the subjective density using underlying time series. Finally Section 4 first describes the dataset and analyses statistical properties, second explains which optimisation's proceeds are used to estimate the models and third studies results on French daily
dataset. Section 5 concludes. Technical results are detailed in the Appendix.

2 Methodology

2.1 Implied risk aversion

The basic investment choice problem for an individual is to determine the optimal allocation of his wealth among the available investment opportunities. We stand in a standard investment theory (see Lucas (1978)). There is a single physical good *S* which may be allocated to consumption or investment and all values are expressed in term of units of this good; there is a risk-free asset, i.e. an asset whose return over the period is known with certainty. Any linear combination of these securities which has a positive market value is called a portfolio. It is assumed that the investor chooses at the beginning of a period the feasible portfolio allocation which maximises the expected value of a Von Neumann-Morgenstern utility function for the end-of-period wealth. The only restriction is the budget constraint. We denote this utility function by $U(.)$, and by W_T the terminal value of the investor's wealth at time T. It is further assumed that U is an increasing strictly concave function of the range of feasible values for W, and that U is twice-continuously differentiable. The only information about the assets that is relevant to the investor's decision is the density probability of W_T.

In addition, it is assumed that:

Hypothesis 1: *Markets are frictionless: there are no transactions costs nor taxes, and all securities are perfectly divisible*.

Hypothesis 2: *There are no-arbitrage opportunities in the markets. All risk-free assets must have the same return between t and T. This return will be denoted by* $r_t\,(T)$ *and is assumed to be known and constant.*

Hypothesis 3: *There are no institutional restrictions on the markets. Short-sales are allowed without restriction.*

As Aït-Sahalia and Lo (1998) write it, the equilibrium price of the risky asset S_t at date t with a T-liquidating payoff $Y(W_T)$ is given by:

$$S_t = E\left[\Psi(W_T)M_{t,T}\right], (1)$$

$$M_{t,T} = \frac{U'(W_T)}{U'(W_t)}, (2)$$

under the true probability, where $M_{t,T}$ is the stochastic discount factor between consumption at dates t and T.

In equilibrium, investor optimally invests all his wealth in the risky stock for all $t<T$ and then consumes the terminal value of the stock at T, $W_T = S_T$.

If we notice by *p(.)* the *subjective density* (SD) of W_T, we may rewrite (1) as:

$$\begin{aligned} S_t &= \int_0^{\infty} \Psi(W_T)\frac{U'(W_T)}{U'(W_t)} p(W_T)dW_T \\ &= e^{-r_t(T)(T-t)} \int_0^{\infty} \Psi(W_T)q(W_T)dW_T \\ &= e^{-r_t(T)(T-t)} E_t^*\left[\Psi(W_T)\right] \end{aligned}$$

with

$$q(W_T) = \frac{M_{t,T}}{\int_0^{\infty} M_{t,T}\, p(W_T)dW_T} p(W_T)\,(3)$$

is called the *state-price density* or *risk neutral density* (RND) which is the equivalent in a continuous-time world of the Arrow-Debreu state-contingent claims in a discrete-time world[2].

A way to specify the preference ordering of all choices available to the investor is the risk-aversion function. A measure of this risk-aversion function is the *absolute risk-aversion function A(.)* of Pratt and Arrow (see Pratt (1964)) given by:

$$A(S) = -\frac{U''(S)}{U'(S)}. (4)$$

By the assumption that U is increasing (*U'(S)>0*) and strictly concave (*U''(S)<0*), function *A(.)* is positive; such investors are called *risk-averse*. An alternative, but related measure of risk aversion is the *relative risk-aversion function*:

[2] Recall that Arrow-Debreu contingent claims pay $1 in a given state and nothing in all other states.

$$R(S) = -\frac{U''(S)}{U'(S)} S. (5)$$

From (3), we can deduce than the ratio *q/p* is proportional to $M_{t,T}$ and we can write:

$$V(S_T) = \frac{q(S_T)}{p(S_T)} = qM_{t,T} = q\frac{U''(S_T)}{U'(S_T)}. (6)$$

where q is a constant independent of the level of *S*.

Differentiating (6) with respect to S_T leads to:

$$V'(S_T) = q\frac{U''(S_T)}{U'(S_T)}$$

and

$$-\frac{V'(S_T)}{V(S_T)} = -\frac{U''(S_T)}{U'(S_T)} = A(S_T)$$

We then may calculate *A(.)* as a function of *p(.)* and *q(.)* and we easily obtain an estimator of the absolute risk-aversion function, which does not depend on the parameter q:

$$A(S_T) = \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)}. (7)$$

At this stage, we need to specify a general form for the utility function and we add the following hypothesis:

Hypothesis 4: *We stand in a state in which investors have preferences characterised by Constant Relative Risk Aversion (CRRA) utility functions (see Merton (1969, 1971)). Those functions have the following*

general form:

$$U(S) = \frac{S^{1-l}}{1-l},\ if\ \ l \neq 1 (8)$$

$$A(S) = \frac{l}{S}, (9)$$

$$\begin{aligned} &U(S) = \ln(S),\ if\ \ l = 1 \\ &A(S) = \frac{1}{S,} \end{aligned} \qquad (10)$$

where l be a nonnegative parameter representing the level of investor's risk aversion.

An estimation of the parameter l will directly give us an idea on the investors' risk aversion level. Once one has supposed a form for the utility function, he must specify a model to extract subjective density p and risk neutral density q. In order to study investor's reactions across time, the risk aversion is to be time-varying. So we replace all previous notations by p_t, q_t, A_t and l_t where t denotes all dates of our dataset. In the next section, first we give an example using the traditional Black-Scholes model, second we explain why Black-Scholes model does not correspond to reality and third we present an extension of Black-Scholes model: Hermite polynomials model which allows for more properties of the data.

2.2 Hermite polynomials expansion vs Black-Scholes

Now, we wish to develop the method for a traditional option pricing model. We have to keep in mind that we need to estimate subjective density p_t and risk-neutral density q_t at each date and then extract parameter l from these estimations.

A large part of the literature concerning options pricing is based on the Black and Scholes (1973) model. Assets returns are lognormally distributed with known mean and variance. The underlying asset S_t, $t \leq T$ follows a Brownian diffusion:

$$dS_t = m_t S_t dt + s_t S_t dW_t, \quad (11)$$

where W_t is a Brownian motion under the subjective probability, m_t is the rate of return of S under the SD and s_t is the volatility; both are supposed to be constant for a certain date t. Harrison and Kreps (1979) show that when hypotheses (1) to (3) hold, there exists a unique risk neutral probability equivalent to the subjective one, under which discounted prices of any asset are martingales. Under this equivalent probability, the underlying asset price S_t is distributed as following:

$$dS_t = (r_t - d_t) S_t dt + s_t S_t dW_t^*, \quad (12)$$

where W_t^* is a standard Brownian motion under the risk neutral probability, d_t denotes the implied dividend at time t and s_t is the volatility which appears to be the same than under the true probability. In the Black-Scholes model, asset price S_t follows a lognormal under both probability[3]. Risk Neutral

[3] Applying Ito's formula to ln(S_t) and (11) gives us:

$$d\ln(S_t) = \frac{dS_t}{S_t} + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\mathrm{var}(dS_t)dt = \left(m - \frac{1}{2}s^2\right)dt + s dW_t$$

Density (RND), $q_t^{BS}(S, s_t)$ and Subjective Density (SD), $p_t^{BS}(S, s_t, m_t)$ only differ in mean and are given by:

$$\mathrm{p}_{\mathrm{t}}^{\mathrm{BS}}(S, s_t, m_t) = \frac{1}{s_t\sqrt{T-t}\sqrt{2pS}}\exp\left[-\frac{\left(\ln(S) - m_t(m_t)\right)^2}{2s_t^2(T-t)}\right], (13)$$

$$\mathrm{q}_{\mathrm{t}}^{\mathrm{BS}}(S, s_t) = \frac{1}{s_t\sqrt{T-t}\sqrt{2pS}}\exp\left[-\frac{\left(\ln(S) - m_t(r_t - d_t)\right)^2}{2s_t^2(T-t)}\right] (14)$$

where

$$m_t(x) = \ln(S_t) + \left(x - \frac{1}{2}s_t^2\right)(T-t).$$

By replacing (13) and (14) and under hypothesis (4) we directly obtain:

$$A_t^{\mathrm{BS}}(S) = \frac{m_t - (r_t - d_t)}{s_t^2 S}. (15)$$

An estimation of parameters m_t and s_t allows us to estimate absolute risk aversion function when the underlying follows (11).

Black and Scholes is based on the fundamental hypothesis that volatility is deterministic, skewness and excess kurtosis are zero. Those hypotheses have been widely reconsidered on the last few years, owing to the fact that option price at maturity is very sensitive to the underlying asset's distribution specifications. Figure 2 shows typical volatility smiles for two dates, May 1995, 5th, date that we can call agitated, and July 1996, 25th, date that we can call flat: we observed that implied volatility at date t is constant neither in strike price neither in maturity; volatility is higher for small strikes, which means that market makers will pay more for a call option on a smaller strike: this feature will appear in the density with a presence of asymmetry; volatility smile for the second date is very U-shape: we will notice a kurtosis effect in the density.

We impose another model for the underlying which allows for skewness and kurtosis. Following Madan and Milne (1994) and Abken, Madan and Ramamurtie (1996), we adopt an Hermite polynomials approximation for the density. Their model operates as follows.

First, we add the following hypotheses to hypotheses (1)-(4):

Hypothesis 5: *The set of all contingent claims is rich enough to form a Hilbert space that is separable and for which an orthonormal basis exists as a consequence. The markets are assumed to be complete.*

Hypothesis 6: *Abken, Madan and Ramamurtie suppose that under a reference measure, the asset price evolves as (11), i.e. as a geometric Brownian motion. Then* S_t *can be written as:*

$$S_T = S_t \exp\left[\left(m_t - \frac{1}{2} s_t^2\right)(T-t) + s_t \sqrt{T-t}z\right] \textit{(16)}$$

where z follows a N(0,1).

Madan and Milne (1994) assume than SD and RND may be written as a product of a change of measure density and reference measure density *n(z)*:

$$\tilde{\mathrm{p}}_{\mathrm{t}}^{\mathrm{HER}}(z) = n_t(z)n(z)\,(17)$$

$$\tilde{\mathrm{q}}_{\mathrm{t}}^{\mathrm{HER}}(z) = u_t(z)n(z)\,(18)$$

where $\tilde{\mathrm{p}}_{\mathrm{t}}^{\mathrm{HER}}(z)$ and $\tilde{\mathrm{q}}_{\mathrm{t}}^{\mathrm{HER}}(z)$ are respectively subjective and risk neutral densities. In our particular case *n(z)* will be a Gaussian distribution of zero mean and unit variance. A basis for the Gaussian reference space may be constructed by using Hermite polynomials which form an orthonormal system for the Hilbert space[4].

As we have carried out for the benchmark model, we wish to estimate time-varying risk aversion function when supposing an Hermite polynomials expansion for the density; therefore, we need to estimate both risk-neutral and subjective densities. Next section is divided in two parts. In a first part, we give the way to estimate risk-neutral model from options prices, and in a second part we show how to use these estimated parameters as observed data to estimate subjective model and extract l_t.

3 Models' specifications

3.1 Risk Neutral Model

To estimate implied volatilities risk neutral parameters we use options prices. A call option (put option) is the right to buy (to sell) the option's underlying asset at some future date -the

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[4] Hermite polynomial of order *k* is defined as follows:

$$f_k(z) = \frac{(-1)^k}{\sqrt{k!}} \frac{\P^k n(z)}{\P z^k} \frac{1}{n(z)} \text{ with } < f_k, f_j >= \int_{-\infty}^{+\infty} f_k(z) f_j(z) n(z) dz = \begin{cases} = 0 \; if\, j \neq k \\ = 1 \; if\, j = k \end{cases}$$

expiration date- at a prespecified price -the striking price. This right has a price today that is a function of the option's specifications. Since under the risk neutral probability discounted prices are martingales, the current option's price may be written as the discounted end-of-period option's payoff expectation. If we denote by $C_e(t,S,K,T)$, a European call price of exercise price K and maturity T, we have:

$$C_e(t,S_t,K,T) = e^{-r_t(T)(T-t)} \int_0^{\infty} \max(S_T - K,0) q_t(S_T) dS_T . (19)$$

As CAC 40 options are American style options, we introduce the approach developed by Melick and Thomas (1997) to price American options. They show that the option's price could be flanked by two bounds representing minimal and maximal value of the price. This method can be applied to any stochastic process if we know the shape of the future underlying's distribution. If we can bound the option's price, we will be able to write it as a weighted sum of the bounds. The idea of the method comes from the martingale's hypothesis of the underlying asset under the risk neutral probability. Low and high bounds for an option call are given by:

$$C_t^u = \max[E_t(S_T) - K, r_t(1) C_e(t,S_t,K,T)] (20)$$

$$C_t^l = \max[E_t(S_T) - K, r_t(T) C_e(t,S_t,K,T)], (21)$$

then the price $C_a(t,S_t,K,T)$ of an American call can be written as:

$$C_a(t,S_t,K,T) = \begin{cases} w_1 C_t^u + (1-w_1) C_t^l \ \ if\, E_t(S_T) \geq K \\ w_2 C_t^u + (1-w_2) C_t^l \ \ if\, E_t(S_T) < K \end{cases} . (22)$$

Let $C_e^{\mathrm{HER}}(t,S_t,K,T,s_t,q_t^*)$ be the price of a European call of strike K and maturity T where q_t^* denotes the vector of parameters that describes the risk neutral density. Under hypotheses(1)-(6), $C_e^{\mathrm{HER}}(t,S_t,K,T,s_t,q_t^*)$ is given by:

$$\begin{aligned} C_e^{\mathrm{HER}}(t,S_t,K,T,s_t,q_t^*) &= e^{-r_t(T)(T-t)} \int_0^{\infty} (S_T - K)^+ \tilde{q}_t^{\mathrm{HER}}(z,s_t,q_t^*) dz \\ C_e^{\mathrm{HER}}(t,S_t,K,T,s_t,q_t^*) &= e^{-r_t(T)(T-t)} \sum_{k=0}^{+\infty} a_{k,t} b_{k,t} \end{aligned} \qquad (23)$$

where S_T is given by (16) and by definition of a basis:

$$a_{k,t} = \int_{-\infty}^{+\infty} \left(S_t \exp\left[(m_t - \frac{1}{2} s_t^2)(T-t) + s_t \sqrt{T-t} z \right] - K \right)^+ f_k(z) n(z) dz \ (24)$$

and $b_{k,t}$, $k=1, 2,...$ represent the implicit price of Hermite polynomial risk $f_k(z)$[5] which needs to be estimated so that $q_t^*=(b_{0,t}, b_{0,t},...)$.

The derivation of expression (23) can be found in Appendix.

Replacing in (18) gives the RND of z:

$$\tilde{q}_t^{\mathrm{HER}}(z, s_t, q_t^*) = \sum_{k=0}^{+\infty} b_{k,t} f_k(z) n(z). (25)$$

For a practical purpose, the sum is truncated up to an arbitrary order L_b. When the sum is truncated up to an order L_b, then the density (25) may lead to some negative values for some given $b_{k,t}$, $k=1, 2,..L_b$. Balistkaia and Zolotuhina (1988) give the positivity constraints when $L_b=6$ and Jondeau and Rockinger (1999) give an ingenious way to implement positivity's constraints when $L_b=4$. For simplifications reasons and since we only need moments up to the fourth order, we restrict our model to $L_b=4$. Madan and Milne (1994) then show that the risk neutral density of the future underlying asset can be written as:

$$q_t^{\mathrm{HER}}(S, s_t, q_t^*) = q_t^{\mathrm{BS}}(S, s_t) P_H(h), (26)$$

where

$$P_H(h) = \left[b_{0,t} - \frac{b_{2,t}}{\sqrt{2}} + \frac{3b_{4,t}}{\sqrt{24}} + (b_{1,t} - 3\frac{b_{3,t}}{\sqrt{6}})h + (\frac{b_{2,t}}{\sqrt{2}} - \frac{6b_{4,t}}{\sqrt{24}})h^2 + \frac{b_{3,t}}{\sqrt{6}}h^3 + \frac{b_{4,t}}{\sqrt{24}}h^4 \right] (27)$$

$$h = \frac{\ln(S) - \left[\ln(S_t) + (r_t - d_t - \frac{1}{2}s_t^2)(T-t) \right]}{s_t \sqrt{T-t}}, (28)$$

and $q_t^{\mathrm{BS}}(S, s_t)$ is given by (14).

One can choose to estimate $b_{k,t}$, $k=1,...,4$ or follow Abken, Madan and Ramamurtie (1996) by imposing $b_{0,t}=1$ $b_{1,t}=0$, $b_{2,t}=0$ and estimate s_t, $b_{3,t}$ and $b_{4,t}$ only (See Appendix for technical details on restrictions on $b_{0,t}$, $b_{1,t}$ $b_{2,t}$ and positivity constraints on $b_{3,t}$ and $b_{4,t}$).

We wish in the next section to estimate the subjective density, in order to compute the absolute risk aversion function (7).

[5] The Hermite polynomials through the fourth order are:

$$f_0(z) = 1, f_1(z) = z, f_2(z) = \frac{1}{\sqrt{2}}(z^2 - 1)$$

$$f_3(z) = \frac{1}{\sqrt{6}}(z^3 - 3z), f_4(z) = \frac{1}{\sqrt{24}}(z^4 - 6z^2 + 3)$$

3.2 Subjective Model

To estimate the SD, we discretize equation (16) after applying Ito's lemma which straight gives us:

$$x_{(k+1)\Delta t} = x_{k\Delta t} + \left(m_{k\Delta t} - \frac{1}{2} s^2_{k\Delta t} \right) \Delta t + s_{k\Delta t} \sqrt{\Delta t}\, e_{(k+1)\Delta t}\,, (29)$$

where $x_{k\Delta t} = \ln(S_{k\Delta t})$ and Dt is a time discretization step (Dt=1/260 for daily data), kDt, $k=1,...N$, are the dates of discretization with $t=NDt$. For example, if data are daily, t will equal one year. After a change

of probability $e_{(k+1)Dt}$ will have the following distribution $\tilde{p}^{\mathrm{HER}}_{k\Delta t}(z)$:

$$\tilde{p}^{\mathrm{HER}}_{k\Delta t}(z) = n(z)\left[1 + \frac{3\hat{b}_{4,k\Delta t}}{\sqrt{24}} - 3\frac{\hat{b}_{3,k\Delta t}}{\sqrt{6}} z - \frac{6\hat{b}_{4,k\Delta t}}{\sqrt{24}} z^2 + \frac{\hat{b}_{3,k\Delta t}}{\sqrt{6}} z^3 + \frac{\hat{b}_{4,k\Delta t}}{\sqrt{24}} z^4 \right] (30)$$

The general idea of the method is that parameters s_{kDt}, $b_{3,kDt}$ and $b_{4,kDt}$ are the same than those estimated in the previous section for the date $t=kDt$ because they are invariant when we switch from risk neutral world to real world. So we can consider them as observed variables. The only parameter to estimate is the drift m_{kDt}; to allow this latter to vary across time, we can write it as:

$$m_{(k+1)\Delta t} = a_0 + a_1 m_{k\Delta t} + b_1 e_{(k+1)\Delta t}\,, (31)$$

where a_0, a_1 and b_1 are to be estimated.

Once we have estimated m_{kDt}, the subjective density $p^{\mathrm{HER}}_{k\Delta t}(S,q)$, where q denotes the vector of parameters to be estimated, that is a_0, a_1 and b_1, of S_{kDt} is known and is given by:

$$p^{\mathrm{HER}}_{k\Delta t}(S,q) = p^{\mathrm{BS}}_{k\Delta t}(S, \hat{s}_{k\Delta t}, m_{k\Delta t})\left[1 + \frac{3\hat{b}_{4,k\Delta t}}{\sqrt{24}} - 3\frac{\hat{b}_{3,k\Delta t}}{\sqrt{6}} h - \frac{6\hat{b}_{4,k\Delta t}}{\sqrt{24}} h^2 + \frac{\hat{b}_{3,k\Delta t}}{\sqrt{6}} h^3 + \frac{\hat{b}_{4,k\Delta t}}{\sqrt{24}} h^4 \right],$$

where h is given by

$$h = \frac{\ln(S) - \left[\ln(S_{k\Delta t}) + (m_{k\Delta t} - \frac{1}{2}\hat{s}^2_{k\Delta t})\Delta t \right]}{\hat{s}_{k\Delta t}\sqrt{\Delta t}},$$

and $p^{\mathrm{BS}}_{k\Delta t}(S, \hat{s}_{k\Delta t}, m_{k\Delta t})$ is given by (13).

The risk aversion function for Hermite polynomials model is then given by:

$$A_t^{\mathrm{HER}}(S) = \frac{p_t^{\mathrm{HER}'}(S)}{p_t^{\mathrm{HER}}(S)} - \frac{q_t^{\mathrm{HER}'}(S)}{q_t^{\mathrm{HER}}(S)} = \frac{l_t}{S}. \quad (32)$$

Analytic form of those functions are given in Appendix.

4 Results

4.1 Data description

We consider the case of the CAC 40 index[6] and short time-to-maturity CAC 40 options[7].

The whole database has been provided by the SBF-Bourse de Paris (Société des Banques Françaises) which produces monthly CD-ROMs including tick-by-tick quotations of the CAC 40 caught every 30 seconds, and all equities options prices quoted on the MONEP tick-by-tick. The database includes time quotation, maturity, strike price, closing and settlement quotes for all calls and puts and volume from January 1995 through June 1997. Short maturity CAC 40 options prices need to be adjusted for dividends. Aït-Sahalia and Lo (1998a) suggest to extract an implied forward underlying asset F_t using the call-put parity on the at-the-money option which requires that the following equation holds:

$$C_{\mathrm{atm}} - P_{\mathrm{atm}} = e^{-r_t(T)(T-t)}(F_t - K_{\mathrm{atm}}) \quad (33)$$

where C_{atm}, P_{atm} and K_{atm} respectively denote the price of the call, the price of the put and the strike at-the-money. Once we have obtained F_t, we may deduce the implied dividend $d_t(T)$ at time t for a maturity T using the arbitrage relation between F_t and S_t

$$F_t = e^{(r_t(T)-d_t(T))(T-t)} S_t \quad (34)$$

Since CAC 40 options data contains many misspriced prices, once needs to filter the data very carefully. First following Aït-Sahalia and Lo (1998a), we drop options with price less than 1/8.

[6] CAC 40 index leans on the major shares of Paris Stock Market. It is constructed from 40 shares quoted on the monthly settlement market and selected in accordance with several requirements (capitalization, liquidity,...). CAC 40 is computed by taking the arithmetical\ average of assets quotations which compose it, weighted by their capitalization.

[7] CAC 40 options are traded on the MONEP (Marché des Options Négociables de Paris). They are american type and there are four expiration dates for each date: 3 months running and a quarterly maturity among March, June, September or December. Two consecutives strike prices are separated by a standard interval of 25 basis points.

Second, for our study, we kept the most liquid maturity which usually appears to be the closest to 30 days yield-to-maturity.

Table 1 shows summary statistics of the CAC 40 index return historical distribution. Negative skewness and positive excess kurtosis show nonnormality of historical distribution, implying a leptokurtic and skewed distribution. Statistic *W* used by Jarque and Bera (1980) to construct a normality test allows to reject normality at 95%.

The Ljung-Box (1978) statistic LB(20) to test heteroskedasticity rejects the homoskedasticity for the square returns. The Ljung-Box (1978) statistic *LB(20)* corrected for heteroskedasticity computed with 20 lags allows to detect autocorrelation returns. Diebold (1988) suggests a Ljung-Box statistic corrected for heteroskedasticity LB_c. We notice that autocorrelation of squared returns is significantly higher than autocorrelation of returns, which implies than large changes tend to be followed by large changes, of either sign.

Table 1: Descriptive statistics of the CAC 40 daily index return for the period from January 1995 to June 1997. Table 1 shows several statistics describing returns series: mean, standard deviation, skewness and excess kurtosis. *LB(20)* is the Ljung-Box statistic to test heteroskedasticity. ρ(h) is the autocorrelation of order *h*. $LB_c(20)$ is the Ljung-Box statistic corrected for heteroskedasticity for the nullity test of the 20 first autocorrelations of returns. Under nullity hypothesis, this statistic is distributed as $\chi^2(2)$ with 20 degrees of freedom. *W* is the Jarque and Bera (1980) statistic that allows to test for normality[8].

[8]Jarque and Bera's statistic is based on empirical skewness, *sk* and kurtosis *kt* given by:

$$sk = \frac{1}{N}\sum_{t=1}^{N}\frac{(x_t - \overline{m})^3}{\overline{s}^3} \text{ et } kt = \frac{1}{N}\sum_{t=1}^{N}\frac{(x_t - \overline{m})^4}{\overline{s}^4}$$

where $\overline{m}$ and $\overline{s}$ represent respectively the empirical mean and empirical standard deviation.

We note by t_1 and t_2 the following statistics:

$$t_1 = \sqrt{N\frac{sk^2}{6}},\ t_2 = \sqrt{N\frac{(kt-3)^2}{24}}.$$

Under the nul hypothesis of normality, the Jarque and Bera’s statistic $W = t_1^2 + t_2^2$ asymptotically follows a $\chi^2(2)$.

	x_t	x_t^2
Number of observations	650	650
Mean	$0.60\ 10^{-3}$	$0.99\ 10^{-4}$
Standard Deviation	$1.00\ 10^{-2}$	$0.16\ 10^{-3}$
Skewness	-0.163	3.913
Excess kurtosis	0.928	21.205
LB(20)	26.740	49.166
r(1)	-0.010	0.010
r(5)	-0.081	-0.031
r(10)	-0.033	0.092
r(20)	-0.022	0.070
$LB_c(20)$	26.692	25.182
W	26.178	13836.563

4.2 Estimations' procedures

A non-linear least squares method is implemented to estimate risk neutral parameters. At each date t, the non-linear least squared estimator (NLLSE) $\hat{b}^*_{NLLSE} = \left\{ s_t, b_{3,t}, b_{4,t}, w_{1,t}, w_{2,t} \right\}$ is obtained so that it minimises the distance between observed and theoretical implied volatilities computed with Hermite polynomials' model (s_i^{BS} for observed ones and s_i^{HER} for theoretical ones):

$$b^*_{NLLSE} = \underset{b^* \in \Theta^*}{\arg\min} \sum_{i=1}^{m_c} \left(s_i^{\mathrm{BS}} - s_i^{\mathrm{HER}}(b^*) \right)^2, (35)$$

where m_c denotes the number of observed call options at date t, $\Theta^* = \left(R^+, D, [0,1], [0,1] \right)$ where D is the domain of ($b_{3,t}$, $b_{4,t}$) for which (25) remains positive for all z (see figure 3).

Subjective model (29)-(31) is estimated by maximum likelihood method. The log-likelihood function L of $x=(x_1,\ldots,x_{NDt})'$ is given by:

$$L(x;b) = \sum_{k=1}^{N} L_{k\Delta t}(x_{k\Delta t})\ (36)$$

where L_{kDt} is the log-likelihood function of x_{kDt}.

The maximum likelihood estimator (MLE) $\hat{b}_{MLE} = \{a_0, a_1, b_1\}$ is obtained so that it maximises the following optimisation problem[9]:

$$\hat{b}_{MLE} = \underset{b \in \Theta}{\arg\max}\left[L(x;b)\right], (37)$$

where $\Theta^* = (R, R, R)$.

To find the implied coefficient of risk aversion l_t, one can solve:

$$l_t = \underset{l \in R^+}{\arg\min} \sum_{r=1}^{M} \left(\frac{p_t^{\text{HER}\prime}(S_r)}{p_t^{\text{HER}}(S_r)} - \frac{q_t^{\text{HER}\prime}(S_r)}{q_t^{\text{HER}}(S_r)} - \frac{l}{S_r} \right), (38)$$

where M is a constant and S_r, $r=1,...,M$ is a range of points around the underlying at date t, S_t.

4.3 Empirical results

In this section, we analyse empirical results.

In figure (4a)-(4b), we show two estimated risk neutral densities for the dates May 1995, 5th with maturity of 56 days and July 1996, 25th with maturity 36 days. The first one corresponds to a so called agitated date during French Presidential Elections and the second corresponds to a quiet date. We notice that asymmetry is higher for the first one. The daily time series for the estimates of the parameters in a risk neutral world are shown in figure (5a)-(6b).

We notice that implied volatilities given by Hermite polynomials' model in figure (5a) appear to be larger than those obtained from Black and Scholes model in figure (5b) which seems to imply that Black-Scholes volatilities are undervalued. The different picks at the beginning of the period may come from the fact that CAC 40 options are much less liquid during 1995 than 1996. We turn to market prices of skewness b_3 in figure (6a); this latter is significantly different from zero during the whole period. Parameter b_3 gives some information about the skewness of the distribution when parameter b_4 gives information about the excess kurtosis which is significantly positive. The skewness appears to be negative along almost all the period which indicates that investors anticipate a decrease more often than an increase in the underlying index. We notice four agitated sub-periods. The first one corresponds to French presidential elections of May 1995. The second one and the

[9] Estimations have been done with the software GAUSS using Optmum routine.

third one respectively in May 1996 and February 1997 are not as so clear and may be due to perturbation in U.S. market. The latest is the French snap elections of May 1997.

During these period, market seems to be agitated which can be seen in the kurtosis. It gives an idea about extreme events.

Figure 7 shows Mean Square Errors (MSE) of parameters[10]. All MSE appear to be less than $8\ 10^{-2}$, that is quite satisfying and confirms the choice of the method. Other properties of the method is that it is computationally fast and it may take into account possible dirty data. Empirical results of these properties can be found in Coutant, Jondeau and Rockinger (1998).

In order to show the consistence of the model, we show in table 2 estimated parameters of the model under the true probability when parameters are supposed to be constant. Volatility parameter is higher than average volatility estimated in a risk neutral world. Parameter b_4 is significantly different from zero which is not the case of b_3.

In table 3, estimation of model (29)-(30) is presented. All parameters appear to be significant and the daily time series of estimated drift $m_{k\Delta\tau}$, $k=1,\dots,N$ from (31) with values of table 3 is given by figure 8.

Table 2: Estimation of the model (29)-(30) when parameters $(\mu_t, s_t, b_{3,t}, b_{4,t}) = (\mu, s, b_3, b_4)$ are supposed to be constant:

	μ	σ	b_3	b_4
BS	0.192 (1.898)	0.161 (29.684)		
Hermite	0.186 (1.708)	0.159 (29.247)	-0.003 (-0.191)	0.177 (3.199)

[10]MSE at date t is calculated as follow:

$$\mathrm{MSE}_t = \frac{1}{m_c - m_b} \sum_{i=1}^{m_c} \left(s_i^{\mathrm{BS}} - s_i^{\mathrm{HER}}(\hat{b}_t^*) \right)^2,$$

with the notations used in (35), m_b is the number of parameters to estimate and $\hat{b}_t^*$ is the vector of estimated parameters at date t.

Table 3: Estimation of the time varying drift μ_t in (31):

α_0	α_1	β_1
-0.113	0.481	0.296
(-3.163)	(2.177)	(5.067)

Figure (8) shows Absolute Risk Aversion functions for several days. First date is 28 February 1995 and CAC 40 moderately rose during this month: implied risk aversion coefficient l_t=4.999 is rather high. Second date is 28 April 1995, the index improved since mid-March and l_t =1.051. The date 15 July 1996, sees a short drop of the CAC 40, l_t =11.404 is very high. Finally last date takes place on 13 November 1996, during a significant growth of the underlying and l_t =3.103. We may conclude from these observations that investor's risk aversion substantially depends on the index's evolution. When CAC 40 goes up, investors have a moderate risk aversion, even they are nearly risk neutral for 13 November 1996.

Figure (9) represents the risk aversion level obtained with (38).

5 Conclusion

In this paper, we have empirically investigated investors' risk aversion coefficient implied in options prices. We showed that this latter could be estimated by the knowledge of a combination of information under risk neutral and subjective probabilities.

We have focused on CAC 40 index options, and we have supposed CRRA utility functions and an Hermite polynomial expansion for risk neutral and subjective densities. This model has the advantage to give directly the skewness and the kurtosis in addition to numerical properties. We first estimated Hermite polynomials' model under a risk neutral probability using options prices, and second injected risk-neutral parameters obtained in an equivalent discretized model under a subjective probability. We then used time series of the CAC 40 index to estimate the subjective density. A relation between densities and their derivatives allowed us to compute all absolute risk aversion functions on the period from 1995 to 1996. Risk aversion function appeared to be time varying and investors' risk aversion is very sensitive to the way underlying asset evolutes. Risk

Aversion coefficient is a good tool to test market-makers reactions to particular events or announcements.

Some future studies could turn on comparing results from several investor's preferences choices and another kind of risk, so that volatility risk for example. In a future research, we will focus on modelling the risk aversion coefficient in order to forecast the true density.

Appendix

Compute derivatives $p_t^{\mathrm{HER}\,\prime}(S)$ and $q_t^{\mathrm{HER}\,\prime}(S)$:

$$\mathrm{q}_{\mathrm{t}}^{\mathrm{HER}\,\prime}(S)=\frac{1}{s_t^2(T-t)\sqrt{2pS}}\exp\left[-\frac{1}{2}\left(\frac{\ln(S)-m_t^*}{s_t\sqrt{T-t}}\right)^2\right]\widetilde{P}_H(h)$$

$$\widetilde{P}_H(h)=-\frac{P_H(h)}{S}-\frac{P_H(h)}{Ss_t\sqrt{T-t}}\frac{\ln(S)-m_t^*}{s_t\sqrt{T-t}}+\frac{Q_H(h)}{Ss_t\sqrt{T-t}}$$

$$m_t^*=\ln(S_t)+\left(r_t-d_t-\frac{1}{2}s_t^2\right)(T-t),$$

where $P_H(.)$ is given by (27), h is given by (28) and

$$Q_H(h)=\frac{3b_{3,t}}{\sqrt{6}}(-1+h^2)+\frac{4b_{3,t}}{\sqrt{24}}(h^3-3h).$$

To obtain $\mathrm{p}_{\mathrm{t}}^{\mathrm{HER}\,\prime}(S)$ just replace m_t^* by:

$$m_t=\ln(S_t)+\left(m_t-\frac{1}{2}s_t^2\right)(T-t)$$

European call in the Hermite polynomials basis:

The price of a European call is given by (19)

$$C_e^{\mathrm{HER}}(t,S_t,K,T,s_t,q_t^*)=e^{-r_t(T)(T-t)}\int_{-\infty}^{+\infty}(S_T-K)^+\,\widetilde{q}_t^{\mathrm{HER}}(z,s_t,q_t^*)dz$$

$$=e^{-r_t(T)(T-t)}\int_{-\infty}^{+\infty}\left(S_t\exp\left[(r_t-d_t-\frac{1}{2}s_t^2)(T-t)+s_t\sqrt{T-t}z\right]-K\right)^+\widetilde{q}_t^{\mathrm{HER}}(z,s_t,q_t^*)dz$$

$$=e^{-r_t(T)(T-t)}\int_{-\infty}^{+\infty}\left(S_t\exp\left[(r_t-d_t-\frac{1}{2}s_t^2)(T-t)+s_t\sqrt{T-t}z\right]-K\right)^+u_t(z,q_t^*)n(z)dz.$$

All functions can be expressed in terms of the basis so that:

$$\left(S_t \exp\left[(r_t - d_t - \frac{1}{2} s_t^2)(T-t) + s_t \sqrt{T-t}z\right] - K\right) = \sum_{k=0}^{+\infty} a_{k,t} f_k(z)$$

$$u_t(z, q_t^*) = \sum_{j=0}^{+\infty} b_{j,t} f_j(z)$$

then

$$C_e^{\mathrm{HER}}(t, S_t, K, T, s_t, q_t^*) = e^{-r_t(T)(T-t)} \int_{-\infty}^{+\infty} \sum_{k=0}^{+\infty} a_{k,t} f_k(z) \sum_{j=0}^{+\infty} b_{j,t} f_j(z) n(z) dz$$

$$= e^{-r_t(T)(T-t)} \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} a_{k,t} b_{j,t} \int_{-\infty}^{+\infty} f_k(z) f_j(z) n(z) dz$$

$$= e^{-r_t(T)(T-t)} \sum_{k=0}^{+\infty} a_{k,t} b_{k,t}$$

Parameters $a_{k,t}$:

Coefficients $a_{k,t}$ for the call price are given by:

$$a_{k,t} = a(k, S_0, x, m, s, t) = \left. \frac{\partial \Phi(u, S_0, x, m, t)}{\partial u^k} \right|_{u=0} \frac{1}{\sqrt{k!}} \quad (39)$$

$$\Phi(u, S_0, x, m, s, t) = S_0 \exp(mt + s\sqrt{t}z) N(d_1(u)) - xN(d_2(u)) \quad (40)$$

Explicitly $a_{k,t}$ are given as follows:

$$d_1 = \frac{1}{s\sqrt{T-t}}\ln(F_t) + \frac{1}{2}s\sqrt{T-t},\ d_2 = d_1 - s\sqrt{T-t},$$
$$a_{0,t} = F_t N(d_1) - KN(d_2),$$
$$a_{1,t} = s\sqrt{T-t}F_t N(d_1) + F_t n(d_1) - Kn(d_2),$$
$$a_{2,t} = \frac{1}{\sqrt{2}}\left[\left(s\sqrt{T-t}\right)^2 F_t N(d_1) + 2s\sqrt{T-t}F_t n(d_1) + F_t n'(d_1) - Kn'(d_2)\right],$$
$$a_{3,t} = \frac{1}{\sqrt{6}}\left[\left(s\sqrt{T-t}\right)^3 F_t N(d_1) + 3\left(s\sqrt{T-t}\right)^2 F_t n(d_1) + 3s\sqrt{T-t}F_t n'(d_1) + F_t n''(d_1) - Kn''(d_2)\right],$$
$$a_{4,t} = \frac{1}{\sqrt{24}}\begin{bmatrix}\left(s\sqrt{T-t}\right)^4 F_t N(d_1) + 4\left(s\sqrt{T-t}\right)^2 F_t n(d_1) + 6\left(s\sqrt{T-t}\right)^2 F_t n''(d_1) \\ +4s\sqrt{T-t}F_t n''(d_1) + F_t n^{(3)}(d_1) - Kn^{(3)}(d_2)\end{bmatrix}$$

where *n(.)* and *N(.)* are the normal and cumulative normal densities.

Restrictions on parameters in Hermite's model:

Let h be:

$$h = \frac{\ln(S_T) - \left[\ln(S_t) + (r_t - d_t - \frac{1}{2}s_t^2)(T-t)\right]}{s_t\sqrt{T-t}},$$

the risk neutral distribution of h is:

$$\tilde{q}_t^{\mathrm{HER}}(z) = n(z)P_H(z),$$

where $P_H(z)$ is given by (27) and $n(z)$ is the Gaussian distribution with mean 0 and variance 1.

$\tilde{q}_t^{\mathrm{HER}}(z)$ must satisfy:

$$\int_{-\infty}^{+\infty}\tilde{q}_t^{\mathrm{HER}}(z)dz = 1,.$$

which implies that

$$\int_{-\infty}^{+\infty}n(z)\left[b_{0,t} - \frac{b_{2,t}}{\sqrt{2}} + \frac{3b_{4,t}}{\sqrt{24}} + (b_{1,t} - 3\frac{b_{3,t}}{\sqrt{6}})z + (\frac{b_{2,t}}{\sqrt{2}} - \frac{6b_{4,t}}{\sqrt{24}})z^2 + \frac{b_{3,t}}{\sqrt{6}}z^3 + \frac{b_{4,t}}{\sqrt{24}}z^4\right]dz = 1,$$
$$\left[b_{0,t} - \frac{b_{2,t}}{\sqrt{2}} + \frac{3b_{4,t}}{\sqrt{24}} + (\frac{b_{2,t}}{\sqrt{2}} - \frac{6b_{4,t}}{\sqrt{24}}) + 3\frac{b_{4,t}}{\sqrt{24}}z^4\right] = 1,$$
$$b_{0,t} = 1$$

for all t.

We also want to impose that the future underlying asset's expectation equals the current future price, that is:

$$E_t(S_T) = S_t e^{(r_t - d_t)(T-t)} \Leftrightarrow E_t(h) = 0,$$

$$\int_{-\infty}^{+\infty} z\tilde{q}_t^{\mathrm{HER}}(z)dz = 0,$$

which gives the restriction for parameter $b_{1,t}$:

$$\int_{-\infty}^{+\infty} zn(z)\left[b_{0,t} - \frac{b_{2,t}}{\sqrt{2}} + \frac{3b_{4,t}}{\sqrt{24}} + (b_{1,t} - 3\frac{b_{3,t}}{\sqrt{6}})z + (\frac{b_{2,t}}{\sqrt{2}} - \frac{6b_{4,t}}{\sqrt{24}})z^2 + \frac{b_{3,t}}{\sqrt{6}}z^3 + \frac{b_{4,t}}{\sqrt{24}}z^4\right]dz = 0,$$

$$\left[(b_{1,t} - 3\frac{b_{3,t}}{\sqrt{6}}) + 3\frac{b_{3,t}}{\sqrt{6}}\right] = 0,$$

$$b_{1,t} = 0$$

for all t.

Finally, third restriction comes from variance which is imposed to be the same under the transformed measure than under the reference measure:

$$\int_{-\infty}^{+\infty} z^2\tilde{q}_t^{\mathrm{HER}}(z)dz = 1,$$

$$\int_{-\infty}^{+\infty} z^2 n(z)\left[b_{0,t} - \frac{b_{2,t}}{\sqrt{2}} + \frac{3b_{4,t}}{\sqrt{24}} + (b_{1,t} - 3\frac{b_{3,t}}{\sqrt{6}})z + (\frac{b_{2,t}}{\sqrt{2}} - \frac{6b_{4,t}}{\sqrt{24}})z^2 + \frac{b_{3,t}}{\sqrt{6}}z^3 + \frac{b_{4,t}}{\sqrt{24}}z^4\right]dz = 1,$$

$$\left[b_{0,t} - \frac{b_{2,t}}{\sqrt{2}} + \frac{3b_{4,t}}{\sqrt{24}} + 3(\frac{b_{2,t}}{\sqrt{2}} - \frac{6b_{4,t}}{\sqrt{24}}) + 15\frac{b_{4,t}}{\sqrt{24}}\right] = 1,$$

$$b_{2,t} = 0$$

for all t.

Positivity's constraints on parameters $b_{3,t}$ and $b_{4,t}$:

Let g and g be the skewness and excess kurtosis respectively. A straight calculus leads to:

$$g_1 = \int_{-\infty}^{+\infty} z^3\tilde{q}_t^{\mathrm{HER}}(z)dz = \sqrt{6}b_3, (41)$$

$$g_2 = \int_{-\infty}^{+\infty} z^4\tilde{q}_t^{\mathrm{HER}}(z)dz - 3 = \sqrt{24}b_4. (42)$$

Then (25) can be rewritten in terms of g and g:

$$\tilde{q}_t^{\mathrm{HER}}(z, s_t, q_t^*) = n(z)\left[1 + \frac{g_1}{6}H_3(z) + \frac{g_2}{24}H_4(z)\right],$$

where $H_j(z) = \sqrt{j!}\,f_j(z)$ is the non standardised Hermite polynomial of order *j*.

Density (25) remains positive when

$$P_H(z) = 1 + \frac{g_1}{6}H_3(z) + \frac{g_2}{24}H_4(z) \geq 0.$$

Jondeau and Rockinger (1999) explain that this is the case if a couple (g, g) lies within the envelope generated by the hyperplane $P_H(z)=0$, with $z \in R$. This envelope is given by the system

$$\begin{cases} P_H(z) = 0, \\ P_H'(z) = 0, \end{cases}$$

with

$$P_H'(z) = \frac{g_1}{6}H_2(z) + \frac{g_2}{24}H_3(z).$$

They find that solving the problem gives explicitly g and g as a function of *z*:

$$\begin{cases} g_1(z) = -24\dfrac{H_3(z)}{d(z)}, \\ g_2(z) = 72\dfrac{H_2(z)}{d(z)}, \end{cases}$$

with

$$d(z) = 4H_3^2(z) - 3H_2(z)H_4(z).$$

After some demanding calculus, Jondeau and Rockinger (1999) find numerically and analytically that the authorised domain for g and g is a steady, continuous and concave curve. The domain for b_3 and b_4 is given by figure 3.

Captions

Figure 1a: Daily CAC 40 index over the period January 1995 to July 1997.

Figure 1b: Daily CAC 40 index returns over the period January 1995 to July 1997.

Figure 2a: CAC 40 volatility smile for the date 05/05/1995 and the maturity 56 days.

Figure 2b: CAC 40 volatility smile for the date 25/07/1996 and the maturity 36 days.

Figure 3: Domain authorised by the skewness and the kurtosis for positivity constraint of an Hermite polynomials' density

Figure 4a: Risk neutral density for the CAC 40 computed with Hermite polynomials for the date 05/05/1995 and the maturity 56 days.

Figure 4b: Risk neutral density for the CAC 40 computed with Hermite polynomials for the date 25/07/1996 and the maturity 36 days.

Figure 5a: Estimation of parameter s_t in Hermite's model under the risk neutral probability.

Figure 5b: Estimation of implied Black's volatilities under the risk neutral probability.

Figure 6a: Estimation of parameter $b_{3,t}$ in Hermite's model under the risk neutral probability.

Figure 6b: Estimation of parameter $b_{4,t}$ in Hermite's model under the risk neutral probability.

Figure 7: Mean Squares Errors for the estimation of risk neutral parameters in Hermite's model.

Figure 8: Graphs of implied absolute risk aversion functions for the dates 28/02/1995, 28/04/1995, 15/07/1996 and 13/11/1996.

Figure 9: Implied risk aversion's coefficients for the period January 1995 to July 1997.

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Shimko, D. (1993), "Bounds of Probability", *Risk*, 6(4), 34-36.

Figure 1a: daily CAC 40 index
over the time period January 1995 to July 1997

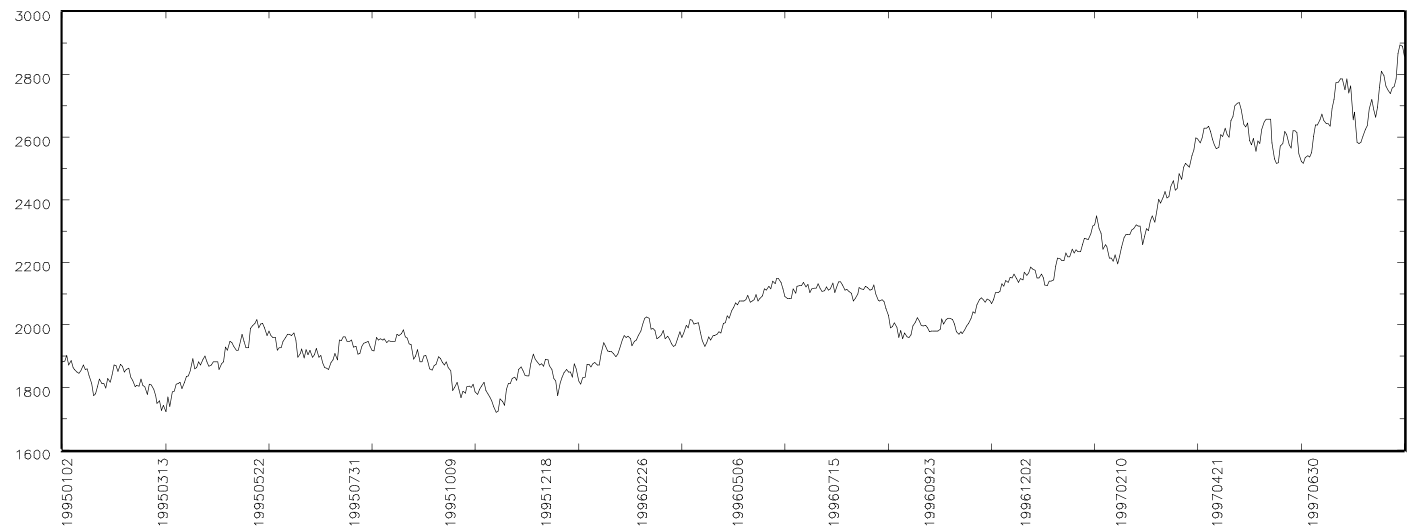


Figure 1b: daily returns on the CAC 40 index
over the time period January 1995 to July 1997

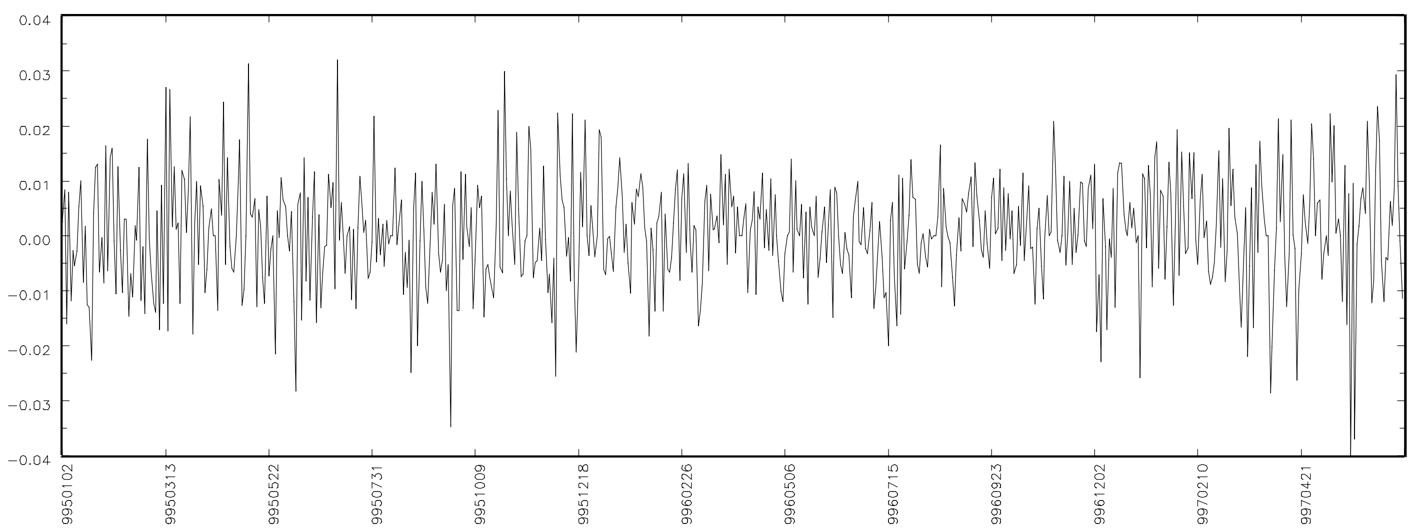


Figure 2a: volatility smile for the date 19950505

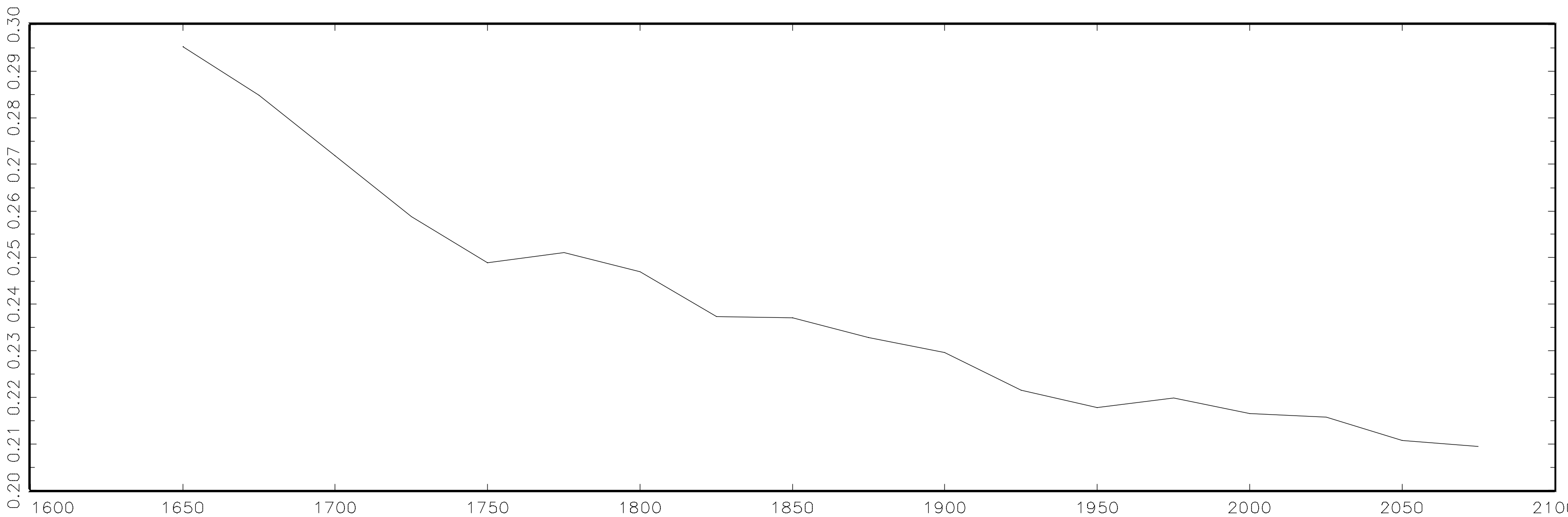


Figure 2b: volatility smile for the date 19960725

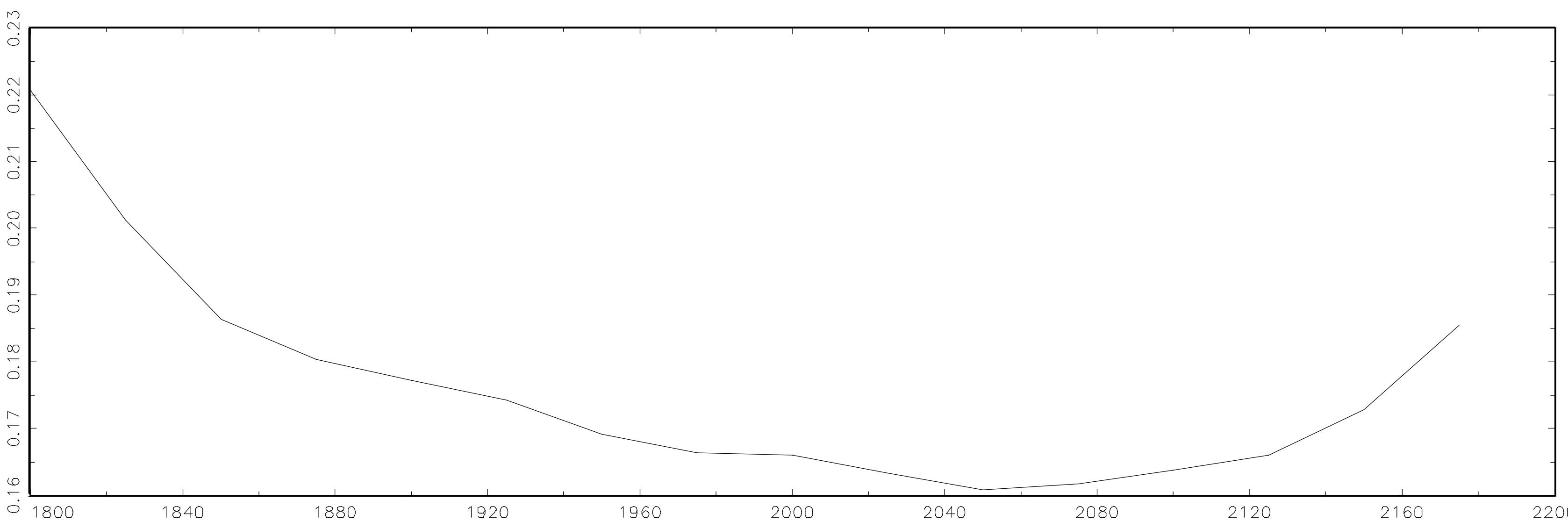


Figure 3: authorized domain for skewness and kurtosis

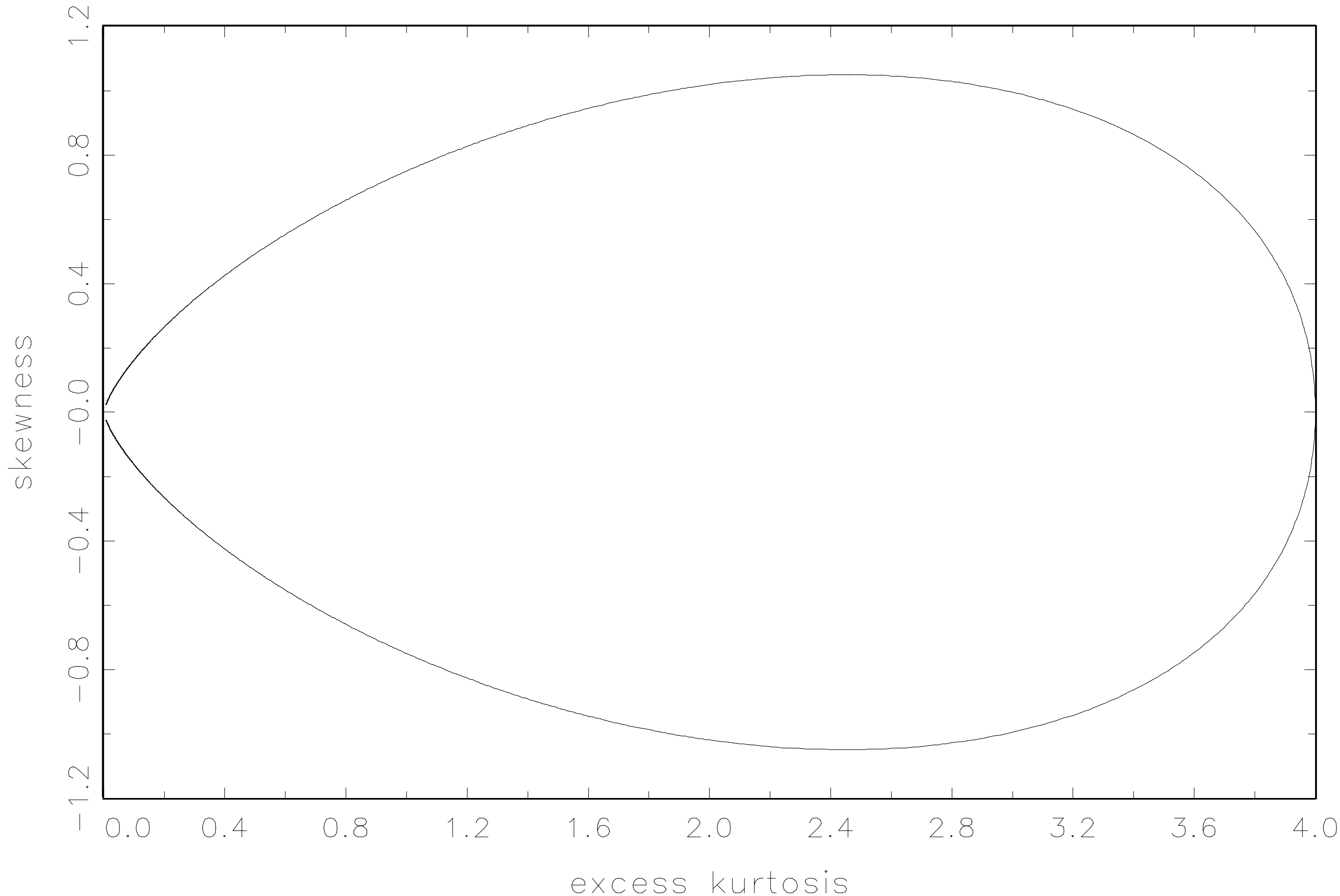


Figure 4a: RND for the date 05/05/1995 and maturity of 56 days

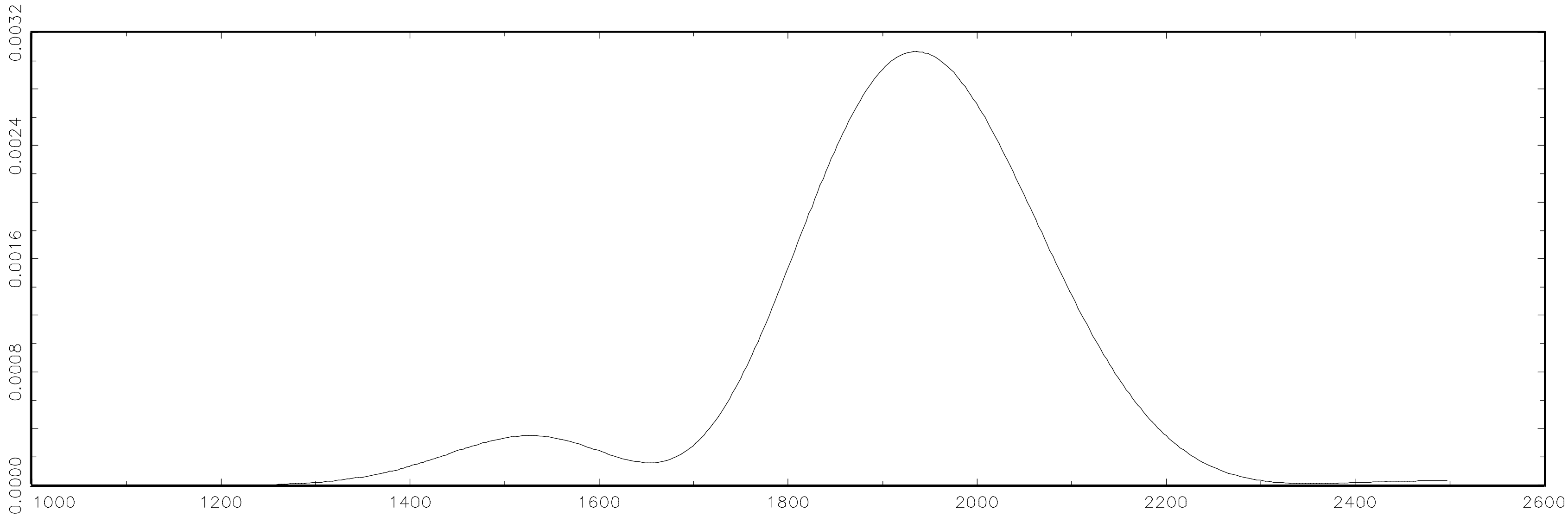
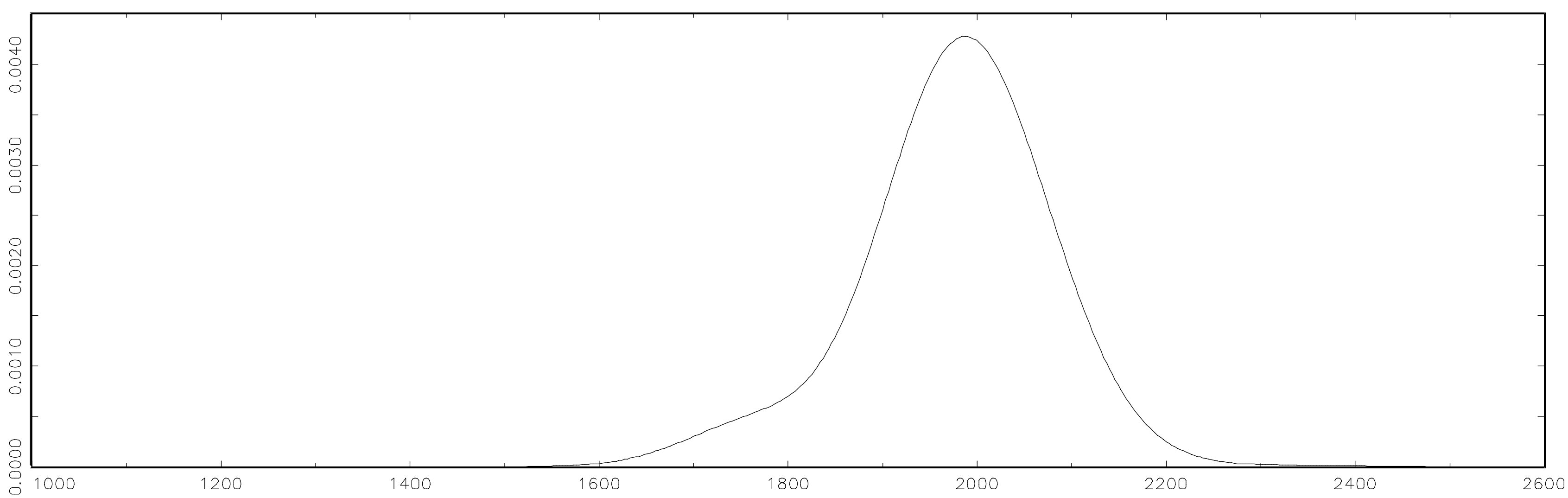


Figure 4b: RND for the date 25/07/1996 and maturity of 36 days



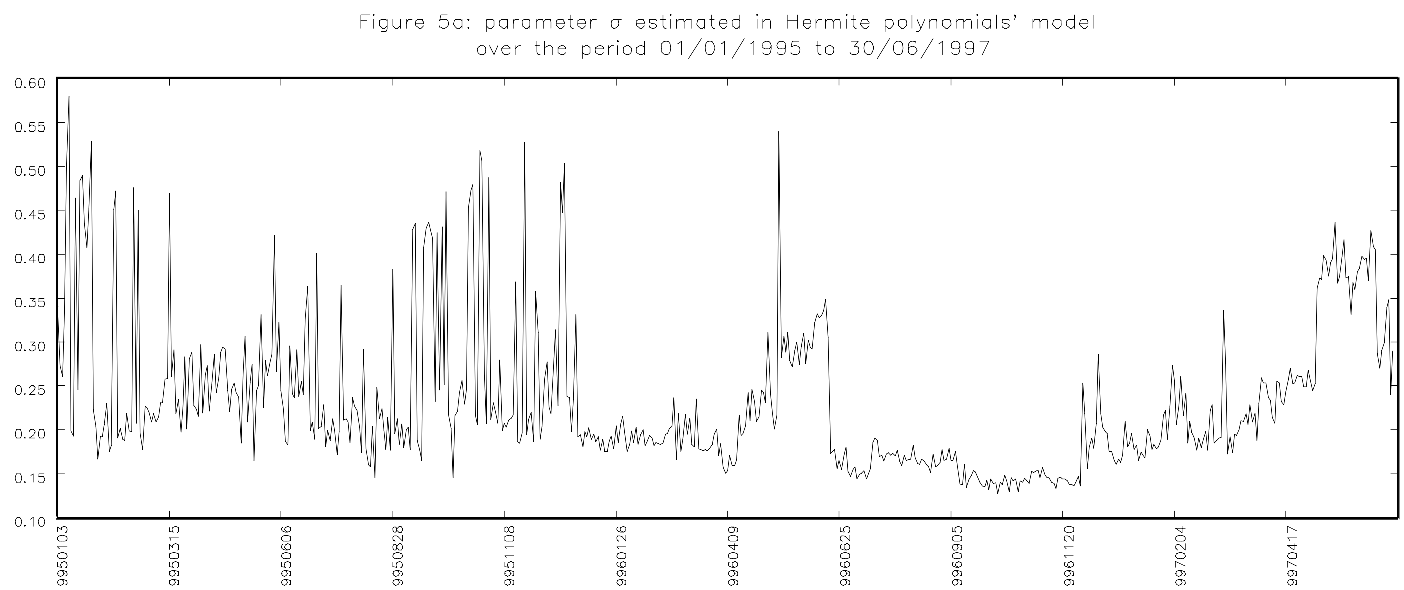


Figure 5a: parameter σ estimated in Hermite polynomials' model
over the period 01/01/1995 to 30/06/1997
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
19950103
19950315
19950606
19950828
19951108
19960126
19960409
19960625
19960905
19961120
19970204
19970417

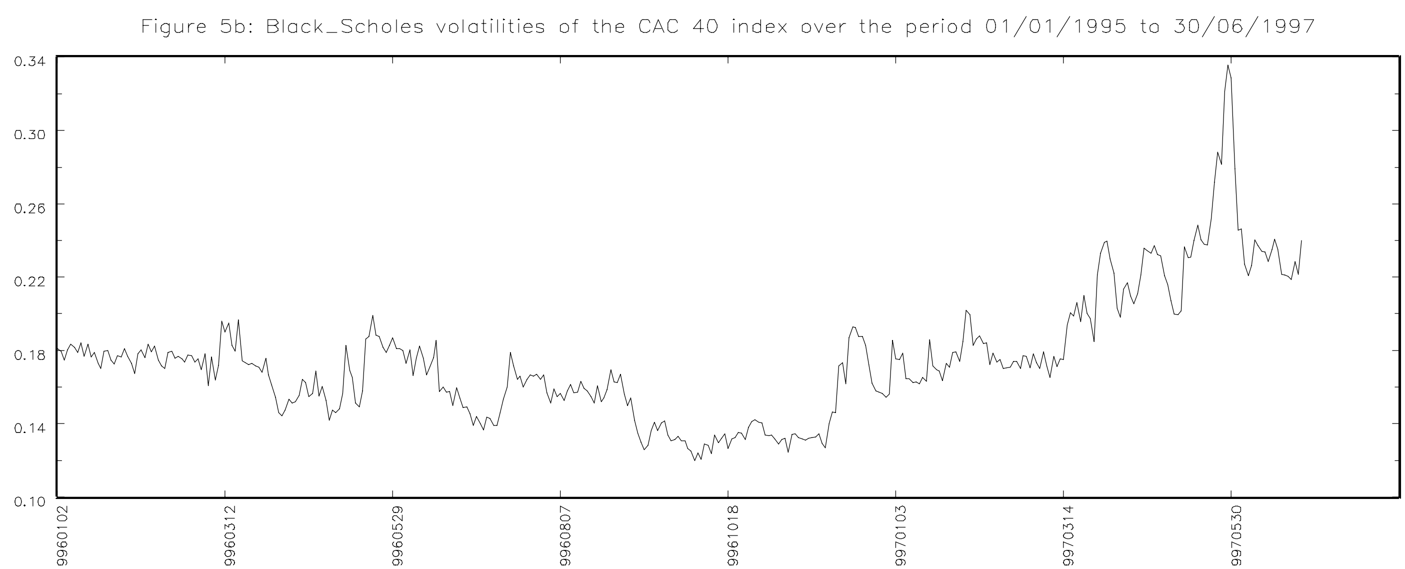


Figure 5b: Black_Scholes volatilities of the CAC 40 index over the period 01/01/1995 to 30/06/1997
0.34
0.30
0.26
0.22
0.18
0.14
0.10
19960102
19960312
19960529
19960807
19961018
19970103
19970314
19970530

Figure 6a: parameter b3 estimated in Hermite polynomials' model
over the period 01/01/1995 to 30/06/1997

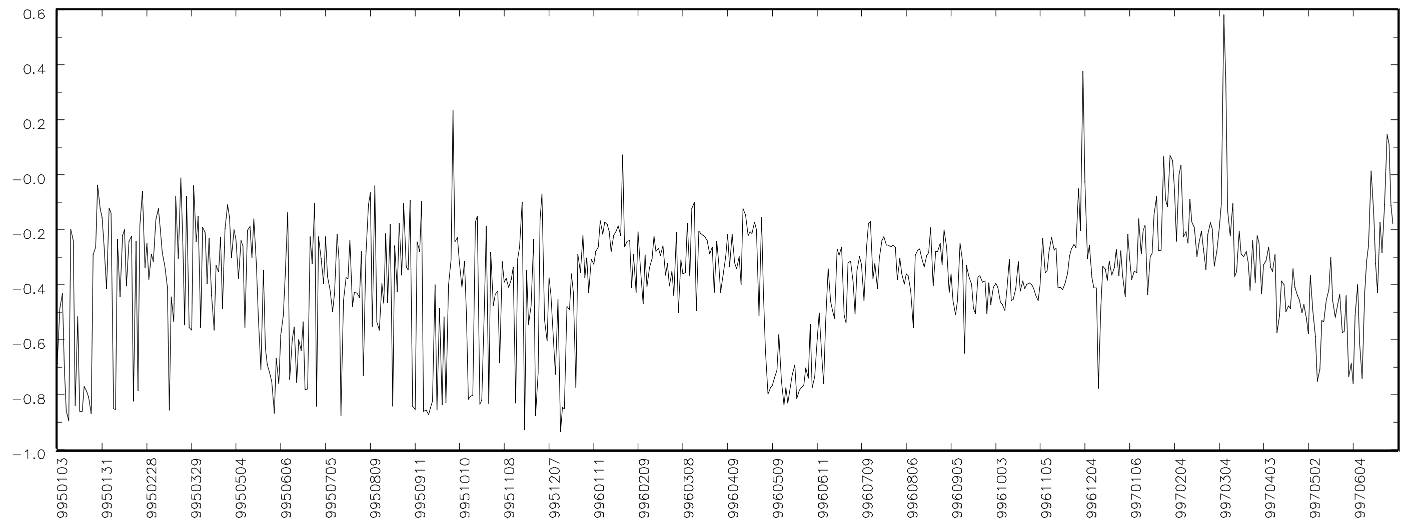
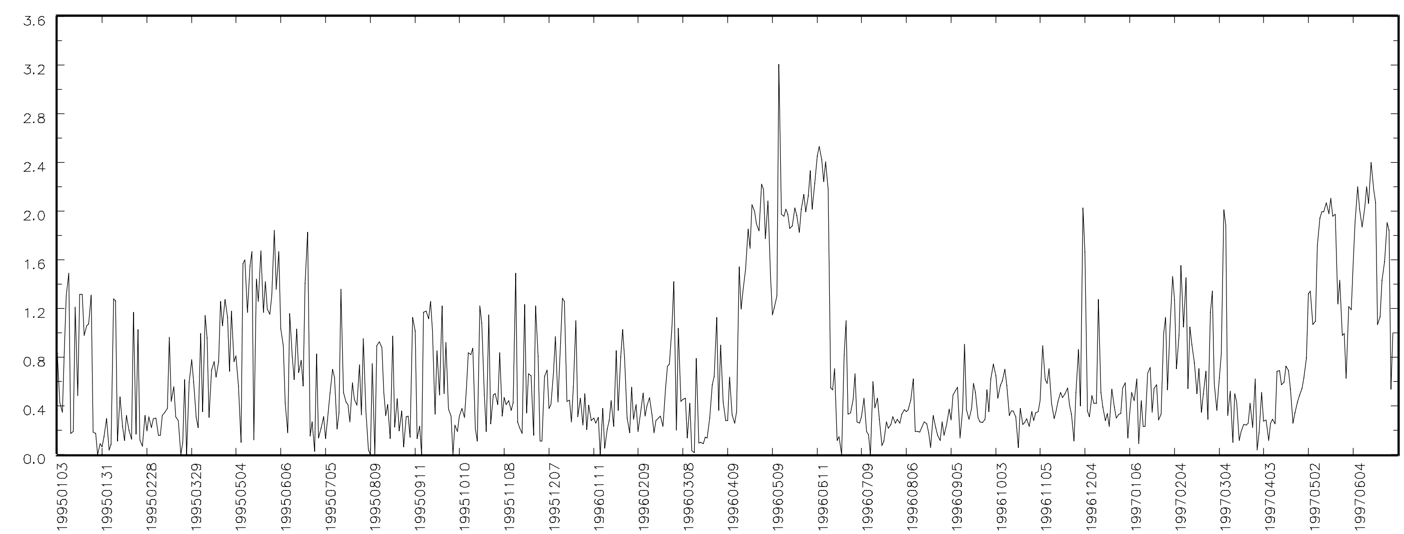


Figure 6b: parameter b4 estimated in Hermite polynomials' model
over the period 01/01/1995 to 30/06/1997



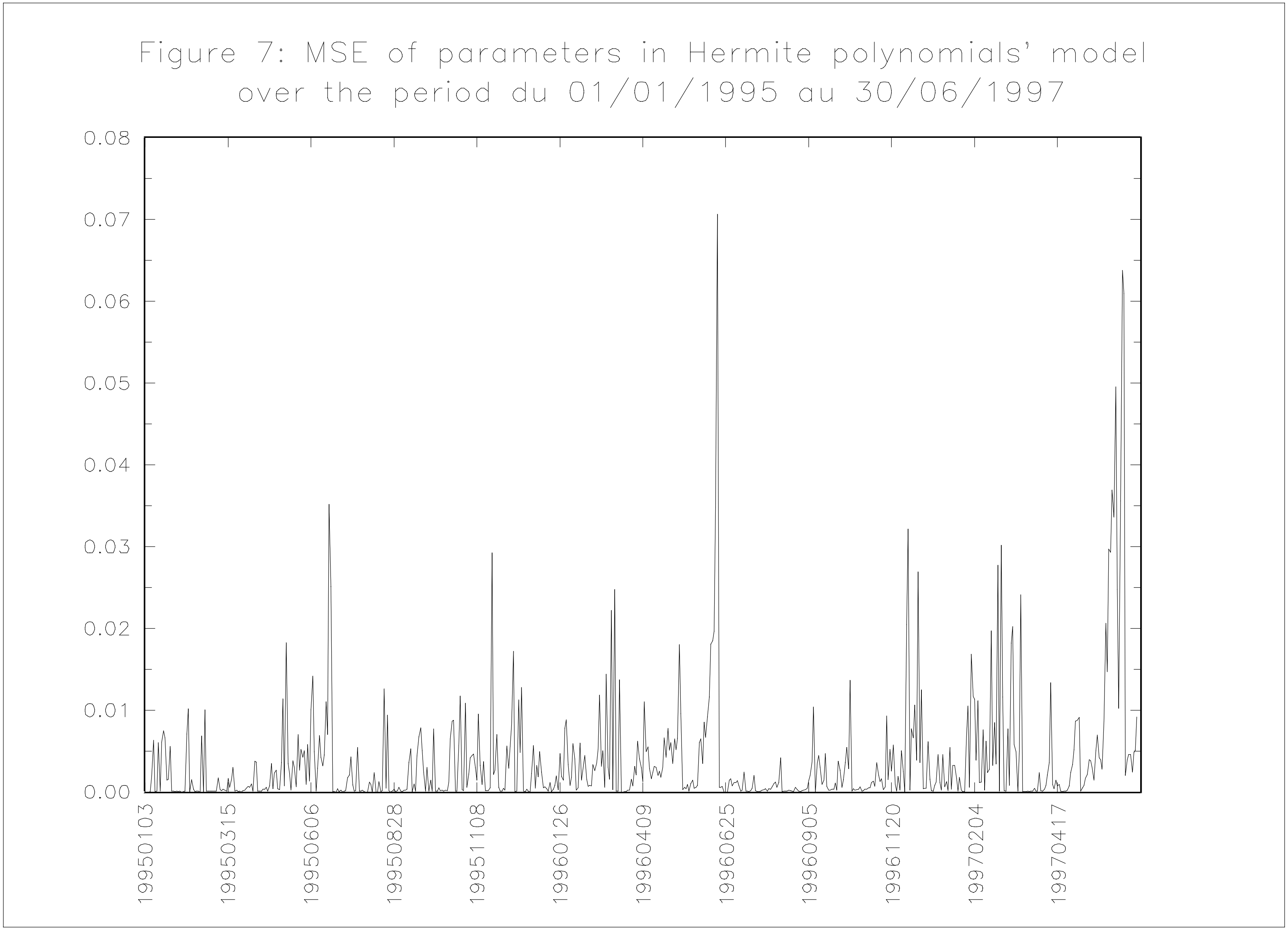


Figure 7: MSE of parameters in Hermite polynomials' model
over the period du 01/01/1995 au 30/06/1997
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
19950103
19950315
19950606
19950828
19951108
19960126
19960409
19960625
19960905
19961120
19970204
19970417

Figure 8a: absolute Risk Aversion function for the date 19950228

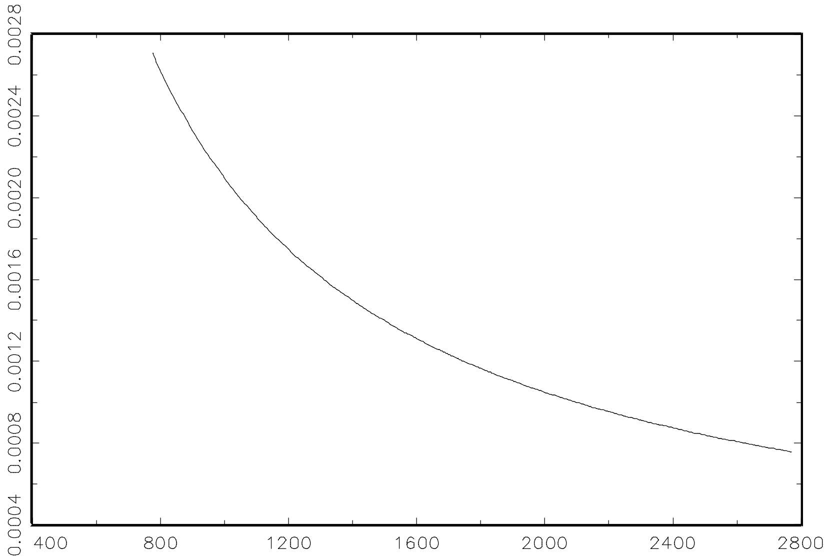


Figure 8b: absolute Risk Aversion function for the date 19950428

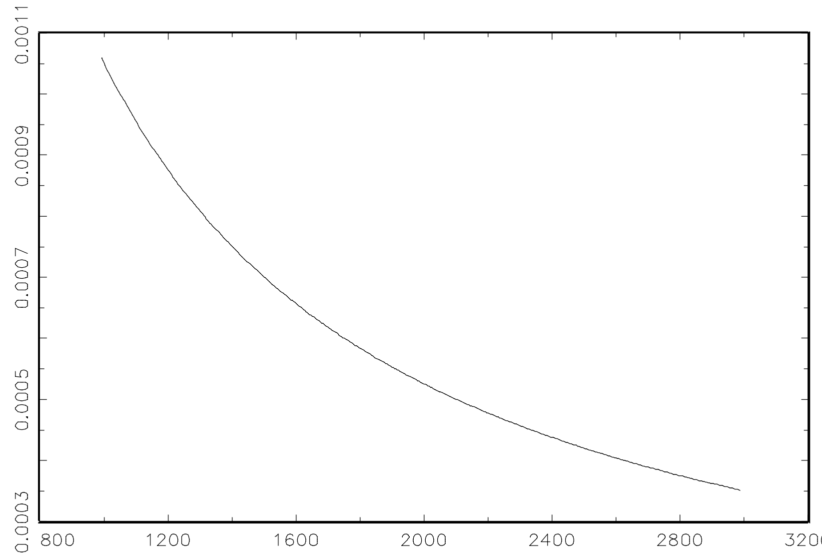


Figure 8c: absolute Risk Aversion function for the date 19960715

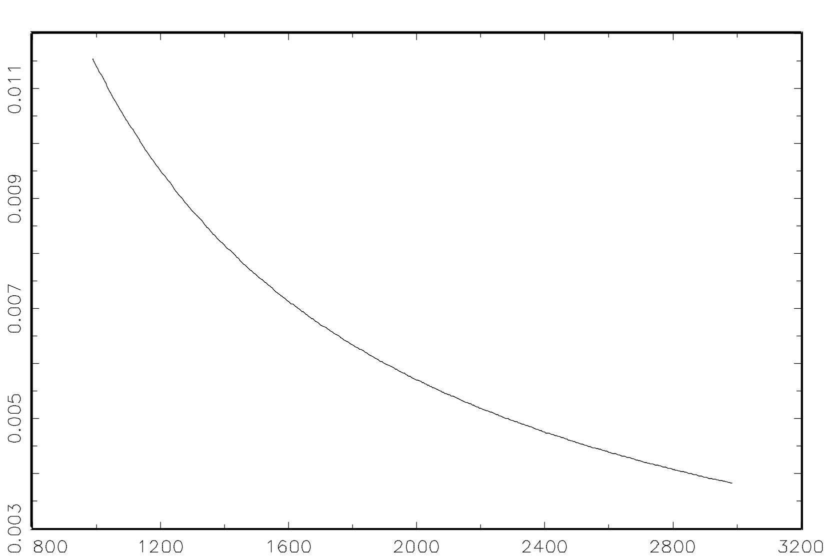


Figure 8d: absolute Risk Aversion function for the date 19961113

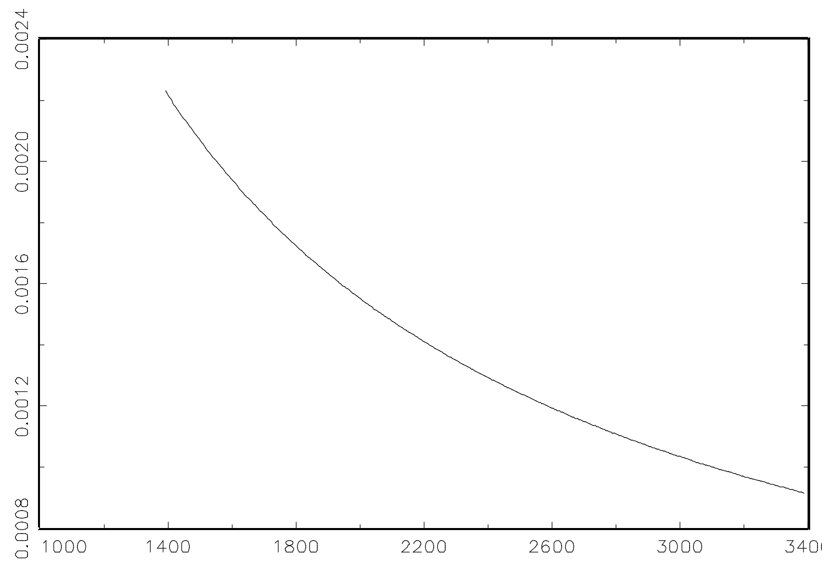
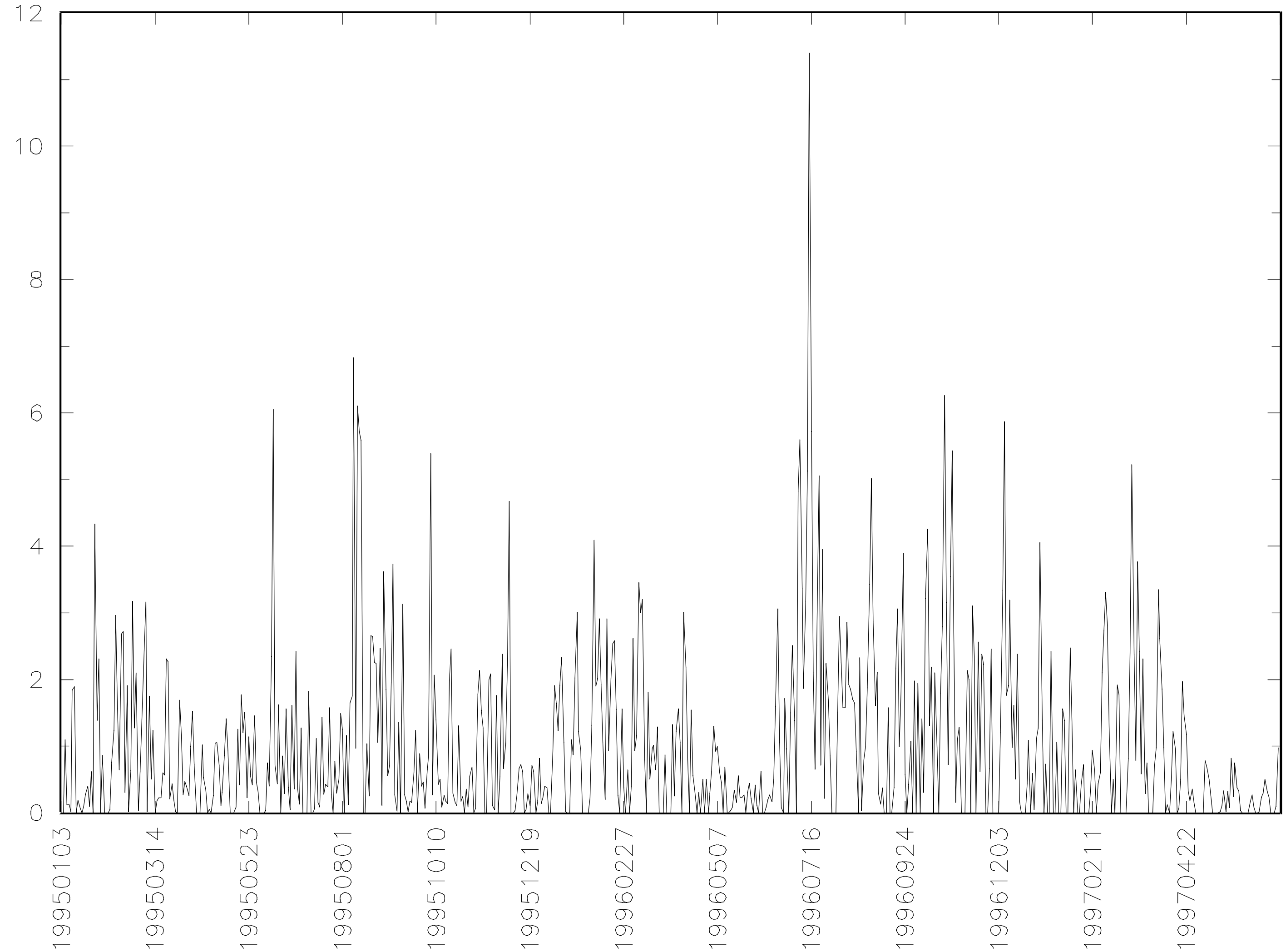


Figure 9: λ estimated with Hermite polynomials
over the period du 01/01/1995 au 30/06/1997



Discussion of the paper by Sophie Coutant, Banque de France:

Implied Risk Aversion in Options Prices

Discussant: Robert Bliss

- Starting point is equation (7) $A(S_T) = \dfrac{p'(S_T)}{p(S_T)} - \dfrac{q'(S_T)}{q(S_T)}$.
- If we know two functions, we can estimate third.
- We can estimate RND $q(S_T)$ from options prices.
- If we want SD $p(S_T)$ we must specify risk aversion funtion $A(S_T)$.
    - Some simply assume investors are risk neutral:
    - $A(S_T) = 0$.
- There is considerable evidence that investors are NOT risk neutral.
- If we can estimate SD (from past data), we can learn about risk aversion function $A(S_T)$.
- This paper makes strong assumptions about both $A(S_T), q(S_T)$, and $p(S_T)$
    - $\begin{array}{l} p_t(S_T) = v_t(S_T) n(S_T) \\ q_t(S_T) = \lambda_t(S_T) n(S_T) \end{array}$
    - Hermite polynomial representation (for $v(S_T)$ and $\lambda(S_T)$?)
        - Sums of Hermite polynomials are general approximating functions.
        - Paper truncates sum at 4^{th} order
            - Is 4^{th} order precise enough? (No discussion here or in Abken *et al.*)
        - Restrict $b_{0,t} = 1,\ b_{1,t} = 0,\ b_{2,t} = 0$.
            - Because Abken *et al.* do (to match RND and SD mean and variance).
            - What is motivation? Abken *et al.* do not explain.
        - Under these restrictions summed Hermite polynomials are no longer a general approximating function.
    - $\begin{array}{l} x_{(k+1)\Delta\tau} = x_{k\Delta\tau} + \mu_{k\Delta\tau}\Delta\tau + \sigma_{k\Delta\tau}\sqrt{\Delta\tau}e_{(k+1)\Delta\tau} \\ \mu_{(k+1)\Delta\tau} = \alpha_0 + \alpha_1\mu_{k\Delta\tau} + \beta_1 e_{(k+1)\Delta\tau} \end{array}$
        - This is a non-mean reverting process: $x_{k\Delta\tau} \to \pm\infty$.
    - $A(S_T) = \dfrac{\lambda_{t?}}{S_T}$ follows from CRRA: $U(S) = \dfrac{S^{1-\lambda}}{1-\lambda}$
        - $\Rightarrow A(S_T) = \dfrac{p'^{Hermite}(S_T)}{p^{Hermite}(S_T)} - \dfrac{q'^{Hermite}(S_T)}{q^{Hermite}(S_T)} = \dfrac{\lambda_t}{S_T}$.

- Given $p^{Hermite}(S_T)$ and $q^{Hermite}(S_T)$, λ_t is estimated w/ least squares.
- Given $p^{Hermite}(S_T)$ and $q^{Hermite}(S_T)$, why impose a functional form on $A(S_T)$?
  - Just compute $A(S_T)$.
  - Or let data suggest appropriate functional form.
    (To test CRRA assumption)

Conclusion

- Paper addresses a difficult but important problem.
  - Critical for using RDNs to assess SDs and market expectations
  - Approach is imaginative.

- Methodology used makes numerous strong, structural assumptions.
  - Restricts possible solution space to particular parsimonious function.
  - If structural assumptions are correct, answers are useful.
  - If structural assumptions are wrong, what do we have?

- Recommendations
  - Provide empirical support for structural assumptions.
  - Or better yet, use methodology to study $A(S_T)$.