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Foundations of the Proposed Modified Supervisory Formula Approach

1. This technical paper describes the modelling framework underlying the Modified Supervisory Formula Approach (MSFA) as proposed in the Basel Committee's recent consultative paper "Revisions to the Basel Securitisation Framework."[1] In contrast to the current Basel securitisation framework's Supervisory Formula Approach (SFA), which assumes a one-year maturity for the underlying pool of securitised loans ('pool'), the MSFA is based on an underlying Expected Shortfall, Mark-to-Market (MtM) framework for setting regulatory capital. This MtM underpinning, along with other key assumptions, is intended to render the MSFA more consistent with the Basel's Internal Ratings-Based (IRB) framework for wholesale exposures.

I. Background: Current Supervisory Formula Approach

2. The conceptual framework underlying the SFA was developed in Gordy-Jones (2003b), and is based on a Value-At-Risk (VAR) approach to regulatory capital along with highly stylised models of the process generating pool credit losses and the mechanism for allocating pool losses among tranches.[2] The capital horizon along with the maturities of the securitisation and underlying pool are assumed to be one year, in effect creating a pure default mode (as opposed to a MtM) modelling approach.

3. Credit losses on the bank's overall portfolio are assumed to be driven by an Asymptotic Single Risk Factor (ASRF) process.[3] This implies that any asset's VAR capital charge equals its expected one-period economic loss conditional on a stress realisation of the global risk factor (ie for a 99.9% confidence level, a realisation of the global risk factor, $\tilde{X}$, equal to its q-percentile value, x^q, where q=0.001).[4]

4. When the underlying pool is infinitely granular, within a one-period setting a well-known limitation of ASRF-based VAR models for securitisation tranches is the so-called 'knife-edge' property: the capital charge for an infinitesimally thin tranche ('tranchelet') is 0% or 100% depending on whether the tranchelet's attachment point exceeds a critical threshold value, equal to the model's implied capital charge for the entire underlying pool. Partly to mitigate this problem, Gordy-Jones (2003b) incorporates model risk as an explicit risk driver through an Uncertainty in Loss Prioritisation (ULP) model. The ULP model presumes uncertainty in how pool credit losses are allocated among tranches. For a hypothetical first-loss or equity tranche, the true economic detachment point is treated as a beta-distributed random variable having a mean of D and variance equal to $\frac{D(1-D)}{1+\tau}$, where τ is a precision parameter. Within the SFA, τ is set equal to 1000.

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1 Basel Committee on Banking Supervision (2012b).

2 To address model risk issues beyond those addressed within the Gordy-Jones (2003b) framework, the SFA incorporates various prudential add-ons (ie the capital floor, the so-called Omega Adjustment, and the 100% capital charge for tranches covering losses below the IRB charge for the underling pool).

3 For a discussion of the ASRF modelling approach, see Gordy (2003a).

4 Henceforth, a 'tilda' over the variable name (eg $\tilde{X}$) denotes a random variable.

5. The ULP model is too complex to allow tractable exact solutions for tranche capital charges, but Gordy-Jones (2003b) develops a closed-form approximation which is the basis for the SFA. Let $\mathrm{K}[\mathrm{D}]$ denote the ULP-implied capital charge for the equity tranche in the preceding paragraph. This charge can be approximated by a function $\widehat{\mathrm{K}}[\mathrm{D}]$ that depends on τ, D and, conditional on $\widetilde{\mathrm{X}} = \mathrm{x}^{\mathrm{q}}$, the expected value and variance of pool losses and the probability of zero of pool losses. Within the SFA, the pool's IRB capital charge ($\mathrm{K}_{\mathrm{IRB}}$) is used to measure the conditional expected value of pool losses. Hence, this input depends on the PDs, LGDs, and EADs of the exposures in the underlying pool and the IRB framework's assumed asset value correlations (AVCs). The other inputs also are related to these same loan-level IRB risk parameters. For an arbitrary tranche with expected attachment point A and detachment point D, the exact ULP charge equals K[D]-K[A], which can be approximated as $\widehat{\mathrm{K}}[\mathrm{D}] - \widehat{\mathrm{K}}[\mathrm{A}]$.

6. Regulators have identified several shortcomings with the SFA associated with the tendency for capital charges to fall off sharply ('cliff effects') as a tranche's notional attachment point increases above $\mathrm{K}_{\mathrm{IRB}}$. Even with the SFA's prudential add-ons, when the attachment point exceeds $\mathrm{K}_{\mathrm{IRB}}$ the SFA often produces very low charges for tranches that, if externally rated, would be well below investment grade. SFA capital charges, particularly for relatively thin mezzanine positions, also can be very sensitive to small changes in $\mathrm{K}_{\mathrm{IRB}}$. Such cliff effects and input sensitivities seem excessive in light of the modelling uncertainties inherent in the SFA. The MSFA attempts to reduce these concerns through three modifications to the Gordy-Jones (2003b) model.

7. First, we relax the one-year maturity assumption. When dealing with maturities greater than one year a MtM-based modelling approach is appropriate for quantifying portfolio risk and capital charges. The SFA makes the implicit assumption that a given tranche will not incur any market value loss until the values for all more-junior tranches have been reduced to zero. Indeed, for given attachment and detachment points the SFA presumes that maturity affects a tranche's capital charge only through the $\mathrm{K}_{\mathrm{IRB}}$ input – presuming that no further adjustments are needed so long as this input fully captures the pool's MtM risk. Conceptually, this treatment is problematic. The MtM modelling approach used herein avoids this unrealistic assumption as well as a related apples-oranges problem in the treatment of expected losses within the SFA and IRB frameworks.[5]

8. Second, in contrast to the SFA's underlying VAR approach, we adopt an Expected Shortfall (ES) approach to estimating regulatory capital.[6] Within an ASRF framework, the ES approach avoids the knife-edge result of VAR-based models by explicitly incorporating the notion that regulators are concerned not only with the probability of a bank's losses exceeding capital, but also with the expected magnitude of any such shortfall.

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5 The apples-oranges issue pertains to the SFA's implicit treatment of expected losses beyond the one year capital horizon. The IRB framework ignores expected default losses beyond one year (in effect, presuming they will be covered by excess spread). However, in practice the credit enhancement associated with a tranche's attachment point (A) generally is available to cover pool credit losses up to A regardless of whether the losses are expected or unexpected. Thus, in practical applications of the SFA there is an apples-oranges problem in that a positive $A - K_{IRB}$ (reflecting a tranche attaching above the pool's IRB capital charge) generally overstates the degree to which the tranche is protected against unexpected pool losses over the life of the transaction. The amount of this overstatement equals the pool's (unconditional) expected losses beyond the capital horizon.

6 See Basel Committee on Bank Supervision (2012b). The Basel Committee's consultative paper on a fundamental review of capital charges for the trading book, published in May 2012 and available at www.bis.org/publ/bcbs219.pdf, also proposes an ES-based capital metric.

9. The third difference relates to the treatment of model risk. Unlike the SFA, the MSFA incorporates model risk by focusing on uncertainty around the true process governing pool losses. Estimated capital charges for securitisation tranches, particularly for mezzanine and junior positions, tend to leverage model risks inherent in the assumed process generating pool losses. In the aftermath of the financial crisis there is recognition of a need to reflect model risks to a greater extent within the Basel's securitisation framework. While in principle the ULP could perhaps incorporate greater model risk through a smaller τ, a key ULP approximation result may break down for τ values materially below 1000.[7] The MSFA framework herein enables setting the τ parameter much lower than 1000 without sacrificing formulaic tractability.

10. The remainder of this paper is organised as follow. Section II reviews the ES metric for calculating an arbitrary asset's capital charge within an ASRF modelling approach and summarises the methodology used to calibrate the ES confidence threshold parameter within the MSFA. In Section III, we derive the key approximation result underpinning the MSFA, through which a tranche's capital charge can be expressed as a closed-form function of (a) the expected value and variance of the pool's lifetime default losses under a particular conditional, risk-neutral probability distribution; (b) the probability of zero lifetime pool losses under the same probability distribution; and (c) a τ parameter that captures regulators' confidence in the underlying regulatory model for quantifying pool credit risk. Section IV then derives the MSFA's equations for estimating these inputs as functions of the IRB risk parameters for the securitised exposures. Lastly, section V presents some concluding remarks.

II. Expected Shortfall Approach to Regulatory Capital Within an ASRF Setting

11. Although retaining the assumption of a one-year capital horizon, the MSFA departs from the VAR-based framework underlying the SFA and, instead, adopts an ES-based approach to regulatory capital. However, before describing this ES approach in some detail, it is necessary to establish some notational conventions. The date t=0 corresponds to the as-of date for which we are estimating a tranche's capital charge (ie 'today's date'), and t=1 corresponds to the end of the capital horizon. Obviously, the estimated capital charge must depend on currently available information at t=0. For notational ease, all monetary variables are normalised relative to the amount of pool principal at t=0 ('current pool principal'). Thus, for example, current pool principal equals 1, and the lifetime loss rate for pool principal can potentially vary between 0 and 1. The maturity of securitisation is denoted M, and is measured in years.

12. For reference we note that under the ASRF modelling approach herein, the VAR-based capital charge for a securitisation tranche be calculated as

(1) $\mathrm{K_{VAR}} = \mathrm{V_0} - \mathrm{E_0^{NP}}\{\tilde{\mathrm{V}}_1 | \tilde{\mathrm{X}}_1 = \mathrm{x_1^{qVAR}}\} \cdot \mathrm{e^{-R}}$

where V0 is the tranche's known value at t=0; $\tilde{\mathrm{V}}1$ is its random value at t=1; $\mathrm{E_0^{NP}}\{.|.\}$ denotes the conditional expectation at t=0 under the natural or physical probability distribution; $\tilde{\mathrm{X}}_1$ is the realised value of the global risk factor at t=1; $\mathrm{x_1^q}$ is the q-percentile of the global risk factor at t=1; and R is the bank's funding rate, which is assumed to equal the fixed risk-free interest

7 See Gordy-Jones (2003a) and Gordy (2005).

rate. Within the IRB framework q is set at 0.001 (consistent with a 99.9% nominal confidence level). The e^{-R} term in (1) discounts the tranche's conditional future expected value (at t=1) back to the current time (t=0).

13. Similarly, a tranche's ES-based capital can be calculated as

$$(2)\quad K_{ES} = V_0 - E_0^{NP}\{\tilde{V}_1 | \tilde{X}_1 \leq x_1^{qES}\} \cdot e^{-R}$$

where x_1^{qES} is the assumed stress threshold for the global risk factor under the ES metric. For notational simplicity, in what follows we will drop the ES subscript from K when referring to ES-based capital charges.

14. To provide consistency with capital charges produced by the IRB wholesale framework, the ES threshold parameter qES is calibrated such that for maturities of one year the ES-based capital charges and IRB capital charges are similar for wholesale loans. For M=1, a loan's wholesale IRB charge per unit of exposures equals

$$(3a)\quad k_{IRB} = LGD \cdot \Phi[(\Phi^{-1}[PD1] - x^{qVAR}\sqrt{AVC})/\sqrt{1 - AVC}]$$

where $\Phi[z]$ is the cumulative distribution function (CDF) for a standard normal random variable, $x^{qVAR} = \Phi^{-1}[0.001]$, and $LGD, PD1, \text{and } AVC$ are the loan's loss rate given default, the loan's one-year default probability, and the loan's asset value correlation, respectively.

15. For M=1, the ES-based charge per unit of exposure is calculated as

$$(3b)\quad k = \left(\frac{LGD}{qES}\right) \cdot \int_{-\infty}^{x^{qES}} \Phi[(\Phi^{-1}[PD1] - x\sqrt{AVC})/\sqrt{1 - AVC}]\, \phi[x]\, dx$$

$$= \left(\frac{LGD}{qES}\right) \cdot \Phi_2\left[\Phi^{-1}[PD1], \Phi^{-1}[qES]; \sqrt{AVC}\right]$$

where $\phi[z]$ is the probability density function for a standard normal random variable, and $\Phi_2[a, b, ; c]$ is the CDF for the standard bivariate normal random variable with correlation parameter c.[8]

16. After some experimentation, the value of qES corresponding to a 99.7 percentile stress threshold (eg $qES = 0.003$) was found to generate ES-based capital charges for one-year wholesale exposures (k) comparable to the IRB framework's 99.9 percentile VAR standard (K_{IRB}). As shown in Chart 1, with qES set at this value and $M = 1$, ES-based charges for non-tranched wholesale loans (inclusive of expected credit losses) are virtually identical to those produced by the IRB wholesale equation.[9] In the remainder of this paper, qES is calibrated to this value.

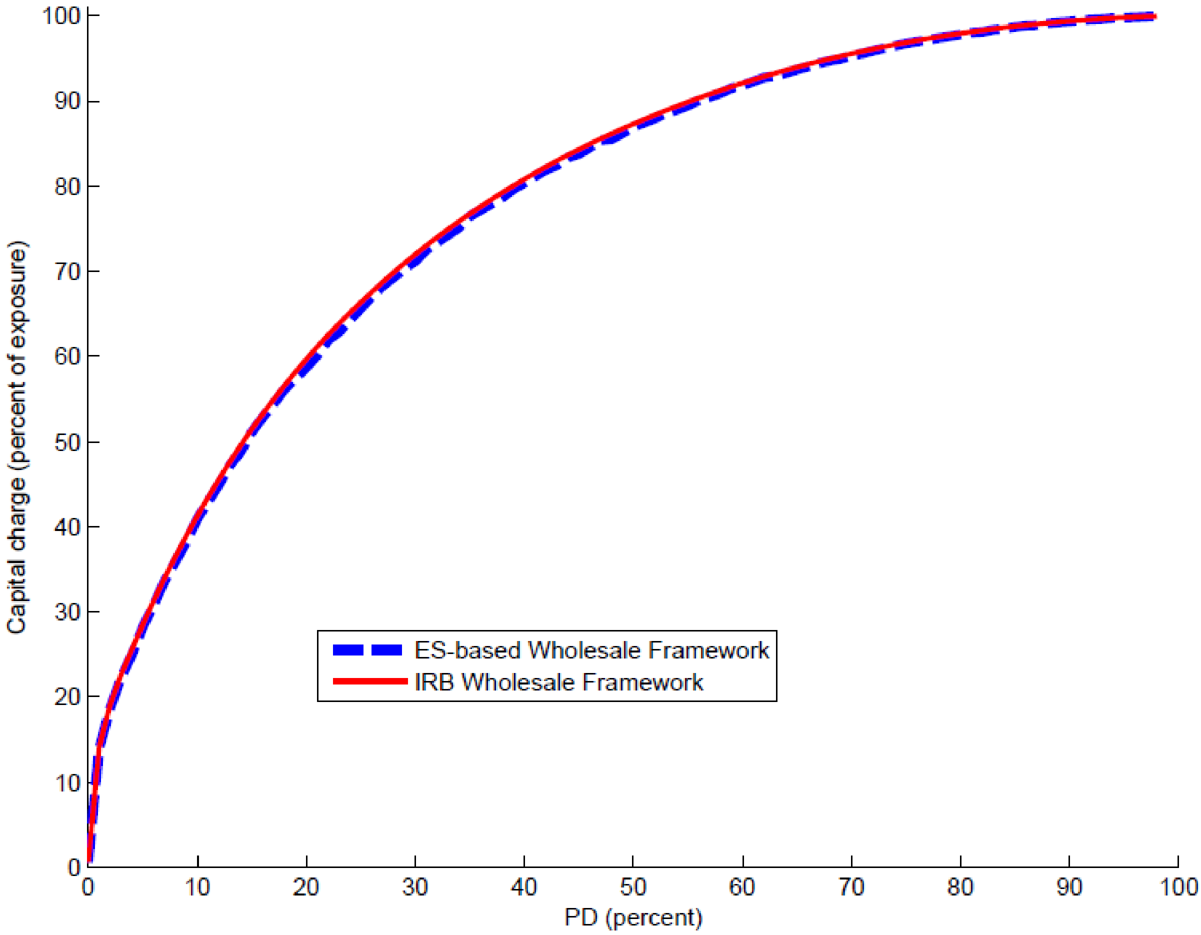
8 The second equality above follows from application of equation (30c) in Andersen-Sidenius (2004/2005).

9 Since required capital is proportional to LGD, for simplicity charts 1 and 2 assume that LGD=100%

Chart 1

Wholesale IRB Capital Charges and ES-based Charges with 99.7% Confidence Threshold

(assumes LGD=100%, M=1 year, and wholesale IRB asset value correlations)

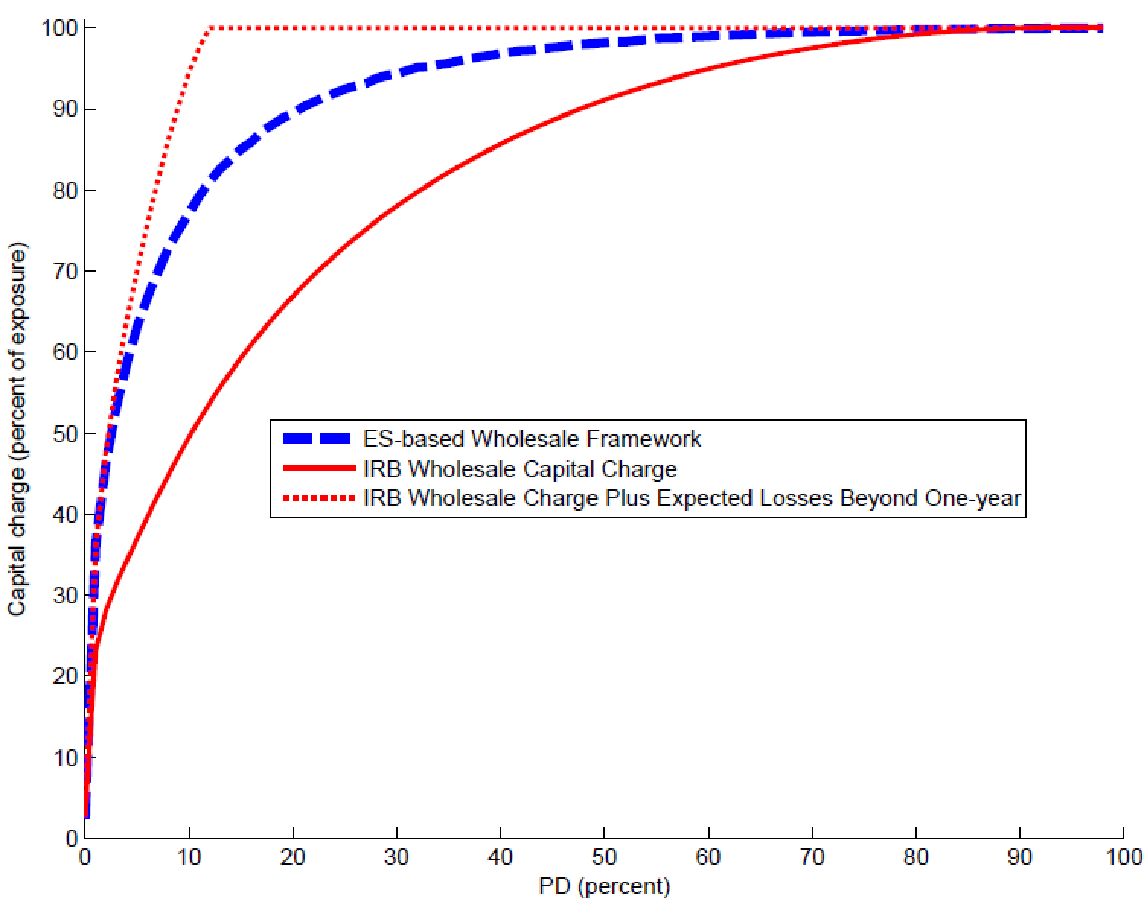


17. Chart 2 presents a similar comparison for $M = 5$. As in the IRB wholesale framework, within the proposed MSFA the maturity input is capped at five years. In this chart, the dashed line depicts capital charges implied by the MSFA's ES-based model, while the dotted line shows charges implied by the IRB framework inclusive of a loan's expected lifetime credit losses. Since tranche capital charges under the MSFA are based on attachment points that represent protection against both unexpected and expected lifetime pool losses, the IRB charge inclusive of lifetime expected credit losses is one possible metric for viewing the MSFA's consistency with the VAR standard underling the IRB approach.[10] This metric avoids the apples-oranges issue noted above. By this yardstick, at a maturity of five years the 99.9% VAR standard embedded in the IRB is somewhat more conservative than the ES-based standard with a 99.7 percentile stress threshold.

Chart 2

Wholesale IRB Capital Charges and ES-based Charges with 99.7% Confidence Threshold

(assumes LGD=100% and M=5 years)



[10] For both the ES-based and IRB-based calculations underlying this chart, we assume the term structure of PDs described in section IV.

18. The comparison is reversed, however, when IRB wholesale capital charges are measured in Pillar 1 terms (as shown by the solid line) – that is, calculated as the sum of unexpected loss and expected default losses over the capital horizon. Under this metric the ES-based charge for a five-year wholesale loan generally exceeds the pool's IRB charge. This relationship arises because IRB Pillar 1 charges assume, in effect, that expected default losses beyond the capital horizon will be covered by margin income or excess spread. This assumption was problematic for many securitisations during the financial crisis, as sharp deteriorations in the underlying pools eroded anticipated excess spread. A key difference between the IRB framework for wholesale exposures and the MSFA framework is that the latter does not provide any capital benefit for excess spread. For maturity exceeding one year, this difference in the treatment of excess spread is one of several reasons why the sum of MSFA charges across all tranches of a securitisation would tend to exceed the IRB charge for the underlying pool, even abstracting from the proposed MSFA's prudential add-ons.

III. Conceptual Framework for Calculating Tranche Capital Charges

19. In this section we develop the MSFA framework for calculating ES capital charges within an MtM setting. The discussion distinguishes between the true, but unknown, process driving pool credit losses and the assumed regulatory model used in calibration. The regulatory model is assumed to be an unbiased, but nevertheless imprecise, estimator of the true model. The key result from this section is a system of equations for approximating a tranche's capital charge in terms of (a) the mean and variance of lifetime pool credit losses under the conditional risk-neutral probability distribution implied by the regulatory model; (b) the conditional risk-neutral probability of zero lifetime pool losses under the regulatory model; and (c) a model risk parameter τ that captures regulators' confidence in the regulatory model.

Pool Cash Flows

20. We assume an underlying pool consisting of N bullet loans where both contractual interest and principal are payable at maturity (t=M). The jth loan's current principal outstanding is denoted θ_j, while it's contractual interest rate is R_j.[11] If the loan defaults at or prior to maturity, the payout is assumed to be $\theta_j \cdot e^{R_j \cdot M} \cdot \left[1 - \widetilde{LGD}_j\right]$, received at maturity. The random loss given default $\widetilde{LGD}_j$ is independent of all other risk factors and has a continuous probability distribution with mean given by $\overline{LGD_j}$ and variance given by $0.25\,\overline{LGD_j} \cdot \left(1 - \overline{LGD_j}\right)$.[12]

21. Lifetime principal losses incurred by the pool ('pool credit losses') equal

(4) $$\tilde{L} = \sum_{j=1}^{N} \theta_j \tilde{I}_j \cdot \widetilde{LGD}_j$$

where the random default indicator $\tilde{I}_j$ equals 1 if the j^{th} loan defaults by t=M, and zero otherwise. Notice that the timing of loan defaults has no impact on the lifetime cash flows for individual loans or the pool as a whole.

22. All cash accumulated within the securitisation special purpose vehicle is distributed to investors. At maturity, the pool's lifetime principal losses are allocated among tranches

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11 Thus, $\sum_{j=1}^{N} \theta_j = 1$.

12 This specification for loss severities is the same as that within the SFA.

based on seniority. Specifically, consider a tranche with notional attachment and detachment points A and D. The remaining principal of this tranche at maturity, denoted $\widetilde{P}_M[A,D]$, is equal to $(D-A)-\min[D-A,\max[0,\widetilde{L}-A]]$. The tranche's total payout at maturity (principal plus interest) then equals $\widetilde{P}_M[A,D]$ multiplied by $e^{RT\cdot M}$, where RT is the tranche's contractual coupon rate. Below, we will make use of the fact that

(5) $$e^{RT\cdot M}\,\widetilde{P}_M[A,D] = e^{RT\cdot M}(\widetilde{P}_M[0,D]-\widetilde{P}_M[0,A]).$$

MtM Framework for Tranche Capital Charges

23. We now set forth the basic MtM framework underlying the MSFA. At t=0, the tranche is assumed to be valued at par (ie $V_0 = D - A$). At t=1, let the tranche's market value be represented as $V_1[A,D;\widetilde{\Omega}_1]$ which depends on the information available to the market participants at that time, denoted $\widetilde{\Omega}_1$. At the beginning of the capital horizon not all information used in setting prices at t=1 will be revealed to market participants (eg defaults occurring during the capital horizon). Thus, as of t=0, the information that will be available at t=1 is a random vector.

24. From (2), the tranche's ES-based capital charge equals

(6) $$K[A,D] = (D-A) - e^{-R}\cdot E_0^{NP}\{V_1[A,D;\widetilde{\Omega}_1] \,\|\, \widetilde{X}_1 \le x_1^{qES}\}$$

where the notation $E_0^{NP}\{Z[\widetilde{\Omega}]\,|\widetilde{X}_1 \le x_1^{qES}\}$ denotes the expected value at t=0 of the random variable $Z[\widetilde{\Omega}]$ under the natural probability distribution for $\widetilde{\Omega}$, conditional on a stressed realisation of the global risk factor $\widetilde{X}_1$ at t=1.

25. We assume the existence of a risk-neutral probability distribution for valuing loans and their derivatives at t=1, which will depend on the realisation of $\widetilde{\Omega}_1$. Thus, the tranche's market value at t=1 equals the expected value of the tranche's cash flow under the risk-neutral distribution, discounted at the risk-free interest rate. Letting $H_1^{RN}[L;\widetilde{\Omega}_1]$ denote the risk-neutral CDF for pool credit losses, upon substituting (5) into (6) we obtain[13]

(7) $$K[A,D] = (D-A) - e^{(RT-R)\cdot M}\cdot E_0^{NP}\left\{\int_A^D H_1^{RN}[L;\widetilde{\Omega}_1]\,dL \,\middle|\, \widetilde{X}_1 \le x_1^{qES}\right\}$$

$$= (D-A) - e^{(RT-R)\cdot M}\cdot \int_A^D G[L]\,dL$$

where $G[L] \equiv E_0^{NP}\{H_1^{RN}[L;\widetilde{\Omega}_1]\,|\widetilde{X}_1 \le x_1^{qES}\}$.

26. The function $G[L]$ represents the expected value at t=0 of the risk-neutral CDF for pool credit losses at t=1, conditional on an ES-stress event. In other words, if a stress event $(\widetilde{X}_1 \le x_1^{qES})$ were to occur over the capital horizon, $G[L]$ represents the probability that market prices at t=1 would imply risk-neutral lifetime pool principal losses not greater than L. Note that this function satisfies the basic properties of a CDF.[14] At some risk of imprecision

[13] Note that for any CDF given by H[z], and associated expectation operator $E\{.\}$, from the definition of $\widetilde{P}_M[0,b]$ we have $E\{\widetilde{P}_M[0,b]\} = E\{b-\min[b,\max[0,\widetilde{L}]]\} = b - b\cdot(1-H[b]) - \int_0^b L\cdot H'[L]\,dL = \int_0^b H(L)\,dL$, where the last equality follows from integration by parts.

[14] That is, $G[z] \ge 0\ \forall z$; $G[1]$=1; and $G[z]$ non-decreasing.

(because we use an ES rather than VAR capital metric), for brevity we shall refer to $G[L]$ as the conditional CDF for pool losses. Similarly, for any function of pool credit losses, $Y[L]$, we shall refer to the expectation $\int_0^1 Y[L] \cdot dG[L]$ as the expected value of Y under the conditional probability distribution.

27. For future reference, E_G and V_G denote the mean and variance implied by $G[L]$, and $h_G \equiv G[0]$.[15] Thus, E_G equals the pool's overall expected credit loss under the conditional probability distribution.

Zero Excess Spread Assumption

28. In equation (7) the tranche coupon rate RT enters in the form of the tranche's contractual excess spread, namely $RT - R$. Within the Basel framework, capital charges are generally calibrated to give no credit for excess spread. For MSFA purposes we do the same; that is, tranche capital charges are calibrated assuming that $RT = R$. Making this substitution into (7), we obtain:[16]

(8) $K[A, D] = K[D] - K[A]$

where $K[z] \equiv K[0, z] = z - \int_0^z G[L]\, dL$.

29. It is worth noting that $K'[z] = 1 - G[z]$ represents the marginal capital charge for a tranchelet with attachment point z. The capital charge against a tranchelet covering the first marginal Euro of pool credit losses equals $K'[0] = 1 - h_G$. While non-negative, this charge will not equal 100% unless there is a zero probability of the pool incurring zero losses under the conditional CDF (ie unless $h_G = 0$). For pools that are extremely granular, such as those characterising most retail securitisations, one would expect that $h_G = 0$. However, for non-granular, it is quite possible that $h_G > 0$.

Key Approximation Formula

30. For reasonable specifications of $G[L]$, exact close-form solutions for (8) generally will be analytically intractable. However, a closed-form approximation formula can be developed using an approach similar to that developed in Gordy-Jones (2003b).

31. Define $F[z] \equiv 1 - \frac{K'[z]}{K'[0]}$ for values of z ranging from zero to one, and note that over this range $F[z]$ has the properties of a CDF. The mean and variance implied by this CDF are given by

(9) $\mu = \int_0^1 z\, dF = 1 - \int_0^1 F[z]dz = (\frac{1}{K'[0]}) \cdot \int_0^1 K'[z]dz = \left(\frac{1}{K'[0]}\right) \cdot E_G$, and

(10) $\sigma^2 = \int_0^1 z^2 dF - \mu^2 = \left(\frac{1}{K'[0]}\right) \cdot \left(V_G + E_G^2\right) - \mu^2$.

32. Next, let $\hat{F}[z] \equiv B[z; \gamma, \delta]$ denote the CDF for a beta distribution with parameters γ and δ, where γ and δ are selected so that the mean and variance of this distribution equal

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15 $E_G = \int_0^1 L\, dG[L]$ and $V_G = \int_0^1 L^2\, dG[L] - E_G^2$.

16 This assumption precludes recognising excess spread on the underlying loans as a form of credit enhancement for any tranche.

μ and σ^2. Using the following relationships, $K[z]$ can be approximated in terms of γ, δ, and h_G:[17]

$$(11) \quad K[z] = \int_0^z K'[x]dx = K'[0](z - \int_0^z F[x]dx)$$

$$= (1 - h_G) \cdot (z - \int_0^z F[x]dx)$$

$$\approx (1 - h_G) \cdot (z - \int_0^z \hat{F}[x]dx)$$

$$= (1 - h_G) \cdot (z - zB[z; \gamma, \delta] + \mu B[z; 1 + \gamma, \delta])$$

$$\equiv \hat{K}[z; \gamma, \delta, h_G]$$

where $\gamma = \mu \cdot \left(\frac{\mu(1-\mu)}{\sigma^2} - 1\right)$ and $\delta = \gamma \cdot \left(\frac{1-\mu}{\mu}\right)$.[18]

Model Risk

33. The above discussion presumes that $G[L]$, the true conditional CDF for pool losses, is known. Of course, in reality this is not the case and capital calibrations must rely on $\hat{G}[L]$, the conditional CDF calculated from some known regulatory model which is a simplification of the true process and subject to errors and uncertainties. In this setting, we need a way to infer the above beta-approximation parameters μ and σ^2 from properties of $\hat{G}[L]$.

34. The MSFA accounts partially for the model risk inherent in calibrating capital charges based on $\hat{G}[L]$. Let $E_{\hat{G}}$ and $V_{\hat{G}}$ represent the mean and variance of pool losses implied by $\hat{G}[L]$, and let $h_{\hat{G}} = \hat{G}[0]$. We assume that $\hat{G}[L]$ is an unbiased estimator of $G[L]$ in the sense that both models imply the same conditional expected value for pool credit losses and the same conditional probability of zero pool credit losses; that is, $E_G = E_{\hat{G}}$ and $h_G = h_{\hat{G}}$.

35. Let $V_{\hat{G}}$ represent the variance of the conditional distribution for pool credit losses that is implied by the regulatory model. To recognise the possibility that the regulatory model understates the tail risk of $G[L]$, we assume that V_G is related to $V_{\hat{G}}$ as

$$(12) \qquad V_G = V_{\hat{G}} + \frac{E_{\hat{G}}(1-E_{\hat{G}}) - V_{\hat{G}}}{\tau}$$

where $\tau > 1$. Since $E_{\hat{G}}(1 - E_{\hat{G}}) > V_{\hat{G}}$, the assumed variance of pool credit losses implied by the true conditional CDF generally will exceed the variance implied by the regulatory model.[19]

36. The τ parameter could be seen as reflecting regulators' confidence in the regulatory model. Higher values of τ correspond to a greater degree of confidence in the regulatory model and, hence, a lower value of V_G for given values of $E_{\hat{G}}$ and $V_{\hat{G}}$. The τ parameter is treated below as a regulatory calibration parameter. Depending on value of τ set by regulators, V_G could assume values between $V_{\hat{G}}$ and $E_{\hat{G}}(1 - E_{\hat{G}})$.

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[17] The fourth line below follows from the properties of the beta distribution. See equation (3) in Gordy (2004).

[18] Since the parameters of the beta distribution must be positive, the approximation requires that $\sigma^2 < \mu(1-\mu)$ and, hence, $V_G < E_G \cdot (1 - E_G)$. This condition will be met for a random variable $\tilde{L}$ that satisfies $0 \leq \tilde{L} \leq 1$ and $0 < E\{\tilde{L}\}$ <1, which is the case here (assuming at least one exposure in the pool has a positive default probability and an expected LGD that is greater than zero but less than one).

[19] The inequality follows from the discussion in preceding footnote.

37. On substituting (12) into (11), the capital charge for a hypothetical tranche with attachment and detachment points A and D can be approximated in terms $E_{\widehat{G}}$, $V_{\widehat{G}}$, and $h_{\widehat{G}}$ as follows:

$$(13) \quad \widehat{K}[A,D] = \widehat{K}[D] - \widehat{K}[A], \text{ where}$$

$$\widehat{K}[z] = (1 - h_{\widehat{G}}) \cdot (z - zB[z;\gamma,\delta] + \mu B[z; 1 + \gamma, \delta]) \ ,$$

$$\gamma = \mu \cdot \left(\frac{\mu \cdot (1-\mu)}{\sigma^2} - 1 \right),$$

$$\delta = \gamma \cdot \left(\frac{1-\mu}{\mu} \right),$$

$$\mu = \frac{E_{\widehat{G}}}{1 - h_{\widehat{G}}},$$

$$\sigma^2 = \frac{\left(V + E_{\widehat{G}}^2 \right)}{1 - h_{\widehat{G}}} - \mu^2, \text{and}$$

$$V = V_{\widehat{G}} + \frac{E_{\widehat{G}} \cdot (1 - E_{\widehat{G}}) - V_{\widehat{G}}}{\tau}.$$

38. It is readily verified that if (13) is applied to all of the individual tranches of a securitisation, the sum of the tranche capital charges is equal to $_{\widehat{G}}$. Thus, for a given $E_{\widehat{G}}$, the above equation system serves to allocate this total charge among the various tranches.

39. Within the MSFA, the τ parameter can have a significant impact on estimated capital charges for mezzanine and senior securitisation positions with attachment points above K_{IRB}. Gordy (2003b) provides insight into plausible ranges for τ when dealing with a particular type of model specification error. Specifically, assume that the true model for pool losses comports with the ARSF framework developed by Pykhtin-Dev (2002, 2003), wherein for each securitisation, pool losses are driven by a common global risk factor and a pool-specific random factor which impacts all of the underlying exposures and is independent of other risks. For homogeneous wholesale pools and M=1, Gordy (2003b) demonstrates that the τ parameter within the Gordy-Jones (2003b) model (which is comparable the MSFA model for M=1) generally can be calibrated so that this model and the Pykhtin-Dev model produce very similar tranche capital charges when other parameters are calibrated consistently so that the two models produce the same capital charge for the overall pool.

40. The value of τ that aligns the tranche charges produced by the two models is shown to depend on the pool's average PD and LGD and the within-pool AVC. Depending on the precise settings for these parameters, the implied τ can take a wide range of values, with estimates apparently quite sensitive to the latter two parameters. In many cases one might expect the average within-pool AVC to exceed the average AVC implied by the IRB formulas, reflecting similar latent characteristics among loans in the pool that are not captured within the IRB framework (eg loans drawn from similar sectors or geographic regions, or subject to similar underwriting biases). Gordy (2003b) presents simulation results showing that for pools with average LGDs between 50% and 100%, and ratios of within-pool AVC to IRB-implied AVC ranging from 1.1 to 2, the implied τ values vary from around 10 to 200.[20] In

[20] See Figure 2 in Gordy (2003b).

comparison, the current SFA sets τ at 1000, while the proposed MSFA sets this parameter at 100.

41. Finally, before concluding this discussion of model risk, it should be noted that when developing (13) we have continued to assume that an underlying ASRF model governs the true process driving pool credit losses. Otherwise, the interpretation of the ES-based capital charge as the conditional expectation in (2) would no longer hold. Thus, implicitly we assume that the model risk inherent in each securitisation is purely idiosyncratic and unrelated to all other risk factors, including model risks associated with other securitisation transactions. Clearly, this is unlikely in practice, as evidenced by the failures of industry-standard risk models across a wide swath of securitisation transactions during the financial crisis. Thus, the MSFA's treatment of model risk is likely to be partial at best, suggesting a need for further prudential add-ons to cover other model risks.[21]

IV. Regulatory MtM Model for Pool Credit Losses

42. This section sets forth the MSFA's regulatory model for pool credit losses and market risk premiums, from which we develop estimates of $E_{\hat{G}}$, $V_{\hat{G}}$, and $h_{\hat{G}}$ that are used in computing tranche capital charges via (13). The specification of this regulatory model was selected to replicate as closely as possible the ASRF model underpinning the IRB wholesale capital functions.

ASRF Model Governing Credit Losses on Individual Loans

43. As in the ASRF model underpinning the IRB framework, credit losses on the bank's overall portfolio are assumed to be driven by a single global risk factor $\widetilde{X}$, and each asset in the bank's portfolio is assumed to represent an infinitesimal share of this portfolio.

44. Defaults on individual loans are governed by a Merton-type model.[22] The jth loan defaults at the end of the capital horizon if the logarithm of the borrower's asset value at t=1 is less than a default threshold $DT1_j$. If the loan does not default at t=1, it defaults at maturity if the logarithm of the borrower's asset value at t=M is less than DTM_j.[23] For expositional ease, below we refer to the logarithm of the borrower's asset value more simply as the 'borrower's asset value.'

45. The jth borrower's asset value, $\widetilde{\Upsilon}_j(T)$, is assumed to evolve according to a geometric Brownian motion process:

$$(14) \quad \widetilde{\Upsilon}_j(t) = \Upsilon_j(0) + \left(m_j - .5\pi_j^2\right) * t + \pi_j \cdot \left(r_j \cdot \widetilde{X}(t) + \widetilde{U}_j(t)\sqrt{1 - r_j{}^2}\right), \text{ where}$$

$\Upsilon_j(0)$ = borrower's asset value at t=0;

---

[21] As with the SFA, the proposed MSFA incorporates several additional prudential add-ons for model risk in the form of a capital floor, an Omega Adjustment, and the 100% capital charge for tranches covering losses below K_{IRB}.

[22] The modelling approach described below is similar to that in Vasicek (1991) and Gordy-Marrone (2012).

[23] We assume two default barriers, one at t=1 and another at t=M, in order to allow the model to be calibrated consistent with historical data on the term structure of default rates.

m_j, π_j = annual mean and standard deviation of the drift in the borrower's asset value;

$\widetilde{X}(t)$ = realised value of the global risk factor at time t, with $\widetilde{X}(0) = 0$;

$\widetilde{U}_j(t)$ = realised value of the idiosyncratic risk factor at time t, with $\widetilde{U}_j(0) = 0$; and

r_j = correlation between the borrower's asset value and the global risk factor.

46. For future reference, we note that when calibrating this model below, the parameter r_j is set equal to the value implied by the IRB framework. Specifically, $r_j = \sqrt{AVC_j}$, where AVC_j is the loan's (positively signed) IRB asset value correlation parameter.

47. The above global risk factor and idiosyncratic risk factors are themselves assumed to be generated by independent geometric Brownian motion processes having zero drifts and unit annualised volatilities. These risk factors can be decomposed temporally in the following manner:

(15) $\widetilde{X}(M) = \widetilde{X}_1 + \widetilde{X}_2\sqrt{M-1}$, and

(16) $\widetilde{U}_j(M) = \widetilde{U}_{1j} + \widetilde{U}_{2j}\sqrt{M-1}$

where $\widetilde{X}_1 \equiv \widetilde{X}(1)$, and $\widetilde{X}_2 \equiv \widetilde{X}(M) - \widetilde{X}(1)$ is the change in the global risk factor from the end of the capital horizon through maturity. The random variables $\widetilde{U}_{1j}$ and $\widetilde{U}_{2j}$ are defined in a similar fashion. In this representation, the random variables $\{\widetilde{X}_1, \widetilde{X}_2, \widetilde{U}_{11}, \widetilde{U}_{21}, \ldots, \widetilde{U}_{1N}, \widetilde{U}_{2N}\}$ are independent and identically distributed standard normal random variables.

48. In solving the regulatory model, we assume that at t=0 market participants know all model parameters, but not future realisations of random risk factors. At t=1, the realisations of $\widetilde{X}_1$ and the $\widetilde{U}_{1j}$ are revealed to market participants, while at t=M the realisations of all remaining random variables are revealed, including loss severities for defaulting loans.

Calibrating the Default Thresholds

49. The default thresholds $DT1_j$ and DTM_j can be determined from the term structure of default probabilities for the jth borrower and other model parameters. Let $PD1_j$ and PDM_jdenote the probability that the jth borrower defaults after 1 year and after M years, respectively. Then $DT1_j$ satisfies the implicit relationship

(17) $$PD1_j = \Phi\left[\frac{DT1_j - Y_j(0) - \left(m_j - .5\pi_j^2\right)}{\pi_j}\right]$$

$$\equiv \Phi\left[DT1_j^*\right].$$

Thus, we can solve for $DT1_j^*$ as

(18) $$DT1_j^* = \Phi^{-1}\left[PD1_j\right].$$

50. Similarly, if we define $TM_j^* \equiv \frac{DTM_j - Y_j(0) - M\cdot\left(m_j - .5\pi_j^2\right)}{\pi_j\sqrt{M}}$, then DTM_j^* satisfies the implicit relationship

(19) $\mathrm{PDM_j} = \mathrm{PD1_j} + \Phi_2[-\mathrm{DT1_j^*}, \mathrm{DTM_j^*}\,;\, -\frac{1}{\sqrt{M}}]$.

Notice that $\mathrm{DT1_j^*}$ and $\mathrm{DTM_j^*}$ can be solved solely as functions of M, $\mathrm{PD1_j}$ and $\mathrm{PDM_j}$, with no need to know $\mathrm{m_j}, \pi_j, \mathrm{r_j}$, or $\Upsilon_j(0)$.

Calibrating the Term Structure of Default Probabilities

51. To allow banks to implement the MSFA using IRB risk parameters for the underlying pool, the MSFA assumes a specific relationship between one-year and M-year probabilities of default. The assumed relationship between PD1 and PDM is given by

$$(20) \quad \mathrm{PDM} = \frac{1}{1 + e^{-\left(x + (5 - 0.15x)\cdot(M^{0.2} - 1)\right)}}$$

where $\mathrm{x} = \log\left(\frac{\mathrm{PD1}}{1-\mathrm{PD1}}\right)$. The derivation of this relationship is summarised in Annex 1.

Calibration of $\mathbf{E_{\hat{G}}}$

52. $E_{\hat{G}}$ represents the pool's expected loss rate (at t=0) under the conditional probability distribution implied by the regulatory model. We will show that $E_{\hat{G}}$ can be calculated using numerical methods. The MSFA employs a simple approximation that is constructed by simulating the exact solution over a broad range of inputs, and then fitting an empirical relationship to these data points. This approximation is shown to be highly accurate under a range of test scenarios.

53. From the definition of $\mathrm{E_{\hat{G}}}$ we have $\mathrm{E_{\hat{G}}} = \mathrm{E_0^{\widehat{NP}}}\{\, \mathrm{E_1^{\widehat{RN}}}\{\tilde{Z}\} |\, \tilde{X}_1 = x_1^{qES}\}$, where $\mathrm{E_0^{\widehat{NP}}}\{\cdot\, |\tilde{X}_1 = x_1^{qES}\}$ denotes the conditional expectation at t=0 under the natural probability distribution implied by the regulatory model, and $\mathrm{E_1^{\widehat{RN}}}\{\tilde{Z}\}$ denotes the expected value at t=1 of an arbitrary random variable $\tilde{Z}$ under the risk-neutral probability distribution implied by the regulatory model. The expectation $\mathrm{E_1^{\widehat{RN}}}\{\tilde{Z}\}$ depends on realisations of the random risk factors $\{\tilde{X}_1, \tilde{U}_{11}, \dots \tilde{U}_{1N}\}$ revealed to market participants at t=1, denoted $\{x_1, u_{11}, \dots, u_{1N}\}$. In the remainder of this section all probability distributions pertain to the regulatory model.

54. Under the natural probability distribution, the risk factors $\{\tilde{X}_1, \tilde{U}_{11}, \dots, \tilde{U}_{1N}\}$ are mutually independent, standard normal random variables. Thus,

$$(21) \quad \mathrm{E_{\hat{G}}} = \left(\frac{1}{qES}\right) \cdot \int_{-\infty}^{x_1^{qES}} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{E_1^{\widehat{RN}}}[\tilde{L}] \cdot \phi[u_{11}] \cdots \phi[u_{1N}]\, du_{11} \dots du_{1N}\right) \cdot \phi[x_1]\, dx_1 .$$

55. Next, recall that the underlying risk factors driving firms' asset values are Brownian motion processes, while the remaining random variables in the model (ie the loss severities for individual loans) are idiosyncratic and, hence, should not earn any risk premium. Thus, $\mathrm{E_1^{\widehat{RN}}}\{\tilde{Z}\}$ can be calculated as the expected value at t=1 of pool credit losses when (a) the mean drift rates $\mathrm{m_j}$ in (14) are replaced with the risk-free rate R and (b) the random vector $\{\tilde{X}_1, \tilde{U}_{11}, \dots, \tilde{U}_{1N}\}$ is set equal to $\{x_1, u_{11}, \dots, u_{1N}\}$.

56. From the linearity of the expectation operator, it follows that

$$(22) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{E_1^{\widehat{RN}}}\{\tilde{L}\} \cdot \phi[u_{11}] \cdots \phi[u_{1N}]\, du_{11} \dots du_{1N}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{E}_1^{\widehat{\mathrm{RN}}}\{\sum_{j=1}^{N} \theta_j \tilde{\mathrm{I}}_j \cdot \widetilde{\mathrm{LGD}}_j\} \cdot \phi[u_{11}] \cdots \phi[u_{1N}]\, du_{11} \ldots du_{1N}$$

$$= \sum_{j=1}^{N} \theta_j \cdot \overline{\mathrm{LGD}}_j \cdot \mathrm{W}[x_1; \mathrm{PD1}_j, \mathrm{PDM}_j, r_j]$$

where $\mathrm{W}[x_1; \mathrm{PD1}_j, \mathrm{PDM}_j, r_j] \equiv \int_{-\infty}^{\infty} \mathrm{E}_1^{\widehat{\mathrm{RN}}}\{\tilde{\mathrm{I}}_j\} \cdot \phi[u_{1j}]\, du_{1j}$ is the expected risk-neutral probability (at t=0) that the jth loan defaults during the life of the securitisation conditional on $\widetilde{\mathrm{X}}_1 = x_1$.

57. Using properties of the bivariate normal distribution, Annex 2 shows that $\mathrm{W}[x_1; \mathrm{PD1}_j, \mathrm{PDM}_j, r_j]$ has a closed-form solution given by

$$(23)\quad \mathrm{W}[x_1; \mathrm{PD1}_j, \mathrm{PDM}_j, r_j] = \left[\frac{\mathrm{DT1}_j^* - r_j \cdot x_1}{\sqrt{1-r_j^2}}\right] + \Phi_2\left[-\frac{\mathrm{DT1}_j^* - r_j x_1}{\sqrt{1-r_j^2}}, \frac{\mathrm{DTM}_j^* - r_j x_1 + \left(\frac{m_j - R}{\pi_j}\right) \cdot (M-1)}{\sqrt{M - r_j^2}}; -\sqrt{\frac{1-r_j^2}{M-r_j^2}}\right].$$

58. In the above expression, the term $m_j - R$ represents the market risk premium built into the jth borrower's asset value process to compensate investors for that borrower's systematic risk (ie the correlation between the borrower's asset values and the global risk factor). We shall assume that these risk premiums are proportional to an assumed constant market price of risk λ:

$$(24)\quad m_j - R = \lambda \cdot r_j \cdot \pi_j.$$ [24]

59. Bohn (2000b) presents evidence suggesting that empirical estimates of λ generally fall in the range 0.3 to 0.5. For MSFA purposes, we set $\lambda = 0.4$. Combining (21)-(24), we obtain

$$(25)\quad E_{\hat{G}} = \sum_{j=1}^{N} \theta_j \cdot \overline{LGD}_j \cdot w_j, \text{ where}$$

$$w_j \equiv \left(\frac{1}{qES}\right) \cdot \int_{-\infty}^{x_1^{qES}} W[x_1; PD1_j, PDM_j, r_j] \cdot \phi[x]\, dx,$$

and $W[x_1; PD1_j, PDM_j, r_j] = \Phi\left[\frac{DT1_j^* - r_j \cdot x_1}{\sqrt{1-r_j^2}}\right] + \Phi_2\left[-\frac{DT1_j^* - r_j \cdot x_1}{\sqrt{1-r_j^2}}, \frac{DTM_j^* - r_j \cdot x_1 + 0.4 r_j \cdot (M-1)}{\sqrt{M-r_j^2}}; -\sqrt{\frac{1-r_j^2}{M-r_j^2}}\right].$

60. The above expression for w_j can be computed using standard numerical methods, but for regulatory capital purposes a simpler expression is desirable. After some numerical experimentation, we find that over a broad range of values for M, $PD1$, and r, the above equation for w_j can be closely approximated with the simpler function.[25]

$$(26)\quad \hat{w}_j = \Phi\left[s_j + \left(0.56 + 0.074 s_j - 0.34 AVC_j^{0.3}\right) \cdot (M-1)^{0.7}\right],$$

[24] This specification can be motivated from CAPM considerations, as in Agrawal, Arora, and Bohn (2004), Kealhofer (2003), and Bohn (2000a).

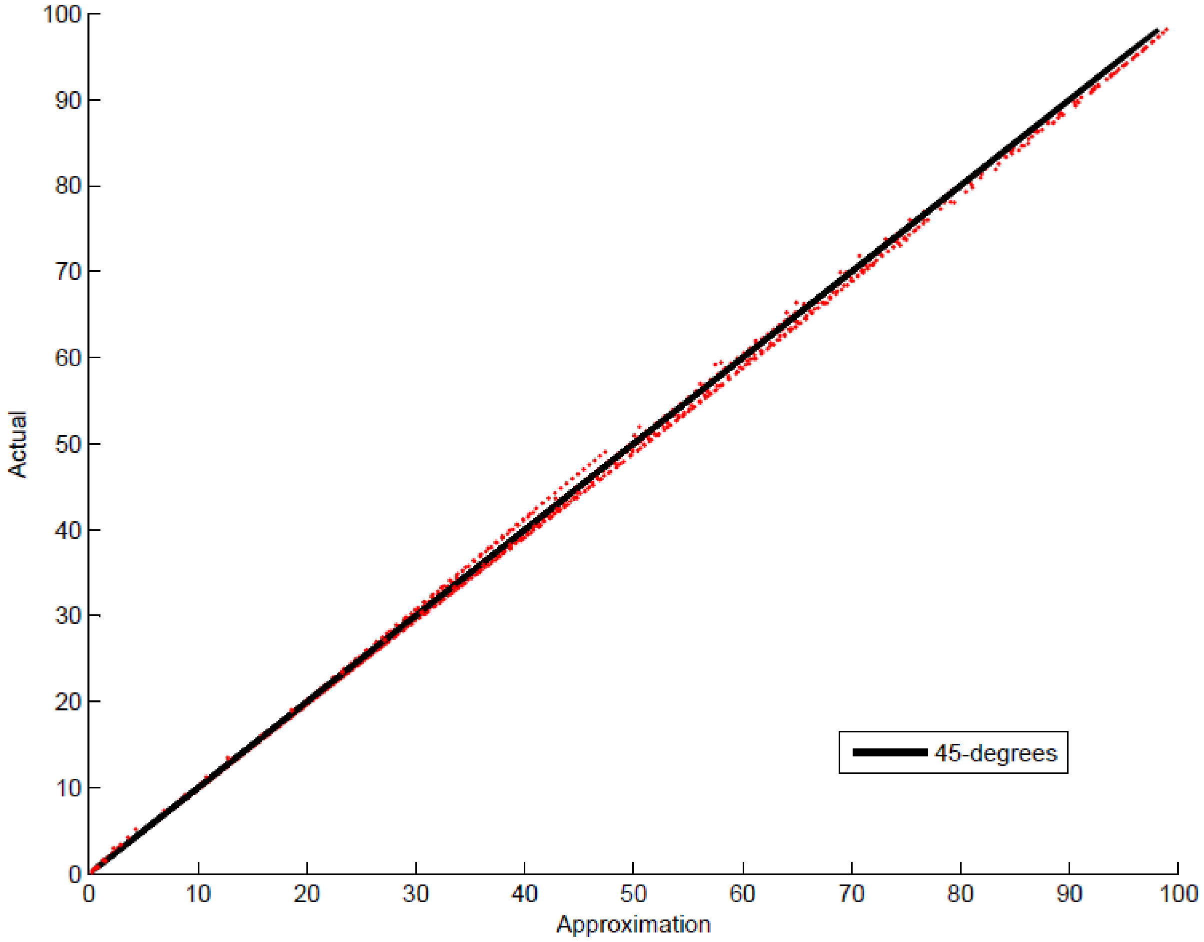
[25] Recall that from (9) that PDM is a function of PD1 and M.

where $s_j = \left(\frac{\Phi^{-1}[PD1_j] + 3.09 r_j}{\sqrt{1 - r_j^2}} \right)$ and $AVC_j = r_j^2$ is the IRB framework's implied AVC for borrower j.

61. Chart 3 summarises the quality of this approximation over the parameter range $M = \{1,2,3,4,5\}$, $\sqrt{AVC} = \{0.05, 0.15, 0.25\}$, and $PD1$ values from 0.03% to 30%.

Chart 3

Actual and Approximated Values of w



62. Within the MSFA, the estimator $\widehat{w}_j$ is used to calculate $E_{\widehat{G}}$ as

(27) $\hat{E}_{\hat{G}} = \sum_{j=1}^{N} \theta_j \cdot \hat{c}_j$, where $\hat{c}_j = \overline{LGD}_j \cdot \widehat{w}_j$.

Calibration of $V_{\hat{G}}$

63. We next derive an expression for the regulatory model's implied conditional risk-neutral variance for pool credit losses, $V_{\widehat{G}}$. Proceeding along the same lines as above,[26]

$$(28)\quad V_{\hat{G}} = \left(\frac{1}{qES}\right) \cdot \int_{-\infty}^{x_1^{qES}} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E_1^{\widehat{RN}}\{\tilde{L}^2\} \cdot \phi[u_{11}] \cdots \phi[u_{1N}]\, du_{11} \ldots du_{1N}\right) \cdot \phi[x_1]\, dx_1 - E_{\hat{G}}^2$$

$$= \left(\frac{1}{qES}\right) \cdot \int_{-\infty}^{x_1^{qES}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E_1^{\widehat{RN}} \left[\left(\sum_{i=1}^{N} \theta_i \cdot \tilde{I}_i \cdot \widetilde{LGD}_i\right)^2\right] \cdot \phi[u_{11}] \cdots \phi[u_{1N}]\, du_{11} \ldots du_{1N} \cdot \phi[x_1]\, dx_1$$

$$- \left(\sum_{j=1}^{N} \theta_j \cdot \overline{LGD}_j \cdot w_j\right)^2$$

$$= \sum_{i=1}^{N} \theta_i^2 (0.25 w_i \cdot \overline{LGD}_i \cdot (1 - \overline{LGD}_i) + w_i \cdot (1 - w_j) \cdot \overline{LGD}_i^2)$$

$$+ \sum_{i=1}^{N} \sum_{j \neq i} COV_{ij} \theta_i \theta_j \overline{LGD_\imath} \cdot \overline{LGD_\jmath}$$

where COV_{ij} is the conditional risk-neutral default covariance between borrowers i and j, which equals

$$(29)\quad COV_{ij} = \left(\frac{1}{qES}\right) \cdot \int_{-\infty}^{x_1^{qES}} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E_1^{\widehat{RN}}\left[(\tilde{I}_i - w_i) \cdot (\tilde{I}_j - w_j)\right] \cdot \phi[u_{11}] \cdots \phi[u_{1N}]\, du_{11} \ldots du_{1N}\right) \phi[x_1]\, dx_1.$$

64. While the above expression for COV_{ij} can be calculated using numerical methods, as before, we seek a simpler closed-form approach. For computational simplicity we are also willing to accept a somewhat conservative estimator for $V_{\widehat{G}}$. To this end, note that the last term in (28) can be bounded from above.[27]

$$(30)\quad \sum_{i=1}^{N} \sum_{j \neq i} COV_{ij} \theta_i \theta_j \overline{LGD_\imath} \cdot \overline{LGD_\jmath}$$

$$\leq \sum_{i=1}^{N} \sum_{j \neq i} \theta_i \theta_j \overline{LGD_\imath} \cdot \overline{LGD_\jmath} \cdot \sqrt{COV_{ii} \cdot COV_{jj}}$$

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[26] Below, we use the assumption that the variance of $\widetilde{LGD}_i$ is $0.25\overline{LGD}_i(1 - \overline{LGD}_i)$ and the observation that $\tilde{I}_i = \tilde{I}_i^2$.

[27] The first inequality below follows from the observation that under the regulatory model $COV_{ij} \leq \sqrt{COV_{ii} \cdot COV_{jj}}$ for $i \neq$j. To see this, consider two hypothetical infinitely-granular, homogeneous pools of loans, one having parameters $PD1_i$ and r_i, and the second having parameters $PD1_j$ and r_j. Under the conditional risk-neutral probability distribution the random variable $\tilde{I}_i$ has a Bernoulli distribution with a mean of w_i, implying a variance equal to $w_i \cdot (1 - w_i)$. Thus, the correlation between the average default rates of the two pools under the conditional risk neutral probability distribution is equal to $COV_{ij}/\sqrt{COV_{ii} \cdot COV_{jj}}$, which implies that $COV_{ij} \leq \sqrt{COV_{ii} \cdot COV_{jj}}$.

$$= \left(\sum_{i=1}^{N} \theta_i \sqrt{COV_{ii}} \cdot \overline{LGD_i}\right)^2 - \sum_{i=1}^{N} COV_{ii} \cdot (\theta_i \cdot \overline{LGD}_i)^2$$

$$\leq \left(\sum_{i=1}^{N} \theta_i \sqrt{COV_{ii}} \cdot \overline{LGD_i}\right)^2$$

where $\mathrm{COV_{ii}}$ is the regulatory model's (positively signed) implied conditional risk-neutral default covariance between two hypothetical loans having PD1 and r values equal to $\mathrm{PD1_i}$ and $\mathrm{r_i}$.[28]

65. From (28) and (30) it follows that

$$(31)\quad V_{\hat{G}} \leq \left(\sum_{i=1}^{N} \theta_i \sqrt{COV_{ii}} \cdot \overline{LGD_i}\right)^2 + \sum_{i=1}^{N} \theta_i^2 (0.25 w_i \overline{LGD}_i \cdot (1 - \overline{LGD}_i) + w_i \cdot (1 - w_i) \cdot \overline{LGD}_i^2).$$

66. Next, we seek a simple expression for approximating the first right-hand-side term in this inequality. Upon calculating $\mathrm{COV_{ii}}$ numerically for PD1 values between .03% and 99.9%, M values from 1 to 5, and four sets of AVC values (corresponding to AVCs implied by the IRB functions for wholesale exposures, residential mortgages, qualifying retail exposures, and other retail exposures) we find that a simple upper bound for $\mathrm{COV_{ii}}$ is given by

$$(32)\quad \widehat{\mathrm{COV}}_{\mathrm{ii}} = 0.09\mathrm{M} \cdot \mathrm{w_i} \cdot (1 - \mathrm{w_i}) \cdot \mathrm{AVC_i}.$$

67. Substituting (32) into (31), and $\mathrm{\hat{w}_i}$ for $\mathrm{w_i}$, we obtain the MSFA equation for estimating $\mathrm{V_{\hat{G}}}$

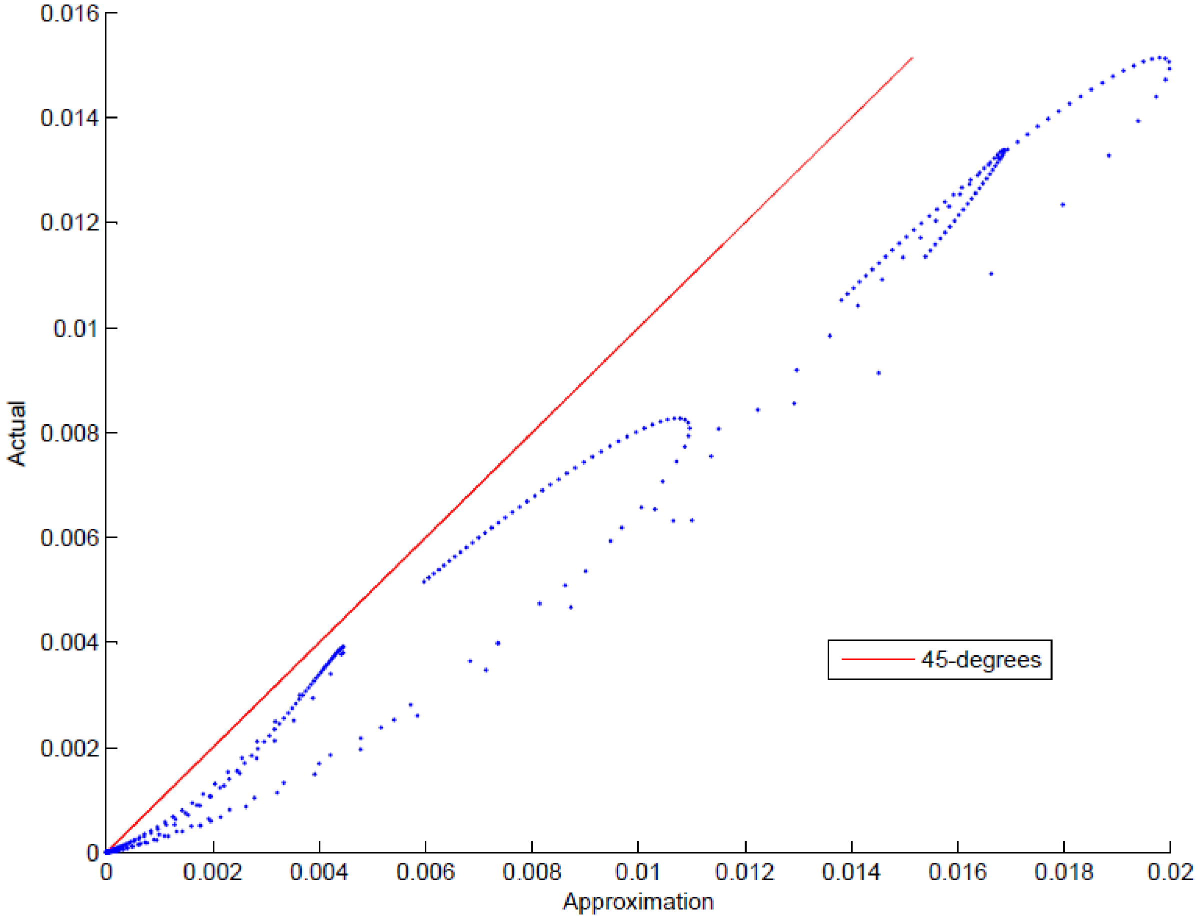
$$(33)\quad \hat{\mathrm{V}}_{\hat{\mathrm{G}}} = \left(\sum_{i=1}^{N} \theta_i \sqrt{\hat{\mathrm{v}}_i}\right)^2 + \sum_{i=1}^{N} \theta_i^2 \left(0.25 \hat{\mathrm{w}}_i \overline{\mathrm{LGD}}_i \cdot (1 - \overline{\mathrm{LGD}}_i) + \hat{\mathrm{w}}_i \cdot (1 - \hat{\mathrm{w}}_i) \cdot \overline{\mathrm{LGD}_i}^2\right)$$

where $\hat{\mathrm{v}}_i \equiv \overline{\mathrm{LGD}}_i^2 \cdot 0.09\mathrm{M} \cdot \hat{\mathrm{w}}_i \cdot (1 - \hat{\mathrm{w}}_i) \cdot \mathrm{AVC_i}$. Chart 4 summarises the overall fit of this approximation.

[28] COV_{ii} is calculated as in (29), but using $PD1_i$ and r_i in place of $PD1_j$ and r_j.

Chart 4

Actual and Approximated Values of $V_{\hat{G}}$



Calibration of $h_{\hat{G}}$

68. The input $h_{\hat{G}}$ represents the conditional probability of zero pool credit losses under the regulatory model. To motivate the MSFA's approach to estimating this input, we first discuss its role in the context of equation (13).

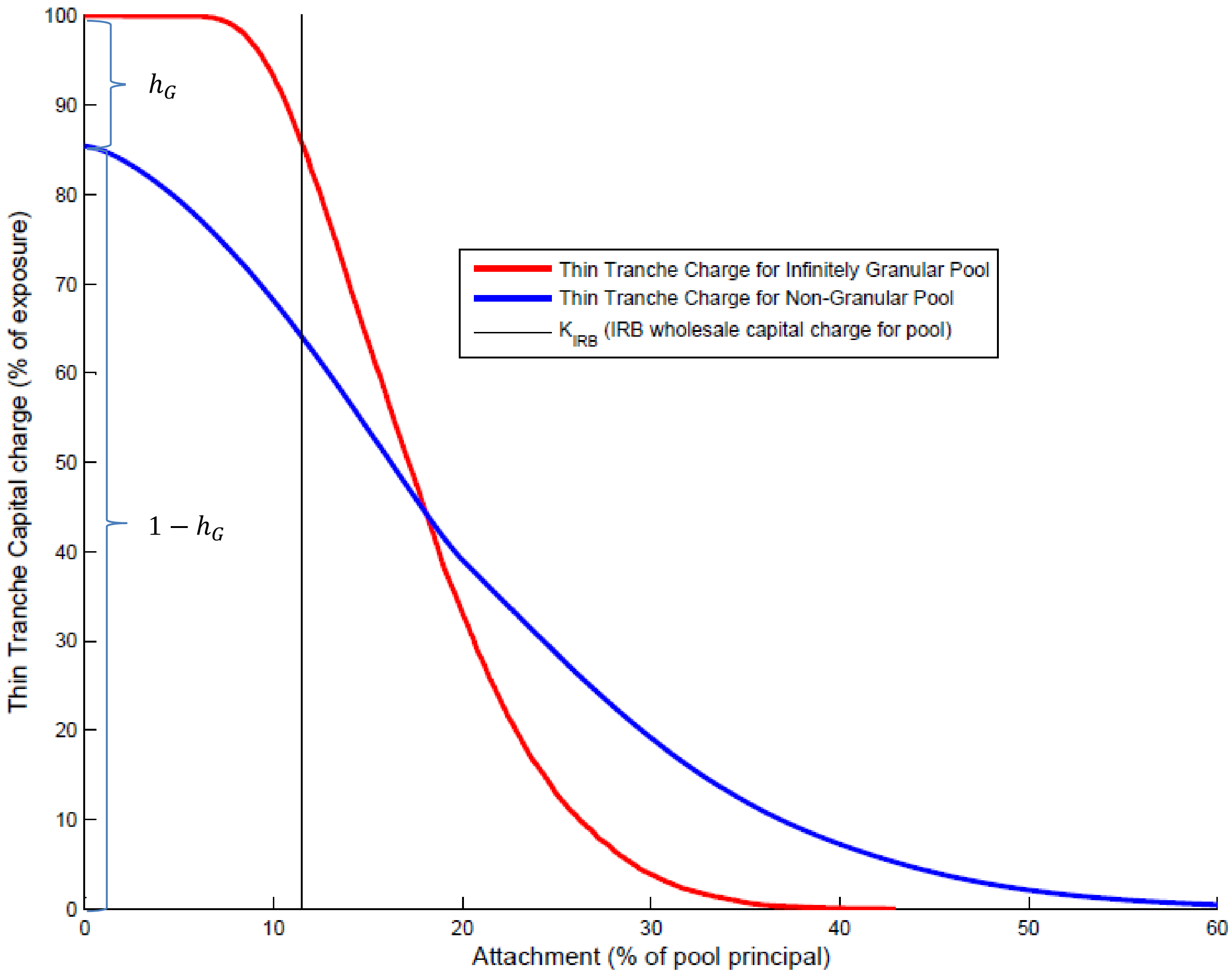
69. Chart 5 illustrates that the value of $h_{\hat{G}}$ can have a substantial impact on MSFA capital charges. The chart shows marginal capital charges (computed via simulation) for infinitesimally thin tranches with attachment points represented by the X axis. For this purpose, we assume no difference between the true model and the regulatory model (ie τ is set at infinity) and no prudential add-ons. The chart examines two homogeneous wholesale pools, one corresponding to an infinitely-granular pool and another consisting of only five loans. The underlying loan-level parameters are $\mathrm{PD1} = 1\%$, $\mathrm{M} = 5$, and $\overline{\mathrm{LGD}} = 50\%$.[29] Geometrically, $1 - h_{\hat{G}}$ denotes the marginal capital charge for a thin tranche having an attachment point just slightly above zero. Since the number of loans in the pool does not affect the pool's overall conditional expected credit loss (ie the corresponding $E_{\hat{G}}$) the area under each curve is the same. The diagram illustrates that if $h_{\hat{G}}$ is underestimated, on average the marginal capital charges for more senior tranchelets will be underestimated as well.

[29] The variance of LGDs is equal to $0.25\overline{LGD} \cdot (1 - \overline{LGD})$.

Chart 5

Simulated Thin Tranche Capital Charges for Infinitely-Granular and Non-Granular Pools

(homogeneous wholesale pools; M = 5 years; $PD1 = 1\%; \overline{LGD} = 50\%$; five loans in non-granular pool)



70. For a homogeneous pool, the conditional risk-neutral probability of default under the regulatory model is equal to $\frac{E_{\hat{G}}}{\overline{LGD}}$, where $\overline{LGD}$ is the expected loss rate given default for loans in the pool. If individual loan defaults were independent under this probability distribution, then we could calculate $h_{\hat{G}}$ as simply $\left(1-\frac{E_{\hat{G}}}{\overline{LGD}}\right)^{N}$, where N is the number of loans in the pool.

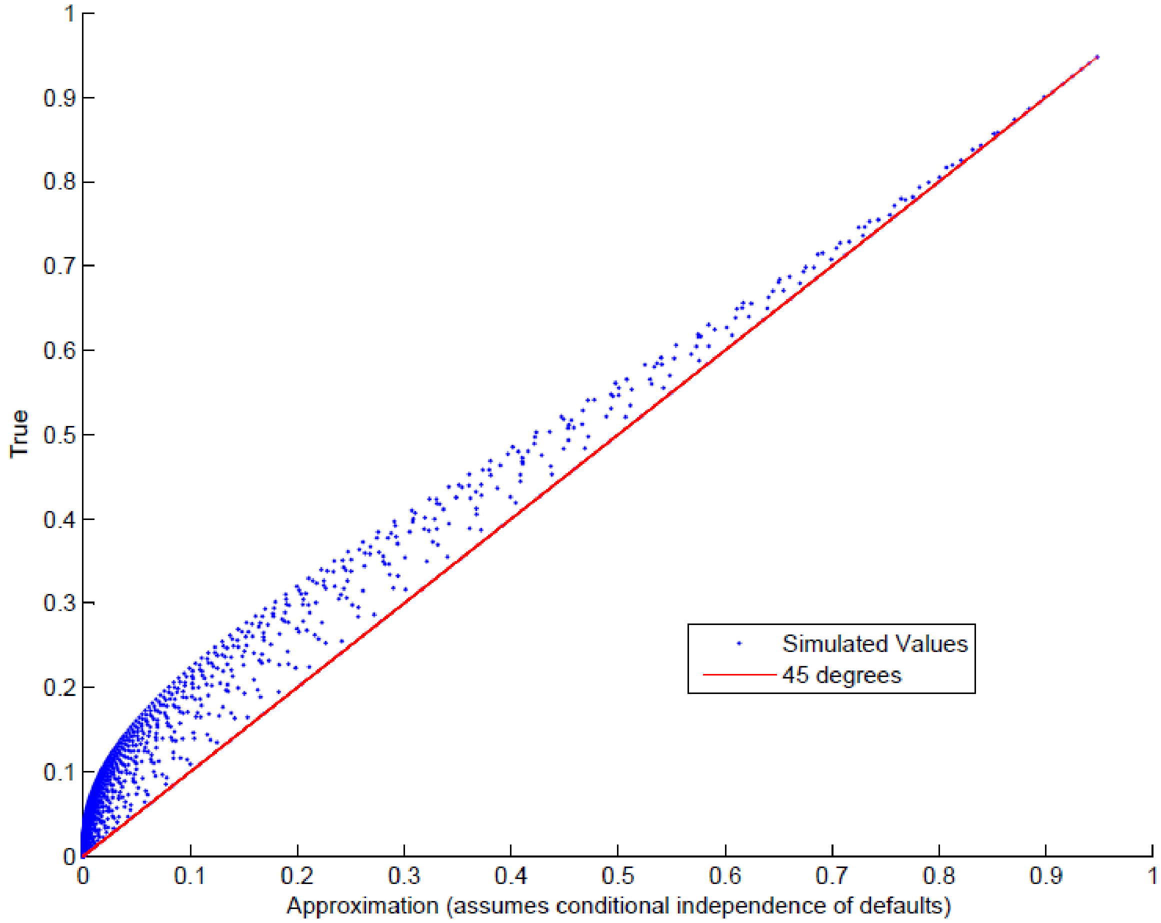
71. However, it is readily apparent that defaults are not independent under the conditional risk-neutral distribution. Conditional on $\tilde{X}_1 \leq x_1^{qES}$, borrower defaults within the capital horizon will be positively correlated owing to the positive dependence of each borrower's asset value on the actual realised value of $\tilde{X}_1$. And, for those borrowers not defaulting within the capital horizon, under the conditional risk-neutral distribution future defaults also will be positively correlated, since borrowers' asset values at maturity will depend positively on $\tilde{X}_2$, the change in the global risk factor between t=1 and t=M. This intuition also suggests that conditional default correlations may tend to increase with M other things the same. Hence, $\left(1-\frac{E_{\hat{G}}}{\overline{LGD}}\right)^{N}$ will tend to be downward biased estimator of $h_{\hat{G}}$, with the magnitude of the bias a positive function of M. Stated differently, even with homogeneous pools, the effective number of loans in the pool will be less than the actual number of loans.

72. Chart 6 illustrates this effect. Using simulations of the regulatory model for a M=5 and a broad range of other parameter inputs, the chart compares actual values of $h_{\hat{G}}$ with those generated from the approximation $h_{\hat{G}} \approx \left(1-\frac{E_{\hat{G}}}{\overline{LGD}}\right)^{N}$. As can be seen, the approximation consistently underestimates the true value of $h_{\hat{G}}$.

Chart 6

Actual and Approximated Values of h_G Assuming Conditional Independence of Defaults

(wholesale pools, $M = 5$ years, N from 1 to 1000, and $PD1$ from 0.03% to 99%)



73. With further investigation (based on simulated data for homogeneous wholesale pools) it was found that the accuracy of this approximation could be improved markedly by adjusting N to compensate for higher average conditional risk-neutral default correlations as M increases:

(34) $\mathrm{h}_{\widehat{\mathrm{G}}} \approx \left(1 - \frac{\mathrm{E}_{\widehat{\mathrm{G}}}}{\mathrm{LGD}}\right)^{\mathrm{N}^*}$, where

$$\mathrm{N}^* = \frac{\widehat{\mathrm{N}}}{\left(1 + 0.0079\mathrm{M}\sqrt{\widehat{\mathrm{N}}}\right)^2}.$$

The variable N^* can be interpreted as the effective number of loans in the pool. Thus, consistent with the above intuition, the effective number of loans in the pool will tend to be a decreasing function of maturity.

74. For maturities of one and five years, Charts 7 and 8 summarise the accuracy of this approximation for homogeneous pools. Note that where prediction errors are visually significant for M=5 years, the approximation generally works to overestimate the true value of $\mathrm{h}_{\widehat{\mathrm{G}}}$. Thus, on average the MSFA is less likely to underestimate average capital charges for more senior tranchelets compared with the simplistic approximation $\mathrm{h}_{\widehat{\mathrm{G}}} \approx \left(1 - \frac{\mathrm{E}_{\widehat{\mathrm{G}}}}{\mathrm{LGD}}\right)^{\mathrm{N}}$.

Chart 7

Actual and Approximated Values of h_G Based on Effective Number of Loans

(wholesale pools, $M = 1$ year, N from 1 to 1000, and $PD1$ from 0.03% to 99%)

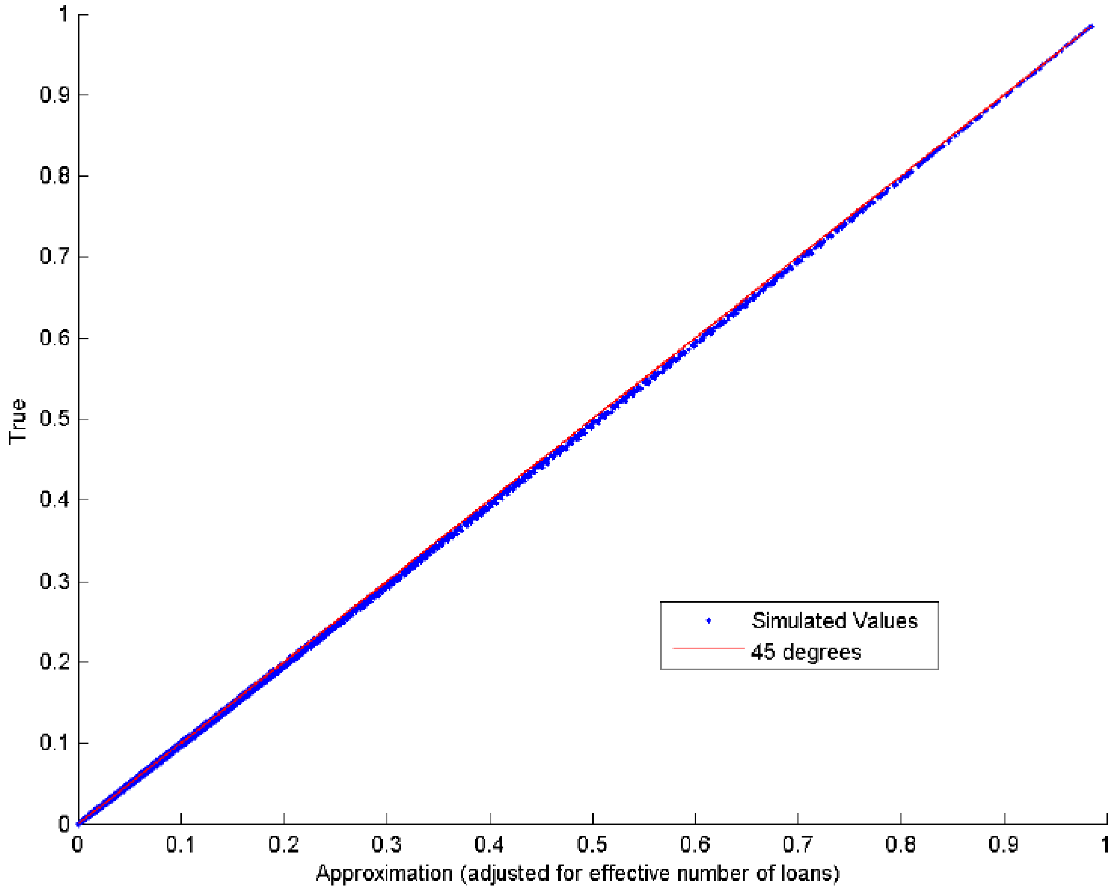
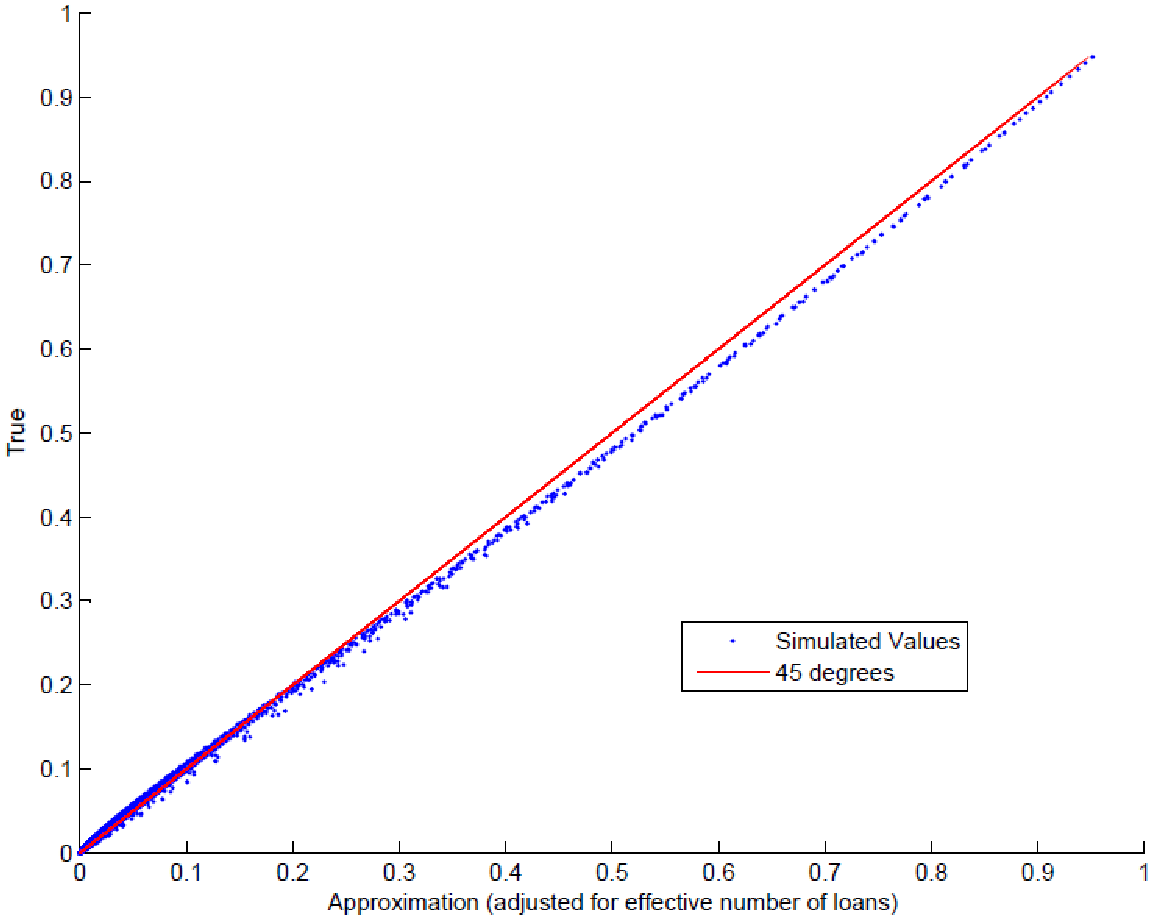


Chart 8

Actual and Approximated Values of h_G Based on Effective Number of Loans

(wholesale pools, $M = 5$ years, N from 1 to 1000, and $PD1$ from 0.03% to 99%)



75. Equation (34) performs well for homogeneous pools and can be adapted readily to deal with non-homogeneous pools. However, as in the current SFA, to avoid issues relating to defaults or near-defaults of non-material loans, within the MFSA we estimate the conditional probability of zero defaults in terms of the pool's exposure-weighted conditional probability of default and exposure-weighted LGD. Specifically, $\mathrm{h}_{\widehat{\mathrm{G}}}$ is estimated as

$$(35) \quad \hat{\mathrm{h}}_{\widehat{\mathrm{G}}} = \left(1 - \frac{\hat{\mathrm{E}}_{\widehat{\mathrm{G}}}}{\mathrm{LGD}}\right)^{\mathrm{N}^*}, \text{ where}$$

$$\mathrm{N}^* = \frac{\widehat{\mathrm{N}}}{\left(1 + 0.0079\mathrm{M}\sqrt{\widehat{\mathrm{N}}}\right)^2},$$

$$\widehat{\mathrm{N}} = \left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \theta_{\mathrm{i}}^2\right)^{-1}, \text{ and}$$

$$\mathrm{LGD} = \sum_{\mathrm{i}} \theta_{\mathrm{i}} \overline{\mathrm{LGD}}_{\mathrm{i}}\ .$$

For a homogeneous pool, this approximation is identical to (34).

V. Concluding Remarks

76. The preceding sections have summarised the modelling assumptions underpinning the proposed MSFA and the methods used to calibrate its inputs. The conceptual framework underlying the MSFA seeks to improve upon that underlying the SFA in a number of areas, particularly in the treatment of maturity effects through the adoption of a MtM modelling approach. However, this comes at the cost of greater complexity, both in derivation and implementation. A particular concern is whether banks seeking to implement the MSFA would have access to the loan-level IRB risk parameters that are the inputs to the MSFA. Even under the current SFA, many banks are challenged in developing estimates of the K_{IRB} input, and in some cases other inputs as well, when they are not the originator or servicer of the securitised loans. This problem will be more acute under the MSFA, where the current proposal would require an estimate of the PD, LGD, and EAD for each loan in the underlying pool.

77. Pending feedback from the industry on this proposal, it may be possible to further simplify the MSFA framework so as to reduce the information requirements on banks. For example, it may be feasible to develop proxies for the MSFA's $\mathrm{E}_{\widehat{\mathrm{G}}}, \mathrm{V}_{\widehat{\mathrm{G}}}, \text{and } \mathrm{h}_{\widehat{\mathrm{G}}}$ inputs in terms of aggregate pool-level statistics, such as a demonstrably conservative estimate of $\mathrm{K}_{\mathrm{IRB}}$, which could provide meaningful implementation benefits to both banks and supervisors.

78. Lastly, it is worth emphasising that the MSFA framework developed herein does not address a number of well-known shortcomings of SFA and IRB framework more generally. The single systematic risk-factor model embedded in the SFA and IRB frameworks remains a key postulate within the MSFA, and is in fact critical to affording tractable solutions or approximations to key inputs. Nevertheless, the assumption is simplistic and restrictive, and may not adequately represent the credit risks within pools containing concentrations of borrowers from multiple sectors, geographic regions, etc. As with the SFA, the MSFA also assumes a highly simplistic structure for how pool losses are allocated among tranches. Many complex structures of the form seen prior to the financial crisis cannot be readily mapped into the simplified MSFA template. Other MSFA shortcomings include reliance on the questionable assumptions that risk factors are normally distributed with fixed variances over time and that the market price of risk is stable over time. While the proposed MSFA

incorporates assumptions and prudential add-ons to offset some potential modelling weaknesses, these may only partially cover the full range of model risks.

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Annex 1

Estimating Term Structure of Default Probabilities from One-year Default Probabilities

Step 1: Estimating Multi-Year Default Probabilities from One-Year Rating Transitions

1. We start from the average one-year letter rating transition matrix reported by Moody's for the time period from 1970 to 2010 (Exhibit 27 in Moody's (2011)), shown in Table 1.1.

2. The two highlighted cells in the table indicate places where adjustments were made: we replaced the actual AAA default rate of 0 with a more conservative value of 0.005% and reduced the AAA withdrawn rating (WR) rate from 3.336% to 3.331% to maintain the total probability of 1 for the AAA row.

3. We transform the matrix by removing the WR column and reallocating the WR probabilities across the other final states. For each pair of initial rating i and final rating j, we adjust the transition probability p_{ij} from i to j according to

(1.1) $$p_{ij}^{(\text{adj})} = \frac{p_{ij}}{1 - p_{i,\text{WR}}}$$

where $p_{i,\text{WR}}$ is the one-year probability of rating withdrawal for initial rating i. Probability $p_{ij}^{(\text{adj})}$ can be interpreted as the probability of transition from rating i to rating j conditional on no rating withdrawal.

4. After this adjustment the rating transition probability matrix takes the form in Table 1.2. Note that within this table we have added an extra row corresponding to the initial default (D) state. The probabilities of default (PD) for time period T, ranging from 2 to 10 years, are taken from the D column of the matrix obtained by raising the adjusted transition probability matrix to power T. Table 1.3 shows the PDs obtained via this procedure.

5. PDs for the finer alphanumeric rating grid are obtained via interpolation. We assign successive integer values to the alphanumeric ratings: Aaa=1, Aa1=2, Aa2=3, …, Caa3=19, Ca=20 and assume that the default probabilities of ratings Aaa, Aa2, A2, Baa2,…, Caa2, Ca are the same as the default probabilities for the alphabetical ratings given in Table 1.3. Linear interpolation is performed for the logarithm of the default likelihood ratios, i.e. $\ln\left(\text{PD}_i(T)/[1-\text{PD}_i(T)]\right)$ at fixed T as function of i where i denotes the initial alphanumeric rating. The outcome of this interpolation procedure is shown in Table 1.4, where the interpolated values are shown in red.

Step 2: Estimating Multi-year Default Probabilities from One-Year PDs

6. The MSFA framework requires specifying each borrower's expected default rate over the remaining life of the securitisation. Under the IRB approach, banks are required to estimate the one-year PDs of their obligors, but not multi-year PDs. Moreover, allowing banks to estimate multi-year PDs may not be prudent. We propose a formula that calculates multi-year PD as a function of one-year PD. The formula is designed to provide a good fit to multi-year PDs shown in Table 1.3 for time periods T from 1 to 5 years and for all alphabetical ratings.

7. Let $x(T)$ denote the logarithm of the default likelihood ratio; that is, $x(T) = \ln\left(\mathrm{PD}(T)/[1-\mathrm{PD}(T)]\right)$. Then the PD for time period T can be expressed as

(1.2) $$\mathrm{PD}(T) = \frac{1}{1+\exp[-x(T)]}.$$

After exploring alternative functional forms for $x(T)$, we adopt the specification

(1.3) $$x(T) = x(1) + [5 - 0.15 \cdot x(1)] \cdot (T^{0.2} - 1)$$

where, from above,

(1.4) $$x(1) = \ln\left(\frac{\mathrm{PD}(1)}{1-\mathrm{PD}(1)}\right).$$

8. Figures 1.1 and 1.2 and Table 1.5 illustrate the quality of the fit. Data points in Figure 1.1 represent the logarithm of default likelihood ratios calculated for all cells of Table 1.3, while the curves $X(T)$ are calculated according to Equation (1.3) using the one-year column of Table 1.3 as the $\mathrm{PD}(1)$ inputs. Figure 1.2 transforms the default likelihood ratios of Figure 1.1 into PDs. Table 1.5 compares $\mathrm{PD}(T)$ of Table 1.3 with $\mathrm{PD}(T)$ calculated according to Equation (1.3) using one-year PDs for all alphabetical ratings from Table 1.3. One can observe that multi-year PDs obtained with the fit function are reasonably close to the ones in Table 1.3.

Table 1.1

	Aaa	Aa	A	Baa	Ba	B	Caa	Ca	D	WR
Aaa	87.395%	8.626%	0.602%	0.010%	0.027%	0.002%	0.002%	0.000%	0.005%	3.331%
Aa	0.971%	85.616%	7.966%	0.359%	0.045%	0.018%	0.008%	0.001%	0.020%	4.996%
A	0.062%	2.689%	86.763%	5.271%	0.488%	0.109%	0.032%	0.004%	0.054%	4.528%
Baa	0.043%	0.184%	4.525%	84.517%	4.112%	0.775%	0.173%	0.019%	0.176%	5.476%
Ba	0.008%	0.056%	0.370%	5.644%	75.759%	7.239%	0.533%	0.080%	1.104%	9.207%
B	0.010%	0.034%	0.126%	0.338%	4.762%	73.524%	5.767%	0.665%	4.230%	10.544%
Caa	0.000%	0.021%	0.021%	0.142%	0.463%	8.263%	60.088%	4.104%	14.721%	12.177%
Ca	0.000%	0.000%	0.000%	0.000%	0.324%	2.374%	8.880%	36.270%	35.451%	16.701%

Table 1.2

	Aaa	Aa	A	Baa	Ba	B	Caa	Ca	D
Aaa	90.406%	8.923%	0.623%	0.010%	0.028%	0.002%	0.002%	0.000%	0.005%
Aa	1.022%	90.118%	8.385%	0.378%	0.047%	0.019%	0.008%	0.001%	0.021%
A	0.065%	2.817%	90.878%	5.521%	0.511%	0.114%	0.034%	0.004%	0.057%
Baa	0.045%	0.195%	4.787%	89.413%	4.350%	0.820%	0.183%	0.020%	0.186%
Ba	0.009%	0.062%	0.408%	6.216%	83.441%	7.973%	0.587%	0.088%	1.216%
B	0.011%	0.038%	0.141%	0.378%	5.323%	82.190%	6.447%	0.743%	4.729%
Caa	0.000%	0.024%	0.024%	0.162%	0.527%	9.409%	68.419%	4.673%	16.762%
Ca	0.000%	0.000%	0.000%	0.000%	0.389%	2.850%	10.660%	43.542%	42.559%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.00%

Table 1.3

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Aaa	0.005%	0.013%	0.024%	0.038%	0.057%	0.082%	0.113%	0.152%	0.200%	0.257%
Aa	0.021%	0.049%	0.085%	0.132%	0.191%	0.264%	0.354%	0.463%	0.593%	0.745%
A	0.057%	0.138%	0.249%	0.393%	0.574%	0.794%	1.056%	1.360%	1.707%	2.096%
Baa	0.186%	0.486%	0.899%	1.421%	2.045%	2.765%	3.573%	4.458%	5.411%	6.423%
Ba	1.216%	2.755%	4.585%	6.652%	8.899%	11.267%	13.706%	16.174%	18.635%	21.061%
B	4.729%	10.078%	15.606%	21.035%	26.204%	31.030%	35.481%	39.552%	43.260%	46.629%
Caa	16.762%	30.671%	41.655%	50.194%	56.836%	62.050%	66.201%	69.556%	72.312%	74.612%
Ca	42.559%	63.016%	73.565%	79.493%	83.148%	85.603%	87.375%	88.725%	89.797%	90.672%

Table 1.4

Rating	Probability of Default									
	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Aaa	0.005%	0.013%	0.024%	0.038%	0.057%	0.082%	0.113%	0.152%	0.200%	0.257%
Aa1	0.010%	0.025%	0.045%	0.071%	0.105%	0.147%	0.200%	0.266%	0.344%	0.438%
Aa2	0.021%	0.049%	0.085%	0.132%	0.191%	0.264%	0.354%	0.463%	0.593%	0.745%
Aa3	0.029%	0.069%	0.122%	0.190%	0.275%	0.381%	0.510%	0.664%	0.844%	1.053%
A1	0.041%	0.098%	0.174%	0.273%	0.398%	0.551%	0.734%	0.951%	1.201%	1.487%
A2	0.057%	0.138%	0.249%	0.393%	0.574%	0.794%	1.056%	1.360%	1.707%	2.096%
A3	0.084%	0.210%	0.382%	0.604%	0.878%	1.207%	1.590%	2.028%	2.518%	3.060%
Baa1	0.125%	0.320%	0.586%	0.927%	1.342%	1.830%	2.389%	3.014%	3.702%	4.448%
Baa2	0.186%	0.486%	0.899%	1.421%	2.045%	2.765%	3.573%	4.458%	5.411%	6.423%
Baa3	0.349%	0.870%	1.557%	2.396%	3.374%	4.474%	5.677%	6.967%	8.327%	9.741%
Ba1	0.652%	1.553%	2.683%	4.015%	5.518%	7.159%	8.907%	10.731%	12.605%	14.507%
Ba2	1.216%	2.755%	4.585%	6.652%	8.899%	11.267%	13.706%	16.174%	18.635%	21.061%
Ba3	1.922%	4.289%	7.003%	9.958%	13.058%	16.218%	19.373%	22.473%	25.483%	28.377%
B1	3.024%	6.617%	10.555%	14.650%	18.761%	22.787%	26.660%	30.339%	33.802%	37.041%
B2	4.729%	10.078%	15.606%	21.035%	26.204%	31.030%	35.481%	39.552%	43.260%	46.629%
B3	7.335%	15.047%	22.486%	29.333%	35.468%	40.889%	45.647%	49.816%	53.474%	56.692%
Caa1	11.210%	21.870%	31.276%	39.275%	45.967%	51.539%	56.189%	60.095%	63.404%	66.232%
Caa2	16.762%	30.671%	41.655%	50.194%	56.836%	62.050%	66.201%	69.556%	72.312%	74.612%
Caa3	27.864%	46.473%	58.498%	66.404%	71.822%	75.717%	78.641%	80.917%	82.741%	84.239%
Ca	42.559%	63.016%	73.565%	79.493%	83.148%	85.603%	87.375%	88.725%	89.797%	90.672%

Table 1.5

T	1Y		2Y		3Y		4Y		5Y	
	Exact	Fit	Exact	Fit	Exact	Fit	Exact	Fit	Exact	Fit
Aaa	0.005%	0.005%	0.013%	0.014%	0.024%	0.025%	0.038%	0.041%	0.057%	0.061%
Aa	0.021%	0.021%	0.049%	0.053%	0.085%	0.098%	0.132%	0.156%	0.191%	0.227%
A	0.057%	0.057%	0.138%	0.140%	0.249%	0.254%	0.393%	0.399%	0.574%	0.575%
Baa	0.186%	0.186%	0.486%	0.449%	0.899%	0.797%	1.421%	1.230%	2.045%	1.750%
Ba	1.216%	1.216%	2.755%	2.776%	4.585%	4.713%	6.652%	6.984%	8.899%	9.550%
B	4.729%	4.729%	10.078%	10.041%	15.606%	15.926%	21.035%	22.069%	26.204%	28.224%
Caa	16.762%	16.762%	30.671%	30.505%	41.655%	42.193%	50.194%	51.793%	56.836%	59.564%
Ca	42.559%	42.559%	63.016%	61.071%	73.565%	71.907%	79.493%	78.785%	83.148%	83.422%

Figure 1.1

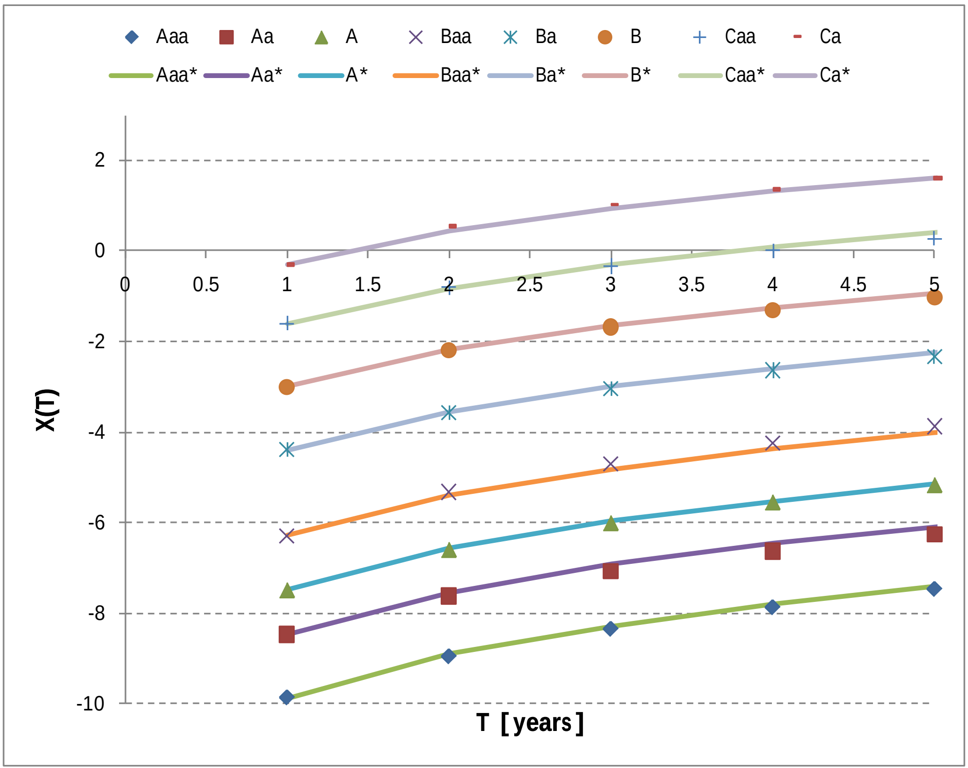
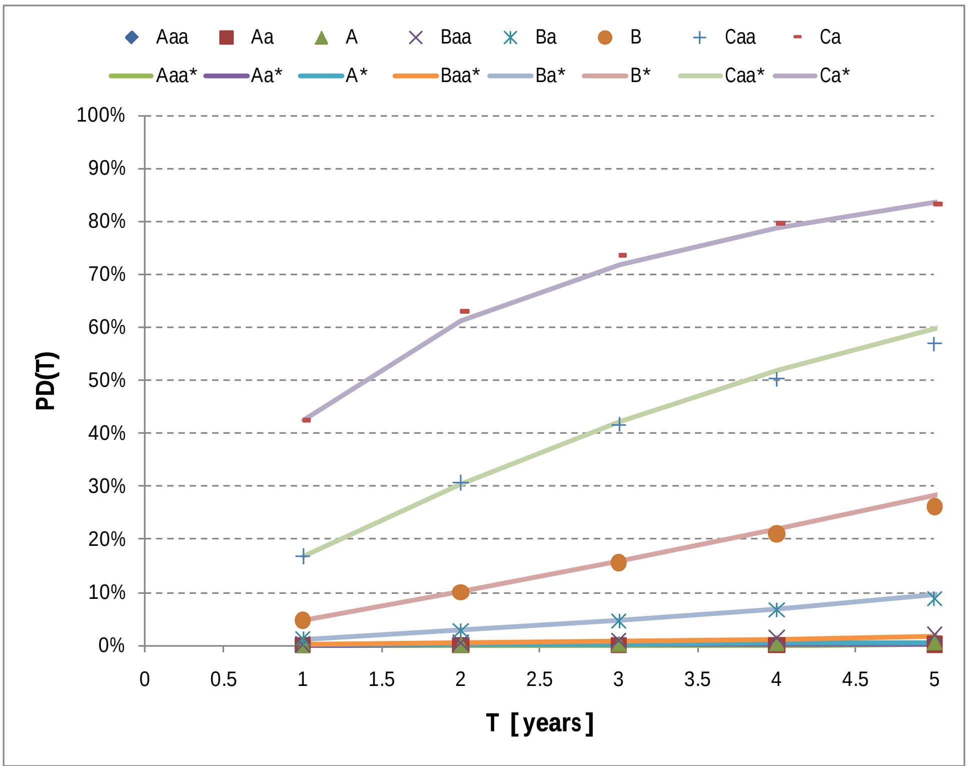


Figure 1.2



Annex 2

Derivation of Solution for $W[x_1; PD1_j, PDM_j, r_j]$

Step 1: Risk-neutral probability (at t=1) that j^th loan defaults by maturity

1. A loan can default either at the end of the capital horizon or at the maturity of the securitisation. Conditional on $\{\tilde{X}_1 = x_1, \tilde{U}_{11} = u_{11}, \ldots, \tilde{U}_{1N} = u_{1n}\}$, the j^th loan defaults under the risk-neutral distribution if and only if (a) the loan defaults at t=1, or (b) loan does not default at t=1, but defaults at t=M.

2. At t=1, the loan has already defaulted the following inequality obtains

(2.1) $$r_j x_1 + u_{1j}\sqrt{1 - r_j^2} \leq DT1_j^*.$$

3. Otherwise (ie if the above inequality does not hold, implying the loan has not defaulted at t=1), then the loan defaults at t=M under the risk-neutral distribution if

(2.2) $$\left(r_j \tilde{X}_2 + \tilde{u}_{2j}\sqrt{1 - r_j^2}\right)\sqrt{M-1} + \left(\frac{R - m_j}{\pi_j}\right) \cdot (M-1) + \left(r_j x_1 + u_{1j}\sqrt{1 - r_j^2}\right) \leq DTM_j^*$$

where $\tilde{X}_2 \text{ and } \tilde{u}_{2j}$ are independent, standard normal random variables.[30] Thus, the risk-neutral probability of the loan defaulting by maturity, given that the loan did not default at t=1, is given by

(2.3) $$Prob\{\text{Default at t} = \text{M} \mid \text{no default at t} = 1\} = \Phi\left[\frac{DTM_j^* - \left(r_j x_1 + u_{1j}\sqrt{1 - r_j^2}\right) + \left(\frac{m_j - R}{\pi_j}\right) \cdot (M-1)}{\sqrt{M-1}}\right].$$

Step 2: Expected risk-neutral probability that loan j defaults conditional on stress event

4. From the definitions of conditional and joint probability distributions, the e*xpected risk-neutral probability that loan j defaults conditional on* $\tilde{X}_1 = x_1$ is the sum of two terms:

A = The natural probability that the loan defaults at t=1; and

B = The probability of no default at t=1 *multiplied by the* natural probability expectation of (2.3) conditional on realisations of u_{1j} for which (2.1) does not hold (for the given x_1).

[30] Note that for $0 \leq t < 1$, the asset value drift under the risk-neutral probability distribution prevailing at t=1 should be set equal to m_j. This is because at $t = 1, Y_j(1)$ is known and was generated from the borrower's actual asset value process.

5. From (2.1), we have

(2.4) $A = \Phi[\xi_j]$, where $\xi_j = \frac{DT1_j^* - r_j x_1}{\sqrt{(1-r_j^2)}}$.

6. From (2.1) and (2.3),

$$(2.5) \quad B = \int_{-\xi_j}^{\infty} \Phi\left[\frac{DTM_j^* - \left(r_j x_1 + u_{1j}\sqrt{1-r_j^2}\right) - \left(\frac{m_j - R}{\pi_j}\right)\cdot(M-1)}{\sqrt{M-1}}\right]\cdot \phi[u_{1j}]\,\mathrm{d}u_{1j}$$

$$= \int_{-\infty}^{-\left(\frac{DT1_j^* - r_j x_1}{\sqrt{1-r_j^2}}\right)} \Phi\left[\frac{DTM_j^* - \left(r_j x_1 - u_{1j}\sqrt{1-r_j^2}\right) - \left(\frac{m_j - R}{\pi_j}\right)\cdot(M-1)}{\sqrt{M-1}}\right]\cdot \phi[u_{1j}]\,\mathrm{d}u_{1j}$$

$$= \Phi_2\left[-\frac{DT1_j^* - r_j x_1}{\sqrt{1-r_j^2}}, \frac{DTM_j^* - r_j x_1 + \left(\frac{m_j - R}{\pi_j}\right)\cdot(M-1)}{\sqrt{M-r_j^2}}; -\sqrt{\frac{1-r_j^2}{M-r_j^2}}\right]$$

where the last line follows from equation (30c) in Andersen-Sidenius (2004/2005).

7. On combining (2.4) and (2.5), we have the desired result

(2.6)

$$\int_{-\infty}^{\infty} E_1^{\widehat{RN}}\{\tilde{I}_j\}\cdot \phi[u_{1j}]\,du_{1j} = \Phi\left[\frac{DT1_j^* - r_j x_1}{\sqrt{1-r_j^2}}\right] + \Phi_2\left[-\frac{DT1_j^* - r_j x_1}{\sqrt{1-r_j^2}}, \frac{DTM_j^* - r_j x_1 + \left(\frac{m_j - R}{\pi_j}\right)\cdot(M-1)}{\sqrt{M-r_j^2}}; -\sqrt{\frac{1-r_j^2}{M-r_j^2}}\right].$$