# Results of the Estimation of Implied PDFs from a Common Dataset

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A part of the June 14, 1999 BIS workshop involved estimation of implied probability density functions (pdfs) by a number of participants using a common dataset. In early April, each participant received settlement price data for options on Eurodollar futures. These are American options that trade on the Chicago Mercantile Exchange. The data covered the 61 trading days from September 1, 1998 through November 30, 1998 for the December 1998 contract. The option strikes and the futures prices were subtracted from 100 (with calls redefined as puts and puts redefined as calls) in order that the probability density functions were estimated in terms of the more intuitive short-term interest rate as opposed to the discount price. These data were chosen for two reasons 1) this particular options market is among the most active in the world 2) the conference was attended by many central bank economists who for monetary policy reasons are eager to learn more about movements in short-term interest rates, and 3) the period was an active one, with the Federal Reserve lowering the federal funds rate by a total of 75 basis points from September through November.[1]

It is believed that the data are of a fairly high quality. Any option that had no open interest, exercises, or volume on a given day was excluded from the dataset. The remaining options were checked to ensure that they satisfied arbitrage restrictions involving monotonicity, slope, concavity, and put-call parity (within ranges that would result from the transactions costs involved in eliminating the arbitrage possibility). For the 61 trading days there was an average of a bit more than 25 option settlement prices per day with an average of roughly 18 unique strikes prices per day.

Each workshop participant was asked to estimate an implied PDF by whatever technique they desired for each of the 61 trading days. The participants were asked to provide a standard set of results, namely the mean, standard deviation, and 11 percentiles.[2] A total of 19 workshop participants submitted estimates, with 14 of those containing all percentiles for the 42 trading days from September 1, 1998 through October 30, 1998. These 14 "complete" submissions were used in the analysis to follow.

Within the 14 submissions, a variety of techniques were used to recover the PDF. Using the taxonomy developed in the background note, 5 of the submissions used some variant of Equation (2), either smoothing the volatility smile and then differentiating twice to recover the PDF or using finite difference methods on the option prices directly to recover the cumulative density function (CDF). Most of the

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[1] The target federal funds rate was lowered by 25 basis points on three occasions - following the regularly scheduled FOMC meetings held on September 29, 1998 and November 17, 1998 and following a conference call meeting of the FOMC on October 15, 1998.

[2] The percentiles were 0.005, 0.001, 0.050, 0.100, 0.250, 0.500, 0.750, 0.900, 0.950, 0.990, 0.995 where percentile x is defined as the eurodollar rate such that there is an x chance of the eurodollar rate falling below that rate.

remaining submissions used a method based on Equation (1), with the great majority of these using the mixture of lognormals assumption. Finally, one submission recovered the PDF as a by-product of specifying that the interest rate futures price follow a jump diffusion process.

**I. Dispersion Across Percentiles**

Given this variety of techniques, a natural first question to consider is the extent to which the different techniques (and different estimation algorithims for a given technique) produce different results. For each of the 42 trading days, the median estimate across the 14 participants for each of the 11 percentiles was calculated. These median percentiles for each trading day were then subtracted from each participants' percentile estimates for that day to create a standard measure of dispersion for each of the 11 percentiles that could be meaningfully aggregated across time. Therefore, for each of the 11 percentiles a total of 588 (14 participants 42 days) deviations were calculated, providing a measure of the dispersion across the estimates. Chart 1 shows the plots for the 11 percentiles and is somewhat discouraging.[3] If all the techniques yielded identical estimates then there would be no deviation from the median estimate of the 11 percentiles for any of the 14 participants on any of the 42 trading days. This would result in 588 zeroes being plotted for each of the percentiles - Chart 1 would show just 11 points that would form a horizontal line at zero on the vertical axis.

Obviously the actual results are nowhere near the ideal of zero dispersion, but a pattern does emerge, namely the dispersion is greater in the lower and upper percentiles than around the 0.500 percentile. As an example, on one trading day one participants' estimate of the 0.005 percentile was almost 1.8 percentage points (180 basis points) away from the median estimate of that percentile. That is, if the median estimate for the 0.005 percentile were a 3-month Eurodollar futures rate of 3.80 percent, one of the participants' estimates for the 0.005 percentile was 2.0 percent  Dispersion around the 0.500 percentile was much smaller, amounting to roughly 12 basis points below and 52 basis points above the median. For example, even on the trading day with the largest dispersion, all of the participants estimates for the 0.500 percentile fell within a range of 64 basis points. (To provide a sense of the magnitude of this dispersion the average estimate of the 0.500 percentile across the 42 trading days at roughly 5.1 percentage points.)

However, a closer examination of the estimates indicated that almost all of the large outliers from the 11 median percentiles came from a single participant.[4] Chart 1a re-plots the data from Chart 1 excluding this participant. As can be seen (the scales on the two charts are identical) the dispersion for the 13 remaining participants is much lower. The largest deviation from any of the median percentiles (again at the 0.005

[3] The plots for the 0.005 and 0.010 percentiles are very close together, as are the plots for the 0.990 and 0.995 percentiles.

[4] The participant was not using an unusual technique, in fact the participant was using the mixture of lognormals assumption, the most popular technique among the 14 complete submissions.

percentile) now amounts to 96 basis points compared to 180 basis points in Chart 1. The range of deviations from the 0.500 percentile now amounts to roughly 25 basis points compared to 64 basis points in Chart 1.

However, the question remains whether the dispersion in Chart 1a is significant in any sense. The answer surely depends on the purpose to which the PDF estimation is being applied. The results shown in Chart 1a indicate that between the 0.100 and 0.900 percentiles there is not that much difference across the techniques. That is, practitioners can have some confidence that the results they report are not overly sensitive to the particular method they use to estimate the PDF. Outside of these percentiles, the sensitivity to the technique increases dramatically. This increase can be a problem for some but not all applications. For example, an analysis for policy-making purposes that uses PDF estimation to provide a 90% confidence interval for market expectations of the future short-term interest rate will not be all that sensitive to the choice of PDF estimation technique. On the other hand, an analysis for a value-at-risk calculation that used PDF estimation to provide a measure of the future short-term interest rate below which there is less than a 1% chance of falling will be quite sensitive to the choice of PDF estimation technique.

Sensitivity of the tail percentiles to the choice of estimation technique is not surprising, given that these regions of the density have few, if any, actively traded options with strike prices in the region. As discussed in Melick and Thomas (1997), outside of the lowest and highest available strike prices there is an infinite variety of probability mass that can be consistent with the observed option prices. Put a little more precisely, for example, below the lowest strike option prices only reveal information about the combination of $\Pr[f < X_L] \cdot E[f \mid f < X_L]$, where $f$ and $X_L$ are the underlying price and lowest strike respectively. As the option price constrains only the value of the product there can be significant variation across the techniques in the two terms of the product. The reported percentiles are only related to $\Pr[f < X_L]$, just one term in the product, so a large dispersion in the tail percentiles across techniques might well be expected. That is, the observed option price provides information about the product, not the two terms, hence two methods could provide very different estimates of one of the terms so long as there were offsetting differences in the estimates for the other term.

A final source of variation relates to the difficult nature of the estimation problem. Those techniques that make use of equation (1) typically involve a nonlinear, constrained optimization. The particular solution algorithm (and the parameters involved in the algorithm such as step size and convergence tolerance) can have dramatic impacts on the estimated PDF. That is, two researchers who both use a mixture of lognormals assumption for the form of the PDF may arrive at much different conclusions depending on the optimization algorithm used. As discussed in McCullough and Vinod (1999) the variation across algorithms can be large, especially for the difficult estimations involved in a PDF recovery.[5] The same sort of problems obtain for those using techniques based on equation (2). The choice of analytic versus numerical derivatives will create variation in the percentiles, even if each researcher is generating a strike price option price mapping (e.g. volatility smile) in exactly the same way.

## II. Effects of Large Events

The dataset also provides the opportunity to assess whether the dispersion in the estimated percentiles increases during periods of large changes in economic conditions. Over the period September 1, 1998 through October 30, 1998 the FOMC lowered the target federal funds rate by 25 basis points on two occasions, September 29 and October 15. The latter cut came as a great surprise to financial market participants. Charts 2 and 3 plot the range of estimates for each of the 11 percentiles on each of the 42 trading days with vertical lines indicating the dates of the changes in the target federal funds rate. Although there is an increase in the dispersion of the percentiles following each of the changes, the increases are not large relative to other increases that do not coincide with FOMC policy changes.[6]

## Conclusion

A very preliminary analysis of the submissions to the common dataset exercise suggests that measures of tail probabilities are quite sensitive to the technique used to estimate a PDF from options prices. However, within the 10th and 90th percentiles, sensitivity to technique is much less of an issue. Finally, a shock to the underlying market does appear to increase the dispersion of the estimates of the PDF percentiles, although the increase is similar to increases seen on other dates where shocks are not readily identified.

## References

McCullough, B.D. and H.D. Vinod (1999), “he Numerical Reliability of Econometric Software”. *Journal of Economic Literature* 37(2) June pp. 633-665.

Melick, William R. and Charles P. Thomas (1997): “Recovering an Asset's Implied PDF from Option Prices: An Application to Crude Oil During the Gulf Crisis”. *Journal of Financial and Quantitative Analysis* 32(1) March pp. 91-115.

5 Future work on the estimates from this common dataset will compare the percentiles of a subset of the participants who are known to be using the same technique (pdf assumption) but different optimization packages and algorithms.

6 These charts include all of the participants, like Chart 1 but unlike Chart 1a.

**Chart 1**
**Scatter Plot of Scaled Percentiles - Sept. 1 - Oct. 30**
**(Estimated Percentile - Median of Estimated Percentile)**

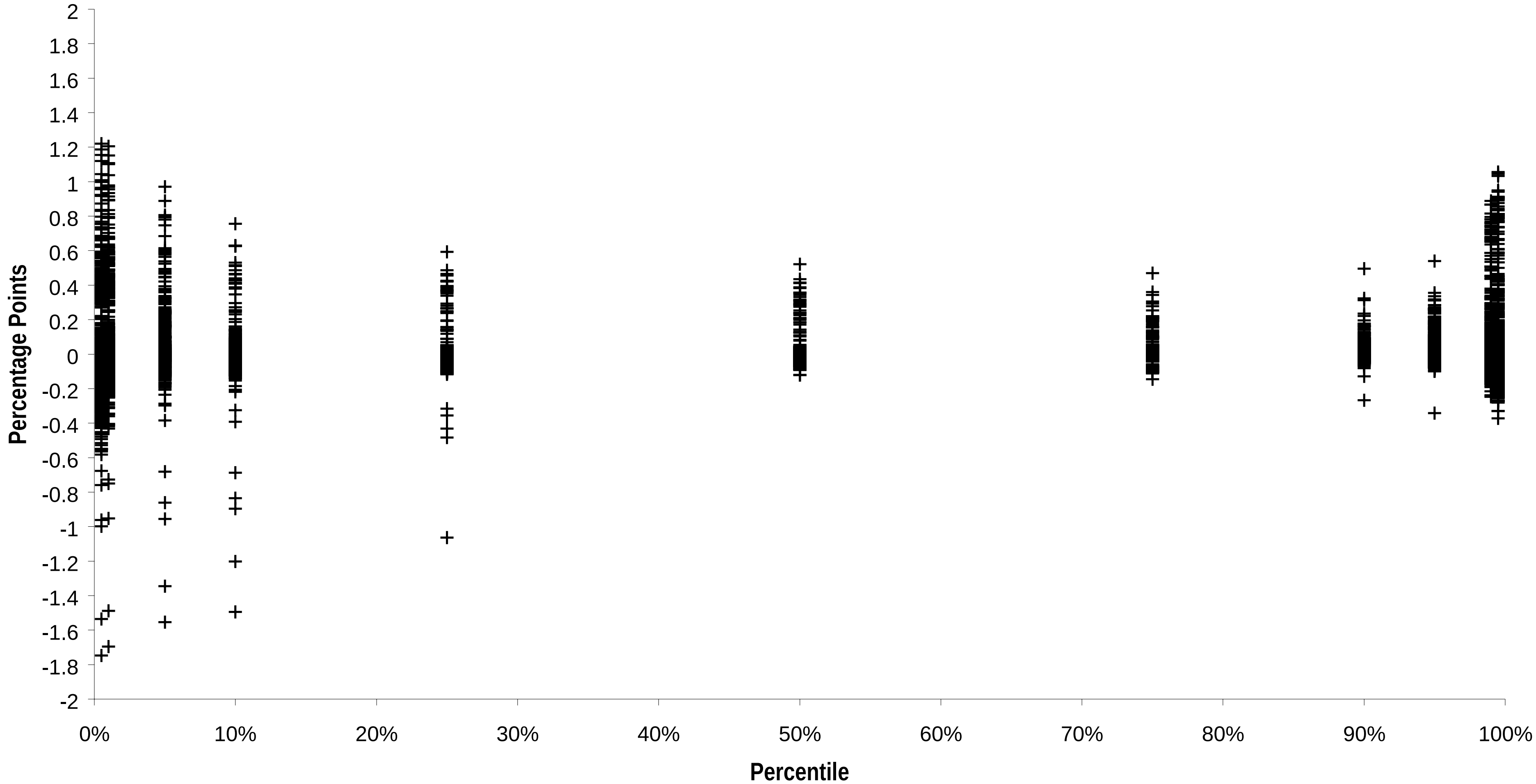
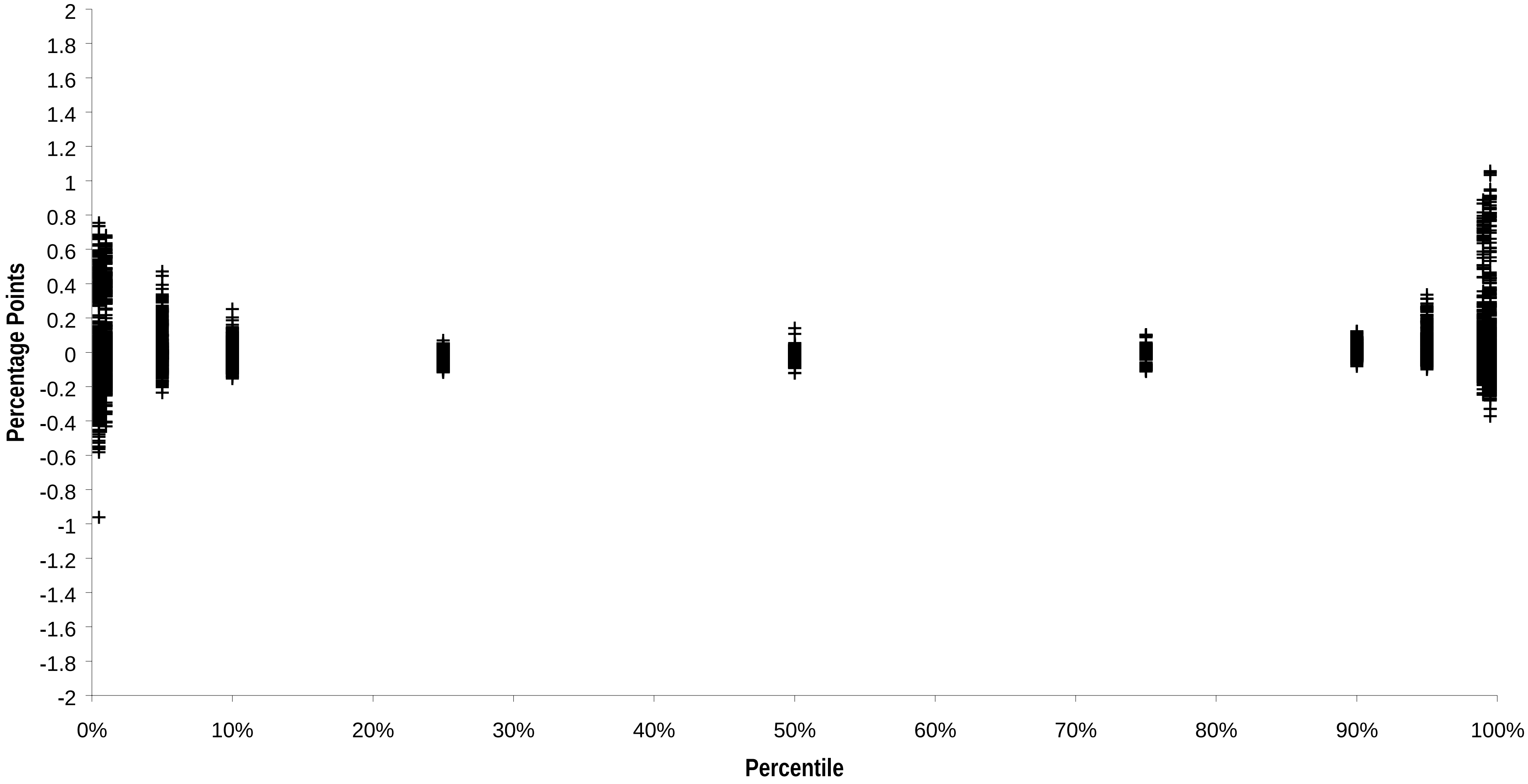


**Chart 1a**
**Scatter Plot of Scaled Percentiles - Sept. 1 - Oct. 30 - Excluding One Participant**
**(Estimated Percentile - Median of Estimated Percentile)**



## Chart 2
## Range of Estimated Percentiles

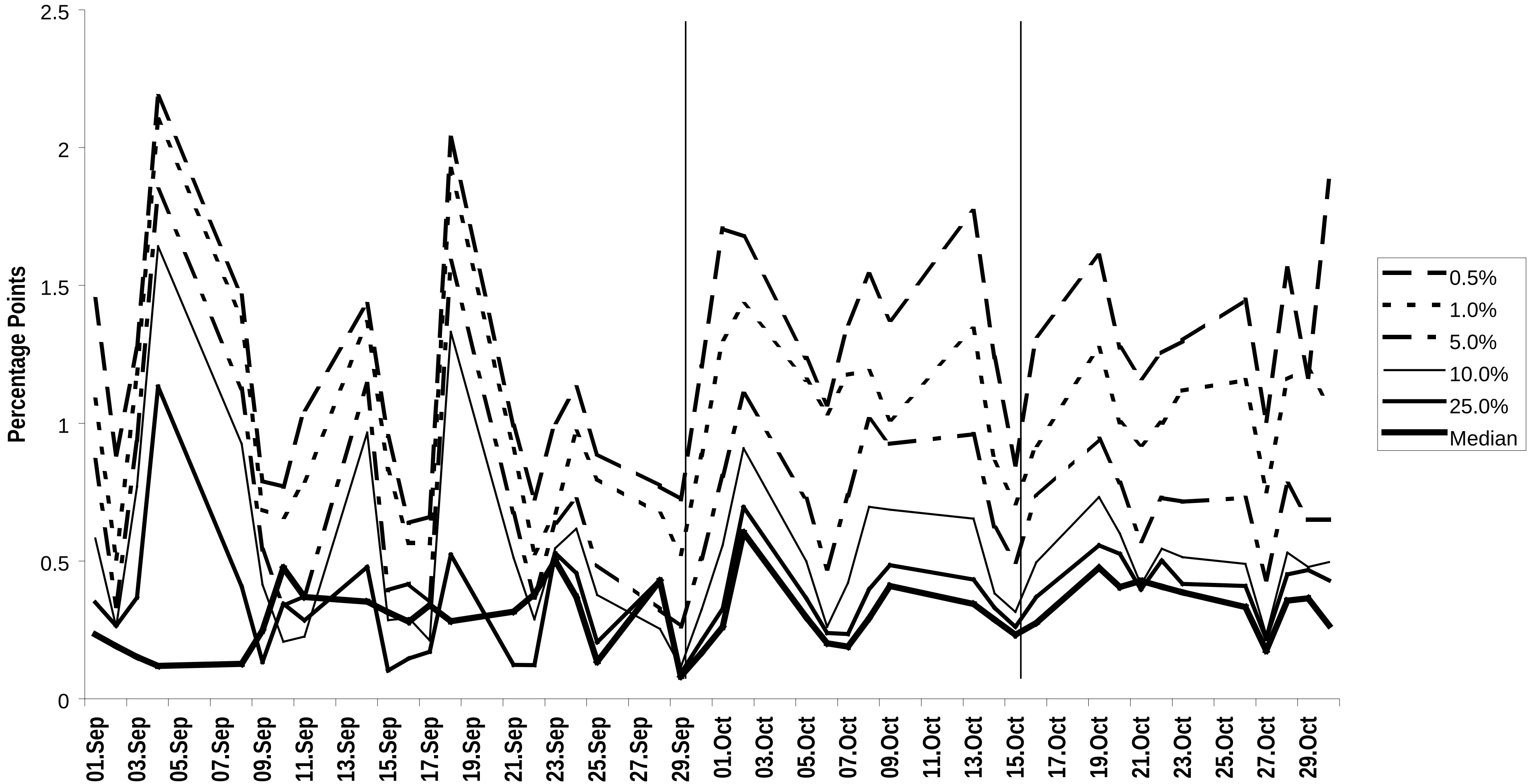


**Chart 3**
**Range of Estimated Percentiles**