Online Appendix to "Mortgage Risk and the Yield Curve"

This online appendix contains all the proofs omitted in the main part of the paper and some additional empirical robustness results. Appendix C.1 provides some preliminary results needed for the proofs given in Appendix C.2. Appendix C.3 provides an extension of our baseline model that jointly prices nominal and real bonds. Appendix C.4 outlines a model with time-varying convexity. Appendix D provides some additional robustness checks for our empirical analysis.

Appendix C Proofs and model extensions

Appendix C.1 Preliminary results

Appendix C.1.1 Properties of useful functions

This appendix introduces three functions necessary to derive our main results and studies their properties.

Lemma 2. *For any $\tau > 0$, the function $F(x) \equiv \frac{1-e^{-x\tau}}{x\tau}$ for all $x \neq 0$ and $F(0) \equiv 1$ is positive, decreasing, and convex for all $x \in \mathbb{R}$. Moreover, for arbitrary $x, y \in \mathbb{R}$, $x \neq y$,*

$$\frac{1-e^{-x\tau}}{x\tau} - \frac{1-e^{-x\tau} - x\tau e^{-x\tau}}{x^2\tau}(y-x) < \frac{1-e^{-y\tau}}{y\tau}. \tag{A-1}$$

Proof. The derivative of F is given by

$$F'(x) = -\frac{1-e^{-x\tau} - x\tau e^{-x\tau}}{x^2\tau}, \tag{A-2}$$

which has the opposite sign as $F_1(x) \equiv 1 - e^{-x\tau} - x\tau e^{-x\tau}$. But $\lim_{x\to 0} F_1(x) = 0$, and $F_1'(x) = x\tau^2 e^{-x\tau}$, which is negative for $x < 0$ and positive for $x > 0$. Hence, $F_1(x) \geq 0$ for all $x \in \mathbb{R}$, and $F'(x) \leq 0$ for all $x \in \mathbb{R}$; F is a decreasing function. However, the limit of F when $x \to \infty$ is $\lim_{x\to\infty} F(x) = 0$. Since F is a decreasing function and it converges to zero when $x \to \infty$, it must be that $F(x) > 0$ for all $x \in \mathbb{R}$.

Regarding convexity, (A-2) implies

$$F''(x) = \frac{2e^{-x\tau}}{x^3\tau}\left[e^{x\tau} - \left(1 + x\tau + \frac{x^2\tau^2}{2}\right)\right],$$

but $1 + z + \frac{z^2}{2}$ are the first three terms of the power series of e^z, and it is well-known that $1 + z + \frac{z^2}{2} < e^z$ for $z > 0$ and $1 + z + \frac{x^2}{2} > e^z$ for $z < 0$. Therefore, $F''(x) > 0$ and thus F is convex.

Finally, convexity of F is equivalent to the function lying above all of its tangents. From (A-2), (A-1) is describing exactly this inequality for the point of tangency x and an arbitrary y: $F(x) + F'(x)(y-x) < F(y)$. □

Lemma 3. *For any $\tau > 0$, the function*

$$G(x,y) \equiv \frac{F(x) - F(y)}{x-y} = \frac{\frac{1-e^{-x\tau}}{x\tau} - \frac{1-e^{-y\tau}}{y\tau}}{x-y}, \tag{A-3}$$

$x, y \in \mathbb{R}$, is symmetric, negative, and increasing in both arguments. Moreover, if $x < x' < y' < y$ while $x + y = x' + y'$, $G(x, y) < G(x', y')$.

Proof. Lemma 2 implies that the numerator of G is positive if and only if $x < y$, hence $G(x, y) < 0$ for all $x, y \in \mathbb{R}$. Symmetry, i.e. $G(x, y) = G(y, x)$, is obvious, and thus for G being increasing, we only need to show that $\frac{\partial G}{\partial x} > 0$ for a fixed y. Differentiating (A-3) w.r.t. x, we have

$$\frac{\partial G}{\partial x} = -\frac{1}{(x-y)^2}\left[\frac{1-e^{-x\tau}}{x\tau} - \frac{1-e^{-x\tau}-x\tau e^{-x\tau}}{x^2\tau}(y-x) - \frac{1-e^{-y\tau}}{y\tau}\right].$$

Lemma 2 also implies that the term inside the bracket is negative, and hence G is increasing in x and y. Moreover, for an arbitrary constant y we have $\lim_{x\to\infty} G(x, y) = 0$, so if G is increasing in x, it must be that for all $x, y \in \mathbb{R}$, $G(x, y) < 0$.

Notice that the last claim of the Lemma is equivalent to showing that fixing $z \equiv \frac{x+y}{2}$, $G(x, y) = G(x, 2z - x)$ is increasing in x whenever $x < z$. Differentiating with respect to x we obtain

$$\frac{dG(x, 2z-x)}{dx} = \frac{d}{dx}\frac{F(x) - F(2z-x)}{x-(2z-x)} = \frac{[F'(x) + F'(2z-x)](x-z) - [F(x) - F(2z-x)]}{2(x-z)^2}$$
$$= \frac{2}{y-x}\left[\frac{F(y) - F(x)}{y-x} - \frac{F'(x) + F'(y)}{2}\right],$$

where in the last equality we substituted y back. Therefore, $G(x, 2z - x)$ is increasing in x if and only if

$$\frac{F(y) - F(x)}{y-x} - \frac{F'(x) + F'(y)}{2} > 0 \tag{A-4}$$

for all $x < y$. Since F is twice differentiable everywhere (see Lemma 2), we can write

$$F(y) - F(x) = \int_x^y F'(t)\,dt = \int_x^y \left[F'(x) + \int_x^t F''(w)\,dw\right]dt = F'(x)(y-x) + \int_x^y\int_x^t F''(w)\,dwdt$$
$$= F'(x)(y-x) + \int_x^y\int_w^y F''(w)\,dtdw = F'(x)(y-x) + \int_x^y F''(w)(y-w)\,dw, \tag{A-5}$$

where the fourth equality is an application of Fubini's theorem. From Lemma 2 we know $F''(x) > 0$ for all $x \in \mathbb{R}$; moreover, the third derivative of F is simply

$$F'''(x) = \frac{dF''(x)}{dx} = -\frac{6e^{-x\tau}}{x^4\tau}\left[e^{x\tau} - \left(1 + x\tau + \frac{x^2\tau^2}{2} + \frac{x^3\tau^3}{6}\right)\right] < 0 \tag{A-6}$$

for all $x \in \mathbb{R}$, thus $F''(x)$ is a positive decreasing function. Therefore, we can write

$$\int_x^y F''(w)(y-w)\,dw > \int_x^y F''(x)(y-w)\,dw = F''(x)\int_x^y (y-w)\,dw = F''(x)\frac{(y-x)^2}{2}. \tag{A-7}$$

Combining (A-5) and (A-7), we obtain

$$\frac{F(y)-F(x)}{y-x}-\frac{F'(x)+F'(y)}{2}=\frac{1}{y-x}\left[F'(x)(y-x)+\int_x^y F''(w)(y-w)\,dw\right]-\frac{F'(x)+F'(y)}{2}$$
$$>\frac{1}{y-x}\left[F'(x)(y-x)+F''(x)\frac{(y-x)^2}{2}\right]-\frac{F'(x)+F'(y)}{2}=\frac{1}{2}\left[F'(x)+F''(x)(y-x)-F'(y)\right].$$

But (A-6) also means that F' is a concave function, i.e., it lies below all its tangents. Therefore, $F'(x)+F''(x)(y-x)-F'(y)>0$, which implies (A-4) and concludes the proof of the lemma. □

Lemma 4. *Fix $\tau>0$. The function*

$$H(x,y)=\frac{1-e^{-x\tau}}{x}+\frac{x(y-x)}{y}\left(-\frac{1-e^{-x\tau}-x\tau e^{-x\tau}}{x^2}\right)-\frac{1-e^{-y\tau}}{y}, \tag{A-8}$$

$x,y\in\mathbb{R}^+$, *satisfies $H(x,y)=0$ if $x=y$ and $H(x,y)>0$ whenever $x\neq y$.*

Proof. If $x=y$, the first and last terms of H are equivalent and the middle one is zero, hence $H(x,y)=0$. Next we differentiate H with respect to y while keeping x fixed to obtain

$$\frac{dH(x,y)}{dy}=\frac{(1-e^{-y\tau}-y\tau e^{-y\tau})-(1-e^{-x\tau}-x\tau e^{-x\tau})}{y^2}=\frac{F_1(y)-F_1(x)}{y^2},$$

where F_1 is defined in the proof of Lemma 2. As shown there, F_1 is increasing on $\mathbb{R}^+$, so $0<x<y$ implies the numerator is positive and thus $\frac{dH(x,y)}{dy}>0$. On the other hand, $0<y<x$ implies the numerator is negative and $\frac{dH(x,y)}{dy}<0$. Therefore, H is decreasing in y before x, reaches zero, then increasing, i.e., is positive for all $y\neq x$. □

Appendix C.1.2 Covariance and autocovariance of D and r

In this appendix we derive the unconditional variance-covariance matrix and the autocovariance of $(D_t,r_t)^\top$ under $\mathbb{P}$ from (1) and (11). Following the standard technique, applying Itô's lemma to $e^{\kappa t}r_t$ and combining it with (1), we obtain

$$d\left(e^{\kappa t}r_t\right)=\left[\kappa e^{\kappa t}r_t+e^{\kappa t}\kappa(\theta-r_t)\right]dt+e^{\kappa t}\sigma dB_t=e^{\kappa t}\kappa\theta dt+e^{\kappa t}\sigma dB_t.$$

Integrating both sides between t and $s>t$, and rearranging gives

$$r_s=\theta\left(1-e^{-\kappa(s-t)}\right)+r_te^{-\kappa(s-t)}+\sigma e^{-\kappa(s-t)}\int_t^s e^{\kappa(v-t)}dB_v. \tag{A-9}$$

Next we want to obtain a similar form for duration. Defining

$$\bar{D}_t=D_t+\frac{\delta_r}{\delta_D-\kappa}r_t, \tag{A-10}$$

we have

$$d\bar{D}_t=dD_t+\frac{\delta_r}{\delta_D-\kappa}dr_t=\delta_D\left(\bar{\theta}_D-\bar{D}_t\right)dt+\bar{\sigma}_DdB_t,$$

where in the last step we use (15) and introduce the notation

$$\bar{\theta}_D = \theta_D + \frac{\delta_r}{\delta_D - \kappa}\theta \text{ and } \bar{\sigma}_D = \eta_y \sigma_y^{\bar{\tau}} + \frac{\delta_r}{\delta_D - \kappa}\sigma = \delta_D \frac{\eta_y \sigma_y^{\bar{\tau}}}{\delta_D - \kappa} = \frac{\delta_D}{\delta_D - \kappa}\frac{\delta_r \sigma}{\kappa}.$$

That is, $\bar{D}_t$ is a Vasicek process with a speed of mean reversion δ_D. Applying the same steps as for the short rate, we obtain

$$\bar{D}_s = \bar{\theta}_D \left(1 - e^{-\delta_D(s-t)}\right) + \bar{D}_t e^{-\delta_D(s-t)} + \bar{\sigma}_D e^{-\delta_D(s-t)} \int_t^s e^{\delta_D(v-t)} dB_v, \tag{A-11}$$

where, importantly, the Brownian increments dB_v are the same as in (A-9).

In what follows, from (A-9) and (A-11) we compute conditional variances and covariances of our random variables. From (A-9), we have

$$Var_t[r_s] = \sigma^2 e^{-2\kappa s} E_t\left[\left(\int_t^s e^{\kappa v} dB_v\right)^2\right] = \sigma^2 e^{-2\kappa s} \int_t^s e^{2\kappa v} dv = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(s-t)}\right), \tag{A-12}$$

where we use that $dB_v \sim N(0, dv)$ i.i.d. over time. Similarly, from (A-11) we obtain

$$Var_t\left[\bar{D}_s\right] = \bar{\sigma}_D^2 e^{-2\delta_D s} E_t\left[\left(\int_t^s e^{\delta_D v} dB_v\right)^2\right] = \bar{\sigma}_D^2 e^{-2\delta_D s} \int_t^s e^{2\delta_D v} dv = \frac{\bar{\sigma}_D^2}{2\delta_D}\left(1 - e^{-2\delta_D(s-t)}\right). \tag{A-13}$$

Finally, for the covariance, we have

$$\begin{aligned} Cov_t\left[r_s, \bar{D}_s\right] &= \sigma \bar{\sigma}_D e^{-(\kappa+\delta_D)s} E_t\left[\int_t^s e^{\kappa v} dB_v \int_t^s e^{\delta_D v} dB_v\right] \qquad \text{(A-14)} \\ &= \sigma \bar{\sigma}_D e^{-(\kappa+\delta_D)s} \int_t^s e^{(\kappa+\delta_D)v} dv = \frac{\sigma \bar{\sigma}_D}{\kappa + \delta_D}\left(1 - e^{-(\kappa+\delta_D)(s-t)}\right). \end{aligned}$$

From (A-10), (A-12), and (A-14) we get

$$\begin{aligned} Cov_t[r_s, D_s] &= Cov_t\left[r_s, \bar{D}_s\right] - \frac{\delta_r}{\delta_D - \kappa} Var_t[r_s] \qquad \text{(A-15)} \\ &= \frac{\sigma \bar{\sigma}_D}{\kappa + \delta_D}\left(1 - e^{-(\kappa+\delta_D)(s-t)}\right) - \frac{\delta_r}{\delta_D - \kappa}\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(s-t)}\right), \end{aligned}$$

and (A-10), (A-12), (A-13), and (A-15) together yield

$$\begin{aligned} Var_t[D_s] &= Var_t\left[\bar{D}_s\right] - 2\frac{\delta_r}{\delta_D - \kappa} Cov_t[r_s, D_s] - \left(\frac{\delta_r}{\delta_D - \kappa}\right)^2 Var_t[r_s] = \frac{\bar{\sigma}_D^2}{2\delta_D}\left(1 - e^{-2\delta_D(s-t)}\right) \\ &- 2\frac{\delta_r}{\delta_D - \kappa}\frac{\sigma \bar{\sigma}_D}{\kappa + \delta_D}\left(1 - e^{-(\kappa+\delta_D)(s-t)}\right) + \left(\frac{\delta_r}{\delta_D - \kappa}\right)^2 \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(s-t)}\right). \qquad \text{(A-16)} \end{aligned}$$

A notable special case that we use to determine the coefficients of the predictive regressions is the unconditional variance-covariance matrix of $(D_t, r_t)^\top$. Taking the limit $(s-t) \to \infty$ in (A-12), (A-15), and (A-16) yields

$$V = \begin{pmatrix} \frac{\delta_r^2 \sigma^2}{2\kappa^2(\kappa+\delta_D)} & \frac{\delta_r \sigma^2}{2\kappa(\kappa+\delta_D)} \\ \frac{\delta_r \sigma^2}{2\kappa(\kappa+\delta_D)} & \frac{\sigma^2}{2\kappa} \end{pmatrix}, \tag{A-17}$$

implying that in general when $\delta_D \neq 0$ the two factors are not collinear.

Finally, to obtain the unconditional covariance between variables at t and $s > t$, using (A-10)-(A-11) we write

$$Cov[D_t, D_s] = Cov\left[D_t, \bar{D}_s\right] - \frac{\delta_r}{\delta_D - \kappa} Cov[D_t, r_s] = \left(Var[D_t] + \frac{\delta_r}{\delta_D - \kappa} Cov[D_t, r_t]\right) e^{-\delta_D(s-t)}$$
$$- \frac{\delta_r}{\delta_D - \kappa} e^{-\kappa(s-t)} Cov[D_t, r_t] = \frac{\delta_r^2 \sigma^2}{2\kappa^2(\kappa+\delta_D)} \frac{\delta_D e^{-\delta_D(s-t)} - \kappa e^{-\kappa(s-t)}}{\delta_D - \kappa} \tag{A-18}$$

and

$$Cov[r_t, D_s] = Cov\left[r_t, \bar{D}_s\right] - \frac{\delta_r}{\delta_D - \kappa} Cov[r_t, r_s] = \left(Cov[r_t, D_t] + \frac{\delta_r}{\delta_D - \kappa} Var[r_t]\right) e^{-\delta_D(s-t)}$$
$$- \frac{\delta_r}{\delta_D - \kappa} e^{-\kappa(s-t)} Var[r_t] = \frac{\delta_r \sigma^2}{2\kappa(\kappa+\delta_D)} \frac{2\delta_D e^{-\delta_D(s-t)} - (\kappa+\delta_D) e^{-\kappa(s-t)}}{\delta_D - \kappa}. \tag{A-19}$$

Appendix C.1.3 Covariance matrix of D, r and r^*

For our extended model, presented in Appendix C.3, we also extend the above result to the variance-covariance matrix of $(D_t, r_t, r_t^*)^\top$ under $\mathbb{P}$ from (1), (11), and (18). Similarly to (A-9), we get

$$r_s^* = \theta^*\left(1 - e^{-\kappa^*(s-t)}\right) + r_t^* e^{-\kappa^*(s-t)} + \sigma^* e^{-\kappa^* s} \int_t^s e^{\kappa^* v} dB_v^*. \tag{A-20}$$

From here, we obtain the conditional variance

$$Var_t[r_s^*] = (\sigma^*)^2 e^{-2\kappa^* s} E_t\left[\left(\int_t^s e^{\kappa^* v} dB_v^*\right)^2\right] = (\sigma^*)^2 e^{-2\kappa^* s} \int_t^s e^{2\kappa^* v} dv = \frac{(\sigma^*)^2}{2\kappa^*}\left(1 - e^{-2\kappa^*(s-t)}\right). \tag{A-21}$$

Next, (A-9) and (A-20), together with $dB_t dB_t^* = \rho dt$, imply

$$Cov_t[r_s, r_s^*] = \sigma\sigma^* e^{-(\kappa+\kappa^*)s} E_0\left[\int_t^s e^{\kappa v} dB_v \int_t^s e^{\kappa^* v} dB_v^*\right] = \rho\sigma\sigma^* e^{-(\kappa+\kappa^*)s} \int_t^s e^{(\kappa+\kappa^*)v} dv$$
$$= \frac{\rho\sigma\sigma^*}{\kappa+\kappa^*}\left(1 - e^{-(\kappa+\kappa^*)(s-t)}\right). \tag{A-22}$$

Finally, from (A-11) and (A-20), we get

$$Cov_t\left[\bar{D}_s, r_s^*\right] = \bar{\sigma}_D\sigma^* e^{-(\kappa^*+\delta_D)s} E_t\left[\int_t^s e^{\delta_D v} dB_v \int_t^s e^{\kappa^* v} dB_v^*\right] \tag{A-23}$$

$$= \rho\bar{\sigma}_D\sigma^* e^{-(\kappa^*+\delta_D)s} \int_t^s e^{(\kappa^*+\delta_D)v} dv = \frac{\rho\bar{\sigma}_D\sigma^*}{\kappa^*+\delta_D}\left(1-e^{-(\kappa^*+\delta_D)(s-t)}\right).$$

Using (A-11) and (A-22), (A-23) then implies

$$Cov_t\left[D_s, r_s^*\right] = \frac{\rho\bar{\sigma}_D\sigma^*}{\kappa^*+\delta_D}\left(1-e^{-(\kappa^*+\delta_D)(s-t)}\right) - \frac{\delta_r}{\delta_D-\kappa}\frac{\rho\sigma\sigma^*}{\kappa+\kappa^*}\left(1-e^{-(\kappa+\kappa^*)(s-t)}\right) \tag{A-24}$$

$$= \frac{\rho\sigma\sigma^*\delta_r}{\kappa\left(\delta_D-\kappa\right)}\left[\frac{\delta_D}{\kappa^*+\delta_D}\left(1-e^{-(\kappa^*+\delta_D)(s-t)}\right) - \frac{\kappa}{\kappa+\kappa^*}\left(1-e^{-(\kappa+\kappa^*)(s-t)}\right)\right],$$

where in the second step we use (15) and the definition of $\bar{\sigma}_D$. After taking the limit $(s-t)\to\infty$, we can combine the above results with (A-17) to obtain the unconditional variance-covariance matrix of $(D_t, r_t, r_t^*)^\top$:

$$V^* = \begin{pmatrix} \frac{\delta_r^2\sigma^2}{2\kappa^2(\kappa+\delta_D)} & \frac{\delta_r\sigma^2}{2\kappa(\kappa+\delta_D)} & \frac{\rho\kappa^*\sigma^*}{(\kappa+\kappa^*)}\frac{\delta_r\sigma}{\kappa(\kappa^*+\delta_D)} \\ \frac{\delta_r\sigma^2}{2\kappa(\kappa+\delta_D)} & \frac{\sigma^2}{2\kappa} & \frac{\rho\sigma\sigma^*}{\kappa+\kappa^*} \\ \frac{\rho\kappa^*\sigma^*}{(\kappa+\kappa^*)}\frac{\delta_r\sigma}{\kappa(\kappa^*+\delta_D)} & \frac{\rho\sigma\sigma^*}{\kappa+\kappa^*} & \frac{(\sigma^*)^2}{2\kappa^*} \end{pmatrix}. \tag{A-25}$$

Finally, to obtain the unconditional covariance between r_t^* and D_s, we write

$$Cov\left[r_t^*, D_s\right] = Cov\left[r_t^*, \bar{D}_s\right] - \frac{\delta_r}{\delta_D-\kappa}Cov\left[r_t^*, r_s\right] = \left(Cov\left[r_t^*, D_t\right] + \frac{\delta_r}{\delta_D-\kappa}Cov\left[r_t^*, r_t\right]\right)e^{-\delta_D(s-t)}$$

$$-\frac{\delta_r}{\delta_D-\kappa}e^{-\kappa(s-t)}Cov\left[r_t^*, r_t\right] = \frac{\delta_r\rho\sigma\sigma^*}{\kappa\left(\delta_D-\kappa\right)}\left(\frac{\delta_D e^{-\delta_D(s-t)}}{\kappa^*+\delta_D} - \frac{\kappa e^{-\kappa(s-t)}}{\kappa+\kappa^*}\right).$$

Appendix C.2 Proofs and derivations

Proof of Lemma 1. For notational simplicity let us write bond prices in the form

$$\frac{d\Lambda_t^\tau}{\Lambda_t^\tau} = \mu_t^\tau dt - \sigma_t^\tau dB_t. \tag{A-26}$$

Substituting (A-26) into intermediaries' budget constraint, (2), we get

$$dW_t = \left[r_t W_t + \int_0^T x_t^\tau \Lambda_t^\tau\left(\mu_t^\tau - r_t\right)d\tau\right]dt - \left[\int_0^T x_t^\tau\Lambda_t^\tau\sigma_t^\tau d\tau\right]dB_t,$$

therefore (3) simplifies to

$$\max_{\{x_t^\tau\}_{\tau\in(0,T]}} \int_0^T x_t^\tau\Lambda_t^\tau\left(\mu_t^\tau - r_t\right)d\tau - \frac{\alpha}{2}\left[\int_0^T x_t^\tau\Lambda_t^\tau\sigma_t^\tau d\tau\right]^2. \tag{A-27}$$

Because markets are complete, by no-arbitrage, there exists a unique market price of interest rate risk across all bonds that satisfies

$$\lambda_t = \frac{E_t\left(\frac{d\Lambda_t^\tau}{\Lambda_t^\tau}\right)/dt - r_t}{\frac{1}{\Lambda_t^\tau}\frac{d\Lambda_t^\tau}{dr_t}\sigma} = \frac{\mu_t^\tau - r_t}{-\sigma_t^\tau}, \tag{A-28}$$

and introducing

$$x_t = \frac{d\left(\int_0^T x_t^\tau \Lambda_t^\tau d\tau\right)}{dr_t} = \int_0^T x_t^\tau \frac{d\Lambda_t^\tau}{dr_t} d\tau = -\frac{1}{\sigma}\int_0^T x_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau, \tag{A-29}$$

for the total exposure of interest rate risk borne by intermediaries, their maximization problem (A-27) reduces to

$$\max_{x_t} \lambda_t x_t - \frac{\alpha\sigma}{2} x_t^2. \tag{A-30}$$

The first order condition of (A-30) together with the market clearing condition (4) determine the equilibrium market price of risk and provides (5). □

Proof of Theorem 1. We conjecture that equilibrium yields in the model defined by (12) and (13) are in the form (14), i.e., bond prices are

$$\Lambda_t^\tau = e^{-[\tau\mathcal{A}(\tau)+\tau\mathcal{B}(\tau)r_t+\tau\mathcal{C}(\tau)D_t]}. \tag{A-31}$$

Applying Itô's Lemma to (A-31), substituting in (12) and (13), and imposing the condition that the bond price drift under $\mathbb{Q}$ must be $r_t\Lambda_t^\tau dt$, we obtain an equation affine in the factors r_t and D_t. Collecting the r_t, D_t, and constant terms, respectively, we get a set of ODEs:

$$1 = \tau\mathcal{B}'(\tau) + \mathcal{B}(\tau) + \kappa\tau\mathcal{B}(\tau) + \delta_r\tau\mathcal{C}(\tau), \tag{A-32}$$

$$0 = \tau\mathcal{C}'(\tau) + \mathcal{C}(\tau) + \delta_D^{\mathbb{Q}}\tau\mathcal{C}(\tau) - \alpha\sigma\sigma_y^{\bar{\tau}}\tau\mathcal{B}(\tau), \text{ and} \tag{A-33}$$

$$0 = \tau\mathcal{A}'(\tau) + \mathcal{A}(\tau) - \kappa\theta\tau\mathcal{B}(\tau) - \delta_0\tau\mathcal{C}(\tau) + \frac{1}{2}\sigma^2\tau^2\mathcal{B}^2(\tau) + \frac{1}{2}\eta_y^2\left(\sigma_y^{\bar{\tau}}\right)^2\tau^2\mathcal{C}(\tau)^2, \tag{A-34}$$

with terminal conditions $\mathcal{A}(0) = \mathcal{C}(0) = 0$ and $\mathcal{B}(0) = 1$. Combining (A-32) and (A-33), we write the following second order ODE for $\mathcal{C}$:

$$0 = \tau\mathcal{C}''(\tau) + 2\mathcal{C}'(\tau) + \left(\kappa + \delta_D^{\mathbb{Q}}\right)\left(\tau\mathcal{C}'(\tau) + \mathcal{C}(\tau)\right) + \left(\kappa\delta_D^{\mathbb{Q}} + \alpha\sigma\sigma_y^{\bar{\tau}}\delta_r\right)\tau\mathcal{C}(\tau) - \alpha\sigma\sigma_y^{\bar{\tau}}. \tag{A-35}$$

Solving (A-35) for $\mathcal{C}$, from there deriving $\mathcal{B}$ and $\mathcal{A}$, and applying the terminal conditions, yields the following solution:

$$\mathcal{C}(\tau) = -\frac{\alpha\sigma\sigma_y^{\bar{\tau}}}{(\kappa+\varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)}\left[\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{1-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}\right], \tag{A-36}$$

$$\mathcal{B}(\tau) = \frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{\varepsilon}{(\kappa+\varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)}\left[\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{1-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}\right], \tag{A-37}$$

and

$$\mathcal{A}(\tau) = \frac{1}{(\kappa+\varepsilon)}\left[\kappa\theta\frac{(\kappa+\varepsilon)-\delta_D^{\mathbb{Q}}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)} - \delta_0\frac{\alpha\sigma\sigma_y^{\bar{\tau}}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}\right]\omega\left((\kappa+\varepsilon)\tau\right) \tag{A-38}$$

$$+\frac{1}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}\left[\kappa\theta\frac{\varepsilon}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)} + \delta_0\frac{\alpha\sigma\sigma_y^{\bar{\tau}}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}\right]\omega\left(\left(\delta_D^{Q}-\varepsilon\right)\tau\right)$$

$$+\frac{1}{2}\sigma^2\frac{\left[(\kappa+\varepsilon)-\delta_D^{\mathbb{Q}}\right]^2+\left[\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2\right]^2}{(\kappa+\varepsilon)^2\left[(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\right]^2}\left[\frac{1}{2}\omega\left(2(\kappa+\varepsilon)\tau\right)-\omega\left((\kappa+\varepsilon)\tau\right)\right]$$

$$+\frac{1}{2}\sigma^2\frac{\varepsilon^2+\left[\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2\right]^2}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)^2\left[(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\right]^2}\left[\frac{1}{2}\omega\left(2\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau\right)-\omega\left(\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau\right)\right]$$

$$+\sigma^2\frac{\left[(\kappa+\varepsilon)-\delta_D^{\mathbb{Q}}\right]\varepsilon-\left[\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2\right]^2}{(\kappa+\varepsilon)\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\left[(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\right]^2}\left[\omega\left(\left(\kappa+\delta_D^{\mathbb{Q}}\right)\tau\right)-\omega\left((\kappa+\varepsilon)\tau\right)-\omega\left(\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau\right)\right],$$

where the function $\omega(.)$ is defined as $\omega(x) = 1 - \frac{1-e^{-x}}{x}$ for all $x \neq 0$ and $\omega(0) = 0$, and where

$$\varepsilon = \frac{\delta_D^{\mathbb{Q}} - \kappa - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\alpha\sigma\sigma_y^{\bar{\tau}}\delta_r}}{2} \tag{A-39}$$

as long as it exists, i.e., the determinant is positive.

Next we pin down the endogenous parameters of the model. First, from (1), (11), and (14) the volatility of the reference yield has to solve

$$\sigma_y^{\bar{\tau}} = \mathcal{B}(\bar{\tau})\,\sigma + \mathcal{C}(\bar{\tau})\,\eta_y\sigma_y^{\bar{\tau}}. \tag{A-40}$$

Moreover, again from (14), we have

$$dy_t^{\bar{\tau}} = \mathcal{B}(\bar{\tau})\,dr_t + \mathcal{C}(\bar{\tau})\,dD_t. \tag{A-41}$$

Plugging (A-41) into (11) and using (1), we get

$$dD_t = \kappa_D(\theta_D - D_t)\,dt + \mathcal{B}(\bar{\tau})\,\eta_y\left[\kappa(\theta - r_t)\,dt + \sigma dB_t\right] + \mathcal{C}(\bar{\tau})\,\eta_y dD_t,$$

and collecting all dD_t terms on the LHS yields

$$\left[1-\eta_y\mathcal{C}(\bar{\tau})\right]dD_t = \left[\kappa_D(\theta_D - D_t) + \kappa\eta_y\mathcal{B}(\bar{\tau})(\theta - r_t)\right]dt + \mathcal{B}(\bar{\tau})\,\sigma dB_t. \tag{A-42}$$

Matching the r_t, D_t, and constant terms in the drift of dD_t from (A-42) with those in (11), we obtain (15).

We make use of the following result:

Lemma 5. *As long as ε exists, we have (i) $\kappa+\varepsilon < \delta_D^{\mathbb{Q}}-\varepsilon$ always; and (ii) ε has the same sign as $\delta_D^{\mathbb{Q}}-\kappa$. Finally, (iii) $\kappa+\varepsilon$ and $\delta_D^{\mathbb{Q}}-\varepsilon$ are always "between" κ and $\delta_D^{\mathbb{Q}}$. That is, if $\kappa<\delta_D^{\mathbb{Q}}$, we have*

$$\kappa < \kappa+\varepsilon < \frac{\kappa+\delta_D^{\mathbb{Q}}}{2} < \delta_D^{\mathbb{Q}}-\varepsilon < \delta_D^{\mathbb{Q}}; \tag{A-43}$$

if $\kappa>\delta_D^{\mathbb{Q}}$, we have

$$\delta_D^{\mathbb{Q}} < \kappa+\varepsilon < \frac{\kappa+\delta_D^{\mathbb{Q}}}{2} < \delta_D^{\mathbb{Q}}-\varepsilon < \kappa. \tag{A-44}$$

Proof. First, notice that (A-39) and (15) together imply ε can be rewritten as

$$\varepsilon = \frac{\left(\delta_D^{\mathbb{Q}}-\kappa\right) - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2}. \tag{A-45}$$

From (A-45), we have

$$\kappa+\varepsilon = \frac{\delta_D^{\mathbb{Q}}+\kappa - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2}$$

and

$$\delta_D^{\mathbb{Q}}-\varepsilon = \frac{\delta_D^{\mathbb{Q}}+\kappa + \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2},$$

and since the square-root is non-negative, we always have

$$\kappa+\varepsilon < \frac{\kappa+\delta_D^{\mathbb{Q}}}{2} < \delta_D^{\mathbb{Q}}-\varepsilon.$$

Second, revisiting (A-45), if $\kappa>\delta_D^{\mathbb{Q}}$, both components of the RHS are negative and thus $\varepsilon<0$. On the other hand, since $4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2>0$, we have

$$\left|\delta_D^{\mathbb{Q}}-\kappa\right| > \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}.$$

Therefore, if $\delta_D^{\mathbb{Q}}>\kappa$, the first component of ε is positive and greater than the second, and thus $\varepsilon>0$.

Third, we can write

$$\kappa+\varepsilon-\delta_D^{\mathbb{Q}} = \frac{-\left(\delta_D^{\mathbb{Q}}-\kappa\right) - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2},$$

and with similar reasoning as above, $\kappa>\delta_D^{\mathbb{Q}}$ implies $\kappa-\delta_D^{\mathbb{Q}}+\varepsilon>0$, i.e. $\delta_D^{\mathbb{Q}}<\kappa+\varepsilon$ and $\delta_D^{\mathbb{Q}}-\varepsilon<\kappa$. Combining the three results gives inequalities (A-43) and (A-44). □

To complete the proof of the Theorem, we need to provide sufficient conditions such that the set of equations given by (15) has a solution. First, we show that all meaningful $\sigma_y^{\bar{\tau}}$

solutions of (A-40) are non-negative. Notice that with the help of (A-3), (A-37) and (A-36) can be written as

$$\mathcal{C}(\tau) = -\alpha\sigma\sigma_y^{\bar{\tau}}\ G\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right) \text{ and} \tag{A-46}$$

$$\mathcal{B}(\tau) = -\left[\frac{e^{-(\kappa+\varepsilon)\tau} - e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)} + \delta_D^{\mathbb{Q}}\ G\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right)\right]. \tag{A-47}$$

Lemma 3 and (A-46) together imply that $\mathcal{C}(\tau)$ and $\sigma_y^{\bar{\tau}}$ have the same sign; therefore, the second term of the RHS of (A-40) is always non-negative. Regarding (A-47), as the function $x \mapsto e^{-x}$ is decreasing, the first term inside the bracket is negative. On the other hand, according to Lemma 3, the second term has the opposite sign as $\delta_D^{\mathbb{Q}}$. But $\delta_D^{\mathbb{Q}}$ must be positive, otherwise the duration process under $\mathbb{Q}$, (13), would explode. Hence, both terms inside the bracket are negative, i.e., $\mathcal{B}(\tau) \geq 0$. Going back to (A-40), we have shown that both components of the RHS are non-negative, and thus in all meaningful solutions $\sigma_y^{\bar{\tau}} \geq 0$. Notice that this also implies $\mathcal{C}(\tau) \geq 0$ for all $\tau \geq 0$, and from (15) it also means $0 < 1 - \eta_y\mathcal{C}(\bar{\tau}) \leq 1$.

Second, a sufficient condition for the existence of a solution to (A-40) is that its LHS is smaller than the RHS for $\sigma_y^{\bar{\tau}} = 0$ but greater than the RHS when $\sigma_y^{\bar{\tau}}$ is large enough. It is easy to see that $\sigma_y^{\bar{\tau}} = 0$ leads to $\mathcal{C}(\tau) = 0$ for all $\tau \geq 0$, and yields

$$\mathcal{B}(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa\tau}.$$

Therefore, the RHS of (A-40) is zero while the LHS equals

$$\mathcal{B}(\bar{\tau})\,\sigma + \mathcal{C}(\bar{\tau})\,\eta_y\sigma_y^{\bar{\tau}} = \frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\sigma > 0.$$

For inequality in the other direction, notice that Lemmas 3 and 5 together imply

$$G\left(\kappa, \delta_D^{\mathbb{Q}}\right) < G\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right) < 0, \tag{A-48}$$

regardless of the order of κ and $\delta_D^{\mathbb{Q}}$. Further, Lemma 3 states that G is increasing in both arguments, therefore

$$G\left(\kappa, \kappa_D^{\mathbb{Q}}\right) < G\left(\kappa, \delta_D^{\mathbb{Q}}\right) < 0, \tag{A-49}$$

where $\kappa_D^{\mathbb{Q}} \equiv \kappa_D - \alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2 \leq \delta_D^{\mathbb{Q}}$ always, because $\kappa_D \leq \delta_D$ holds due to (15) and $\mathcal{C}(\tau) \geq 0$. Combining (A-46), (A-48), and (A-49), we obtain

$$0 < \mathcal{C}(\tau) < -\alpha\sigma\sigma_y^{\bar{\tau}}G\left(\kappa, \kappa_D^{\mathbb{Q}}\right). \tag{A-50}$$

We also approximate $\mathcal{B}(\tau)$ from above with the help of Lemma 3. First, if we assume $\kappa < \delta_D^{\mathbb{Q}}$, which also implies $\varepsilon > 0$ and $\kappa < \delta_D^{\mathbb{Q}} - \varepsilon$ according to Lemma 5, since G is negative and increasing in both arguments, we have

$$G(\kappa+\varepsilon, \kappa) < G\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right) < 0.$$

Rearranging and using (A-37), we obtain

$$\mathcal{B}(\tau)=\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\,\tau}-\varepsilon\,G\left(\kappa+\varepsilon,\delta_D^{\mathbb{Q}}-\varepsilon\right)<\frac{1-e^{-\kappa\tau}}{\kappa\tau}. \tag{A-51}$$

If, on the other hand, $\delta_D^{\mathbb{Q}}<\kappa$, we rewrite (A-37) as

$$\mathcal{B}(\tau)=\frac{1-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}+\left(\kappa+\varepsilon-\delta_D^{\mathbb{Q}}\right)\,G\left(\kappa+\varepsilon,\delta_D^{\mathbb{Q}}-\varepsilon\right).$$

Notice that Lemma 5 in this case yields $\kappa+\varepsilon-\delta_D^{\mathbb{Q}}>0$, and since G is negative, we get

$$\mathcal{B}(\tau)<\frac{1-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}<\frac{1-e^{-\delta_D^{\mathbb{Q}}\tau}}{\delta_D^{\mathbb{Q}}\tau}\leq\frac{1-e^{-\kappa_D^{\mathbb{Q}}\tau}}{\kappa_D^{\mathbb{Q}}\tau}, \tag{A-52}$$

where in the last two steps we use $\delta_D^{\mathbb{Q}}-\varepsilon>\delta_D^{\mathbb{Q}}\geq\kappa_D^{\mathbb{Q}}$. Combining (A-51) and (A-52), we obtain that under any circumstances we have

$$\mathcal{B}(\tau)<\max\left\{\frac{1-e^{-\kappa\tau}}{\kappa\tau},\frac{1-e^{-\kappa_D^{\mathbb{Q}}\tau}}{\kappa_D^{\mathbb{Q}}\tau}\right\}, \tag{A-53}$$

and, together with (A-50),

$$\mathcal{B}(\tau)\,\sigma+\mathcal{C}(\tau)\,\eta_y\sigma_y^{\bar{\tau}}<\left[\max\left\{\frac{1-e^{-\kappa\tau}}{\kappa\tau},\frac{1-e^{-\kappa_D^{\mathbb{Q}}\tau}}{\kappa_D^{\mathbb{Q}}\tau}\right\}-\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2G\left(\kappa,\kappa_D^{\mathbb{Q}}\right)\right]\sigma. \tag{A-54}$$

We want to give a sufficient condition for the LHS of (A-40) to be larger than the RHS when $\sigma_y^{\bar{\tau}}$ is large enough to make $\kappa_D^{\mathbb{Q}}=0$, that is, $\left(\sigma_y^{\bar{\tau}}\right)^2=\frac{\kappa_D}{\alpha\eta_y}$. For this it is sufficient if we make $\sigma_y^{\bar{\tau}}$ larger than the RHS of (A-54), which, after some algebra, is equivalent to

$$\sigma_y^{\bar{\tau}}>\left[1-\frac{\kappa_D}{\kappa}\left(\frac{1-e^{-\kappa\tau}}{\kappa\tau}-1\right)\right]\sigma,$$

because $\kappa_D^{\mathbb{Q}}=0$ makes the RHS of (A-52) equal to 1. Taking squares of both sides and using $\left(\sigma_y^{\bar{\tau}}\right)^2=\frac{\kappa_D}{\alpha\eta_y}$ again, after some algebra we obtain

$$\alpha<\frac{\kappa_D}{\eta_y\left(\frac{\kappa+\kappa_D}{\kappa}-\frac{\kappa_D}{\kappa}\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2\sigma^2}. \tag{A-55}$$

Defining $\bar{\alpha}$ as the RHS of (A-55), which is certainly positive, (A-40) has at least one solution whenever $0\leq\alpha<\bar{\alpha}$. □

Corollary 1. *Bond return volatility, σ_t^{τ}, is positive and increasing in maturity: $d\sigma_t^{\tau}/d\tau>0$.*

Proof. Applying Itô's lemma to (A-31) and using (1), (11), and (A-26), we obtain that bond return volatility is given by satisfies

$$\sigma_t^{\tau}=\sigma\tau\mathcal{B}(\tau)+\eta_y\sigma_y^{\bar{\tau}}\tau\mathcal{C}(\tau). \tag{A-56}$$

On the other hand, (15) and (A-32) together yield

$$1 = \tau\mathcal{B}'(\tau) + \mathcal{B}(\tau) + \frac{\kappa}{\sigma}\left[\sigma\tau\mathcal{B}(\tau) + \eta_y\sigma_y^{\bar{\tau}}\tau\mathcal{C}(\tau)\right].$$

Combining it with (A-56) we obtain

$$\sigma_t^\tau = \frac{\sigma}{\kappa}\left(1 - \left[\tau\mathcal{B}'(\tau) + \mathcal{B}(\tau)\right]\right),$$

and differentiating w.r.t. τ we get

$$\frac{d\sigma_t^\tau}{d\tau} = -\frac{\sigma}{\kappa}\frac{d}{d\tau}\left[\tau\mathcal{B}'(\tau) + \mathcal{B}(\tau)\right] = -\frac{\sigma}{\kappa}\left[\tau\mathcal{B}''(\tau) + 2\mathcal{B}'(\tau)\right].$$

Bond return volatility is increasing in maturity if we show that the terms in brackets are negative. Differentiating (A-36) twice, after some algebra we obtain

$$\frac{d}{d\tau}\left[\tau\mathcal{B}'(\tau) + \mathcal{B}(\tau)\right] = -\frac{\left[\kappa - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\right](\kappa + \varepsilon)}{(\kappa + \varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)}\left[e^{-(\kappa+\varepsilon)\tau} - e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau}\right] - \kappa e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau}.$$

Since $\kappa < \kappa + \varepsilon < \delta_D^{\mathbb{Q}} - \varepsilon$, we have $e^{-(\kappa+\varepsilon)\tau} - e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau} > 0$, and the coefficient of this term is also positive. Therefore, the total RHS is negative, and we conclude that $\frac{d\sigma_t^\tau}{d\tau} > 0$. Moreover, it is easy to confirm that $\sigma_t^0 = 0$. Thus, bond return volatility is positive and increasing across maturities. □

Proof of Propositions 1 and 2. The excess return over horizon $(t, t+h)$ on a maturity-τ bond is

$$rx_{t,t+h}^\tau = \log\Lambda_{t+h}^{\tau-h} - \log\Lambda_t^\tau + \log\Lambda_t^h. \tag{A-57}$$

To express the RHS, we start by applying Itô's lemma to $\log\Lambda_t^\tau$ and using (A-26) to obtain

$$d\log\Lambda_t^\tau = \left[\mu_t^\tau - \frac{1}{2}\left(\sigma_t^\tau\right)^2\right]dt - \sigma_t^\tau dB_t. \tag{A-58}$$

Next we consider the change in the log price of a bond, which at time t has time to maturity τ, over a horizon of h. The price of this bond at time $s \in [t, t+h]$ is given by $\Lambda_s^{t+\tau-s}$, hence (A-58) implies

$$\log\Lambda_{t+h}^{\tau-h} - \log\Lambda_t^\tau = \int_t^{t+h} d\log\Lambda_s^{t+\tau-s} = \int_t^{t+h}\left[\mu_s^{\tau-(s-t)} - \frac{1}{2}\left(\sigma_s^{\tau-(s-t)}\right)^2\right]ds - \int_t^{t+h}\sigma_s^{\tau-(s-t)}dB_s. \tag{A-59}$$

Similarly, setting $\tau = h$ in (A-59), we obtain

$$-\log\Lambda_t^h = \log\Lambda_{t+h}^0 - \log\Lambda_t^h = \int_t^{t+h}\left[\mu_s^{h-(s-t)} - \frac{1}{2}\left(\sigma_s^{h-(s-t)}\right)^2\right]ds - \int_t^{t+h}\sigma_s^{h-(s-t)}dB_s.$$

Substituting the last two expressions into (A-57), the excess return on the bond over a horizon h becomes

$$rx^{\tau}_{t,t+h} = \int_{t}^{t+h} \left(\mu_s^{\tau-(s-t)} - \mu_s^{h-(s-t)}\right) ds + \frac{1}{2}\int_{t}^{t+h} \left(\sigma_s^{h-(s-t)}\right)^2 ds - \frac{1}{2}\int_{t}^{t+h} \left(\sigma_s^{\tau-(s-t)}\right)^2 ds$$
$$+ \int_{t}^{t+h} \sigma_s^{h-(s-t)} dB_s - \int_{t}^{t+h} \sigma_s^{\tau-(s-t)} dB_s. \quad \text{(A-60)}$$

We want to examine the regression coefficients when we regress $rx^{\tau}_{t,t+h}$ on state variables r_t and D_t. As the volatilities $\sigma_s^{t+\tau-s}$ and σ_s^{t+h-s} expressions do not depend on r_t or D_t, and the Brownian increments dB_s, $s \in [t, t+h]$, are independent of state variables at time t, we can ignore the last four terms in (A-60). On the other hand, using (10) and (A-28) we have

$$\mu_s^{\tau-(s-t)} = r_s + \alpha\sigma_y^{\bar{\tau}}\sigma_s^{\tau-(s-t)} D_s \text{ and } \mu_s^{h-(s-t)} = r_s + \alpha\sigma_y^{\bar{\tau}}\sigma_s^{h-(s-t)} D_s,$$

which implies

$$\int_{t}^{t+h} \left(\mu_s^{\tau-(s-t)} - \mu_s^{h-(s-t)}\right) ds = \alpha\sigma_y^{\bar{\tau}} \int_{t}^{t+h} \left(\sigma_s^{\tau-(s-t)} - \sigma_s^{h-(s-t)}\right) D_s ds. \quad \text{(A-61)}$$

When running a multivariate regression of $rx_{t,t+h,\tau}$ on D_t and r_t in the form

$$rx^{\tau}_{t,t+h} = \beta^{\tau,h}_{0,D,r} + \left(\beta_1^{\tau,h}, \beta_2^{\tau,h}\right)(D_t, r_t)^{\top} + \epsilon_{t+h},$$

the vector of theoretical coefficients becomes $\left(\beta_1^{\tau,h}, \beta_2^{\tau,h}\right)^{\top} = V^{-1}Cov\left[rx^{\tau}_{t,t+h}, (D_t, r_t)^{\top}\right]$, where V is given in (A-17). Combining (A-17)-(A-19) with (A-60), (A-61), and the discussion in between, after some algebra we obtain

$$\beta_1^{\tau,h} = \alpha\sigma_y^{\bar{\tau}} \int_{t}^{t+h} \left(\sigma_s^{\tau-(s-t)} - \sigma_s^{h-(s-t)}\right) e^{-\delta_D(s-t)} ds \text{ and} \quad \text{(A-62)}$$

$$\beta_2^{\tau,h} = -\frac{\delta_r}{\delta_D - \kappa}\alpha\sigma_y^{\bar{\tau}} \int_{t}^{t+h} \left(\sigma_s^{\tau-(s-t)} - \sigma_s^{h-(s-t)}\right)\left(e^{-\kappa(s-t)} - e^{-\delta_D(s-t)}\right) ds. \quad \text{(A-63)}$$

We use the Leibniz integral rule to express how $\beta_1^{\tau,h}$ and $\beta_2^{\tau,h}$ change as a function of τ:

$$\frac{d\beta_1^{\tau,h}}{d\tau} = \alpha\sigma_y^{\bar{\tau}} \int_{t}^{t+h} \frac{d\sigma_s^{\tau-(s-t)}}{d\tau} e^{-\delta_D(s-t)} ds \text{ and} \quad \text{(A-64)}$$

$$\frac{d\beta_2^{\tau,h}}{d\tau} = \delta_r\alpha\sigma_y^{\bar{\tau}} \int_{t}^{t+h} \frac{d\sigma_s^{\tau-(s-t)}}{d\tau} \frac{e^{-\kappa(s-t)} - e^{-\delta_D(s-t)}}{\kappa - \delta_D} ds. \quad \text{(A-65)}$$

Corollary 1 implies that $d\sigma_s^{\tau-(s-t)}/d\tau > 0$, and $0 < e^{-\delta_D(s-t)}$ trivially. Therefore, each term of the integral in (A-64) is positive, which implies $d\beta_1^{\tau,h}/d\tau > 0$. Similarly, $e^{-\kappa(s-t)} > e^{-\delta_D(s-t)}$

if and only if $\kappa < \delta_D$. Thus, the integral in (A-65) is negative, and $d\beta_2^{\tau,h}/d\tau < 0$. On the other hand, when $\tau = h$, we have $\beta_1^{h,h} = \beta_2^{h,h} = 0$. Thus, $\beta_1^{\tau,h}$ is positive and increasing across maturities, and $\beta_2^{\tau,h}$ is negative and decreasing across maturities.

Next we turn to a univariate regression of $rx_{t,t+h}^{\tau}$ on D_t in the form $rx_{t,t+h}^{\tau} = \beta_{0,D}^{\tau,h} + \beta^{\tau,h} D_t + \epsilon_{t+h}$. Similarly to (A-62) and (A-63), from (A-17) and (A-61), after some algebra, we obtain

$$\beta^{\tau,h} = \frac{Cov\left[rx_{t,t+h}^{\tau}, D_t\right]}{Var\left[D_t\right]} = \alpha\sigma_y^{\bar{\tau}} \int_t^{t+h} \left(\sigma_s^{\tau-(s-t)} - \sigma_s^{h-(s-t)}\right) \frac{\delta_D e^{-\delta_D(s-t)} - \kappa e^{-\kappa(s-t)}}{\delta_D - \kappa} ds. \quad \text{(A-66)}$$

Applying the Leibniz rule, we get

$$\frac{d\beta^{\tau,h}}{d\tau} = \alpha\sigma_y^{\bar{\tau}} \int_t^{t+h} \frac{d\sigma_s^{\tau-(s-t)}}{d\tau} \frac{\delta_D e^{-\delta_D(s-t)} - \kappa e^{-\kappa(s-t)}}{\delta_D - \kappa} ds.$$

Corollary 1 implies $d\sigma_s^{\tau-(s-t)}/d\tau > 0$, and it is easy to confirm that $\frac{\delta_D e^{-\delta_D x} - \kappa e^{-\kappa x}}{\delta_D - \kappa} \geq 0$ iff $0 \leq x \leq \bar{h} \equiv \frac{\log \delta_D - \log \kappa}{\delta_D - \kappa}$. Therefore, as long as h is sufficiently small such that $h \leq \bar{h}$, each term in the integral of the RHS is positive, and $d\beta^{\tau,h}/d\tau > 0$. On the other hand, setting $\tau = h$ we get $\beta^{h,h} = 0$. Thus, $\beta^{\tau,h}$ is positive and increasing across maturities.

Finally, from (14), the effect of duration on yields is given by $\mathcal{C}(\tau)$. From the Proof of Theorem 1, $\mathcal{C}(\tau) \geq 0$. Moreover, from (A-36) it is easy to show that

$$\lim_{\tau \to 0} \mathcal{C}(\tau) = \lim_{\tau \to \infty} \mathcal{C}(\tau) = 0,$$

with $\mathcal{C}(\tau)$ increasing for small but decreasing for large τ values, which implies that the effect is either increasing across maturities if T is small, or first increasing then decreasing if T is sufficiently large. This completes the proof. □

Proof of Proposition 3. This time we look at how regression betas $\beta^{\tau,h}$ and $\beta_1^{\tau,h}$ change when we alter the horizon h. We start by $\beta_1^{\tau,h}$. From (A-62), after some algebra, the Leibniz rule yields

$$\frac{d\beta_1^{\tau,h}}{dh} = \alpha\sigma_y^{\bar{\tau}} \left[\sigma_s^{\tau-h} e^{-\delta_D h} - \int_t^{t+h} \frac{d\sigma_s^{h-(s-t)}}{dh} e^{-\delta_D(s-t)} ds\right]. \quad \text{(A-67)}$$

To save space, instead of studying the exact value of the RHS of (A-67), we find tractable upper and lower thresholds for it, and analyze how those behave. All our statements can be shown analyically, or be confirmed numerically.

Since $d\sigma_s^{h-(s-t)}/dh > 0$ and $0 < e^{-\delta_D h} < e^{-\delta_D(s-t)}$ for all $s \in (t, t+h)$, it can be confirmed that (A-67) implies

$$\frac{d\beta_1^{\tau,h}}{dh} < \alpha\sigma_y^{\bar{\tau}} \left[\sigma_s^{\tau-h} e^{-\delta_D h} - e^{-\delta_D h} \int_t^{t+h} \frac{d\sigma_s^{h-(s-t)}}{dh} ds\right] = \alpha\sigma_y^{\bar{\tau}} e^{-\delta_D h} \left(\sigma_s^{\tau-h} - \sigma_s^h\right). \quad \text{(A-68)}$$

On the other hand, we also have $e^{-\delta_D(s-t)} \leq 1$ for all for all $s \in (t, t+h)$, thus

$$\frac{d\beta_1^{\tau,h}}{dh} > \alpha\sigma_y^{\bar{\tau}} \left[\sigma_s^{\tau-h} e^{-\delta_D h} - \int_t^{t+h} \frac{d\sigma_s^{h-(s-t)}}{dh} ds \right] = \alpha\sigma_y^{\bar{\tau}} \left(\sigma_s^{\tau-h} e^{-\delta_D h} - \sigma_s^h \right). \tag{A-69}$$

But the RHSs of (A-68) and (A-69) both converge to $\sigma_t^\tau > 0$ when $h \to 0$ and to $-\sigma_t^\tau$ when $h \to \tau$, and change continuously and monotonically in between because $\sigma_s^{\tau-h}$ and $e^{-\delta_D h}$ both decrease in h while σ_s^h increases. Thus, $d\beta_1^{\tau,h}/dh > 0$ for small h, then becomes negative for large h values. In particular, it is easy to confirm that $\exists!\ \hat{h} \in (0, \tau/2)$ that satisfies $e^{-\delta_D \hat{h}} \sigma_s^{\tau-\hat{h}} = \sigma_s^{\hat{h}}$, which, together with (A-68) and (A-69), implies $\frac{d\beta_1^{\tau,h}}{dh} > 0$ for all $h \in \left[0, \hat{h}\right)$ and $\frac{d\beta_1^{\tau,h}}{dh} < 0$ for all $h \in (\tau/2, \tau]$. Since (A-62) also implies $\beta_1^{\tau,0} = \beta_1^{\tau,\tau} = 0$, we conclude that $\beta_1^{\tau,h}$ is positive and hump-shaped in h. With similar argument we could also show that $\beta_2^{\tau,h}$ goes from zero to zero and is negative and U-shaped in between.

Finally we study $\beta^{\tau,h}$. From (A-66), after some algebra, the Leibniz rule yields

$$\begin{aligned} \frac{d\beta^{\tau,h}}{dh} &= \frac{\alpha\sigma_y^{\bar{\tau}}}{\delta_D - \kappa} \left[\sigma_s^{\tau-h} \left(\delta_D e^{-\delta_D h} - \kappa e^{-\kappa h} \right) - \int_t^{t+h} \frac{d\sigma_s^{h-(s-t)}}{dh} \left(\delta_D e^{-\delta_D (s-t)} - \kappa e^{-\kappa (s-t)} \right) ds \right] \\ &= \frac{\delta_D}{\delta_D - \kappa} \frac{d\beta_1^{\tau,h}}{dh} - \frac{\kappa}{\delta_D - \kappa} \alpha\sigma_y^{\bar{\tau}} \left[\sigma_s^{\tau-h} e^{-\kappa h} - \int_t^{t+h} \frac{d\sigma_s^{h-(s-t)}}{dh} e^{-\kappa(s-t)} ds \right], \end{aligned}$$

where the term in the last bracket is similar to that in (A-67) but κ replacing δ_D. As (A-68) and (A-69) imply

$$\alpha\sigma_y^{\bar{\tau}} \left(e^{-\delta_D h} \sigma_t^{\tau-h} - \sigma_t^h \right) < \frac{d\beta_1^{\tau,h}}{dh} < \alpha\sigma_y^{\bar{\tau}} e^{-\delta_D h} \left(\sigma_t^{\tau-h} - \sigma_t^h \right),$$

analogously, we also must have

$$e^{-\kappa h} \sigma_t^{\tau-h} - \sigma_t^h < \sigma_s^{\tau-h} e^{-\kappa h} - \int_t^{t+h} \frac{d\sigma_s^{h-(s-t)}}{dh} e^{-\kappa(s-t)} ds < e^{-\kappa h} \left(\sigma_t^{\tau-h} - \sigma_t^h \right).$$

The last two results together imply

$$\frac{\delta_D e^{-\delta_D h} - \kappa e^{-\kappa h}}{\delta_D - \kappa} \sigma_t^{\tau-h} - \frac{\delta_D - \kappa e^{-\kappa h}}{\delta_D - \kappa} \sigma_t^h < \frac{1}{\alpha\sigma_y^{\bar{\tau}}} \frac{d\beta^{\tau,h}}{dh} < \frac{\delta_D e^{-\delta_D h} - \kappa e^{-\kappa h}}{\delta_D - \kappa} \sigma_t^{\tau-h} - \frac{\delta_D e^{-\delta_D h} - \kappa}{\delta_D - \kappa} \sigma_t^h.$$

Taking the limit $h \to 0$, both the LHS and RHS of the above inequality converge to σ_t^τ, which means we must have $\lim_{h\to 0} d\beta^{\tau,h}/dh = \alpha\sigma_y^{\bar{\tau}}\sigma_t^\tau > 0$. By a continuity argument, then there exists $\bar{h} \in (0, \tau/2)$ (which actually satisfies $\bar{h} < \hat{h}$) such that $\frac{d\beta^{\tau,h}}{dh} > 0$ for all $h \in \left(0, \bar{h}\right)$. Thus, $\beta^{\tau,h}$ increases in h for small h, and after that it decreases. Since (A-66) also implies $\beta^{\tau,0} = \beta^{\tau,\tau} = 0$, we conclude that $\beta^{\tau,h}$ is hump-shaped in h.

We conclude our proof with noting that the term "hump-shaped" is not exact in the sense that $\beta^{\tau,h}$ is not necessarily positive for all h. In fact, while $\beta^{\tau,h}$ starts out as positive and increasing in h, and goes to zero when $h \to \tau$, numerical examples show that it can go negative when h is sufficiently close to τ. The reason for this is that, as we argue above, $\beta^{\tau,h}$ is the difference of two hump-shaped functions that take the value of zero at both extremes. Thus, depending on the position of these two humps compared to each other, the overall effect can be either hump-shaped or wave-formed. $\square$

Proof of Proposition 4. From (14), (A-36), and (A-37), bond yield volatility is given by

$$\begin{aligned}\sigma_y^\tau &= \mathcal{B}(\tau)\sigma + \mathcal{C}(\tau)\eta_y\sigma_y^{\bar\tau} \\ &= \frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau}\sigma - \frac{\varepsilon+\alpha\eta_y\left(\sigma_y^{\bar\tau}\right)^2}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}\left[\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{1-e^{-\left(\delta_D^Q-\varepsilon\right)\tau}}{\left(\delta_D^Q-\varepsilon\right)\tau}\right]\sigma.\end{aligned} \tag{A-70}$$

Due to the complexity the feedback mechanism introduces into the endogenous parameters, we cannot compute the exact effect of convexity $-\eta_y$ on yield volatilities in closed form. Instead, we derive its effect by considering (A-70) around $\alpha = 0$. From (A-70) we write $\sigma_y^\tau \approx h_0(\tau) + \alpha\eta_y h_1(\tau)$, where

$$h_0(\tau) \equiv \left(\mathcal{B}(\tau)\sigma + \mathcal{C}(\tau)\eta_y\sigma_y^{\bar\tau}\right)|_{\alpha=0} \text{ and } h_1(\tau) \equiv \frac{1}{\eta_y}\frac{d\left(\mathcal{B}(\tau)\sigma + \mathcal{C}(\tau)\eta_y\sigma_y^{\bar\tau}\right)}{d\alpha}|_{\alpha=0}.$$

We start with h_0. It is straightforward from (A-39) and (15) that taking the limit $\alpha \to 0$ yields

$$\lim_{\alpha\to 0}\varepsilon = 0 \text{ and } \lim_{\alpha\to 0}\delta_D^Q = \lim_{\alpha\to 0}\delta_D = \kappa_D, \tag{A-71}$$

hence

$$h_0(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa\tau}\sigma. \tag{A-72}$$

and as a special case, the volatility of the reference-maturity yield is

$$\lim_{\alpha\to 0}\sigma_y^{\bar\tau} = h_0(\bar\tau) = \frac{1-e^{-\kappa\bar\tau}}{\kappa\bar\tau}\sigma. \tag{A-73}$$

Second, differentiating (A-70) with respect to α, we get

$$\begin{aligned}\frac{1}{\sigma}\frac{d\sigma_y^\tau}{d\alpha} &= -\frac{d(\kappa+\varepsilon)}{d\alpha}\frac{1-e^{-(\kappa+\varepsilon)\tau}-(\kappa+\varepsilon)\tau e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)^2\tau} \\ &\quad -\left(\varepsilon+\alpha\eta_y\left(\sigma_y^{\bar\tau}\right)^2\right)\frac{d}{d\alpha}\left(\frac{\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau}-\frac{1-e^{-\left(\delta_D^Q-\varepsilon\right)\tau}}{\left(\delta_D^Q-\varepsilon\right)\tau}}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}\right) \\ &\quad -\left(\frac{d\varepsilon}{d\alpha}+\eta_y\left(\sigma_y^{\bar\tau}\right)^2+2\alpha\eta_y\sigma_y^{\bar\tau}\frac{d\sigma_y^{\bar\tau}}{d\alpha}\right)\frac{\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau}-\frac{1-e^{-\left(\delta_D^Q-\varepsilon\right)\tau}}{\left(\delta_D^Q-\varepsilon\right)\tau}}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}.\end{aligned} \tag{A-74}$$

As

$$\frac{d(\kappa+\varepsilon)}{d\alpha} = \frac{d\varepsilon}{d\alpha} = \frac{\varepsilon}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}\frac{d\delta_D^Q}{d\alpha} - \frac{\kappa\eta_y\left(\sigma_y^{\bar\tau}\right)^2}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)},$$

we get

$$\lim_{\alpha\to 0}\frac{d\left(\kappa+\varepsilon\right)}{d\alpha}=\lim_{\alpha\to 0}\frac{d\varepsilon}{d\alpha}=\frac{\kappa\eta_y}{\kappa_D-\kappa}\lim_{\alpha\to 0}\left(\sigma_y^{\bar{\tau}}\right)^2=\frac{\kappa\eta_y\sigma^2}{\kappa_D-\kappa}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2, \tag{A-75}$$

where in the last step we used (A-73). Hence, after some algebra, (A-74) yields

$$\frac{1}{\sigma}\lim_{\alpha\to 0}\frac{d\sigma_y^{\tau}}{d\alpha}=\frac{\kappa_D\eta_y\sigma^2}{\left(\kappa_D-\kappa\right)^2}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2\left[\frac{1-e^{-\kappa\tau}}{\kappa\tau}-\frac{\kappa_D-\kappa}{\kappa_D}\frac{1-e^{-\kappa\tau}-\kappa\tau e^{-\kappa\tau}}{\kappa\tau}-\frac{1-e^{-\kappa_D\tau}}{\kappa_D\tau}\right]$$
$$=\frac{\kappa_D\eta_y\sigma^2}{\left(\kappa_D-\kappa\right)^2}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\sigma\right)^2 H\left(\kappa,\kappa_D\right),$$

and thus

$$h_1\left(\tau\right)=\frac{\kappa_D\sigma^3}{\left(\kappa_D-\kappa\right)^2}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2 H\left(\kappa,\kappa_D\right),$$

where H is defined in (A-8). But $H\geq 0$ always according to Lemma 4, hence $h_1\left(\tau\right)\geq 0$, which implies that bond yield volatilities are increasing in negative convexity: $\frac{d\sigma_y^{\tau}}{d\eta_y}>0$.

Third, we trivially verify that

$$\lim_{\tau\to 0}\sigma_y^{\tau}=\sigma \text{ and } \lim_{\tau\to\infty}\sigma_y^{\tau}=0,$$

independent of η_y. Therefore, the effect of negative convexity on yield volatilities tends to zero at very short and very long maturities, and hence it must be hump-shaped.

Regarding bond return volatilities, given by $\sigma_y^{\tau}\tau$, we have $\lim_{\tau\to 0}\sigma_y^{\tau}\tau=0$ and

$$\lim_{\tau\to\infty}\sigma_y^{\tau}\tau=\frac{\delta_D}{\left(\kappa+\varepsilon\right)\left(\delta_D^{Q}-\varepsilon\right)}\sigma=\frac{\sigma}{\kappa},$$

again independent of η_y, where the last equality is due to (A-39). Hence, the effect of negative convexity on bond return volatilities tends to zero at very short and very long maturities. However, as η_y increases σ_y^{τ}, it also increases $\sigma_y^{\tau}\tau$, and thus the effect must be hump-shaped. ☐

Corollary 2 (Background calculations for Section 5.3)**.** *The theoretical R^2s of univariate regressions of the duration factor D_t on the short rate factor r_t, the long-term yield $y_t^{\bar{\tau}}$, and the slope $y_t^{\bar{\tau}}-r_t$ are given by*

$$R^2_{D,r}=1-\frac{\delta_D}{\kappa+\delta_D}, \tag{A-76}$$

$$R^2_{D,y}=1-\frac{\delta_D}{\kappa\left(1+\frac{\mathcal{C}(\bar{\tau})}{\mathcal{B}(\bar{\tau})}\frac{\delta_r}{\kappa}\right)^2+\delta_D}, \textit{ and} \tag{A-77}$$

$$R^2_{D,y-r}=1-\frac{\delta_D}{\kappa\left(1+\frac{\mathcal{C}(\bar{\tau})}{\mathcal{B}(\bar{\tau})-1}\frac{\delta_r}{\kappa}\right)^2+\delta_D}. \tag{A-78}$$

Proof. When running a linear regression in the form $Y_t = \alpha + \gamma X_t + \epsilon_t$, the theoretical slope coefficient is simply $\gamma = Cov\left[X_t, Y_t\right]/Var[X_t]$, whereas the R^2 of the regression is

$$R^2 = \frac{\gamma^2 Var[X_t]}{Var[Y_t]} = \frac{Cov^2\left[X_t, Y_t\right]}{Var[X_t]Var[Y_t]}. \tag{A-79}$$

Applying this to $Y_t = D_t$ and $X_t = r_t$ and using (A-17) yields (A-76). Applying (A-79) to $Y_t = D_t$ and $X_t = y_t^{\bar{\tau}}$ and combining it with (14), we obtain

$$R^2_{D,y} = \frac{Cov^2\left[y_t^{\bar{\tau}}, D_t\right]}{Var[y_t^{\bar{\tau}}]Var[D_t]} = \frac{\left(\mathcal{B}\left(\bar{\tau}\right)Cov\left[r_t, D_t\right] + \mathcal{C}\left(\bar{\tau}\right)Var\left[D_t\right]\right)^2}{\left(\mathcal{B}^2\left(\bar{\tau}\right)Var[r_t] + 2\mathcal{B}\left(\bar{\tau}\right)\mathcal{C}\left(\bar{\tau}\right)Cov[r_t, D_t] + \mathcal{C}^2\left(\bar{\tau}\right)Var[D_t]\right)Var[D_t]}.$$

Equation (A-17) and rearranging gives (A-77). Finally, applying (A-79) to $Y_t = D_t$ and $X_t = y_t^{\bar{\tau}} - r_t$ and combining it with (14), we obtain

$$R^2_{D,y-r} = \frac{\left(\left(\mathcal{B}\left(\bar{\tau}\right) - 1\right)Cov\left[r_t, D_t\right] + \mathcal{C}\left(\bar{\tau}\right)Var\left[D_t\right]\right)^2}{\left(\left(\mathcal{B}\left(\bar{\tau}\right) - 1\right)^2 Var[r_t] + 2\left(\mathcal{B}\left(\bar{\tau}\right) - 1\right)\mathcal{C}\left(\bar{\tau}\right)Cov[r_t, D_t] + \mathcal{C}^2\left(\bar{\tau}\right)Var[D_t]\right)Var[D_t]}.$$

Using (A-17) and rearranging gives (A-78). □

Appendix C.3 Extended model with nominal and real bonds

In this Appendix we present an extension of the baseline model of Section 1 that allows for the pricing of real bonds. At each date t, there exist a continuum of zero-coupon nominal and real bonds with time to maturity $\tau \in (0, T]$. We assume that nominal bonds with time to maturity τ are in a total net supply of s_t^{τ}, and real bonds, for simplicity, are in zero net supply.

As before, Λ_t^{τ} denotes the time-t price of a zero-coupon nominal bond paying one dollar at maturity $t + \tau$, with yield $y_t^{\tau} = -\frac{1}{\tau}\log\Lambda_t^{\tau}$. Analogously, $\Lambda_t^{\tau*}$ denotes the time-t price of a zero-coupon real bond paying off at maturity $t + \tau$, with yield $y_t^{\tau*} = -\frac{1}{\tau}\log\Lambda_t^{\tau*}$. We assume that the dynamics of the nominal short rate r_t under the physical probability measure is given by (1). In addition, we assume there is an exogenously given real short rate that under $\mathbb{P}$ follows (18), and the evolution of the inflation index is given by (19), with instantaneous correlations $dB_t dB_t^* = \rho dt$, $dB_t dB_t^{\pi} = \rho_{\pi} dt$, and $dB_t^{\pi} dB_t^* = \rho_{\pi *} dt$. While we fix the diffusion of I_t exogenously, we derive its drift endogenously so that in the equilibrium of the model there are no arbitrage opportunities between real and nominal bonds. We also assume that there exist two money-market accounts, a nominal and a real one, whose dynamics are given by

$$dM_t = r_t M_t dt \text{ and } dM_t^* = r_t^* M_t^* dt. \tag{A-80}$$

Further, we write nominal bond prices in the form (A-26), and we assume (and later confirm) that under $\mathbb{P}$ real bond prices follow

$$\frac{d\Lambda_t^{\tau*}}{\Lambda_t^{\tau*}} = \mu_t^{\tau*}dt - \sigma_t^{\tau*}dB_t - \sigma_{t*}^{\tau*}dB_t^*. \tag{A-81}$$

Bonds are held by financial institutions who are competitive and have mean-variance preferences over the instantaneous change in the value of their bond portfolio given by (3). If x_t^{τ}

denotes the quantity they hold in maturity-τ nominal bonds and $x_t^{\tau*}$ denotes the quantity of maturity-τ real bonds at time t, investors' budget constraint becomes

$$dW_t = r_t\left(W_t - \int_0^T x_t^\tau \Lambda_t^\tau d\tau - \int_0^T x_t^{\tau*} I_t \Lambda_t^{\tau*} d\tau\right) dt + \int_0^T x_t^\tau \Lambda_t^\tau \frac{d\Lambda_t^\tau}{\Lambda_t^\tau} d\tau + \int_0^T x_t^{\tau*} I_t \Lambda_t^{\tau*} \frac{d\left(I_t \Lambda_t^{\tau*}\right)}{I_t \Lambda_t^{\tau*}} d\tau. \tag{A-82}$$

Equation (A-82), for simplicity, assumes that the amount institutions do not spend on bonds are invested in the nominal money market account—this is without loss of generality as in equilibrium they would be indifferent between investing in the real and nominal riskless assets.

We solve for an equilibrium of the model in three steps. First, we determine the drifts of our random processes under the risk-neutral measure in general. Second, we solve financial institutions' optimization problems to determine the equilibrium market prices of risks. Finally, we conjecture and verify equilibrium bond prices.

Because markets are complete, by no-arbitrage, there exist unique market prices of the three types of risk, and it must be the case that Λ_t^τ/M_t, $I_t M_t^*/M_t$, and $I_t \Lambda_t^{\tau*}/M_t$ are all martingales under $\mathbb{Q}$ for all τ. For this, analogously to (A-26), (A-81) and (19) we write

$$\frac{d\Lambda_t^\tau}{\Lambda_t^\tau} = \mu_t^{\tau\mathbb{Q}} dt - \sigma_t^\tau dB_t^{\mathbb{Q}}, \tag{A-83}$$

$$\frac{d\Lambda_t^{\tau*}}{\Lambda_t^{\tau*}} = \mu_t^{\tau*\mathbb{Q}} dt - \sigma_t^{\tau*} dB_t^{\mathbb{Q}} - \sigma_{t*}^{\tau*} dB_t^{*\mathbb{Q}}, \text{ and} \tag{A-84}$$

$$\frac{dI_t}{I_t} = \mu_t^{\pi\mathbb{Q}} dt - \sigma^\pi dB_t^{\pi\mathbb{Q}}. \tag{A-85}$$

First, $\frac{\Lambda_t^\tau}{M_t}$ being a martingale simply means that $\mu_t^{\tau\mathbb{Q}} = r_t$. Second, combining (A-85) with (A-80), we obtain

$$\frac{d\left(\frac{I_t M_t^*}{M_t}\right)}{\frac{I_t M_t^*}{M_t}} = \left(\mu_t^{\pi\mathbb{Q}} + r_t^* - r_t\right) dt - \sigma^\pi dB_t^{\pi\mathbb{Q}},$$

which is a martingale if and only if

$$\mu_t^{\pi\mathbb{Q}} = r_t - r_t^*, \tag{A-86}$$

i.e., the ex-ante Fisher relation holds in the model. Finally, using (A-80), (A-84), and (A-85), we get

$$\frac{d\left(\frac{I_t \Lambda_t^{\tau*}}{M_t}\right)}{\frac{I_t \Lambda_t^{\tau*}}{M_t}} = \left[\mu_t^{\tau*\mathbb{Q}} + \mu_t^{\pi\mathbb{Q}} + \rho_\pi \sigma_t^{\tau*}\sigma^\pi + \rho_{\pi*}\sigma_{t*}^{\tau*}\sigma^\pi - r_t\right] dt - \sigma_t^{\tau*} dB_t^{\mathbb{Q}} - \sigma_{t*}^{\tau*} dB_t^{*\mathbb{Q}} - \sigma^\pi dB_t^{\pi\mathbb{Q}}.$$

$I_t \Lambda_t^{\tau*}/M_t$ is a martingale if and only if its drift under $\mathbb{Q}$ is zero, which together with (A-86) implies

$$\mu_t^{\tau*\mathbb{Q}} + \rho_\pi \sigma_t^{\tau*}\sigma^\pi + \rho_{\pi*}\sigma_{t*}^{\tau*}\sigma^\pi = r_t^*. \tag{A-87}$$

Next we return to institutions' optimization problems. Using Itô's Lemma and substituting (A-26), (19), and (A-81) into intermediaries' budget constraint, (A-82), we get

$$dW_t = \left[r_t W_t + \int_0^T x_t^\tau \Lambda_t^\tau \left(\mu_t^\tau - r_t\right) d\tau + \int_0^T x_t^{\tau *} I_t \Lambda_t^{\tau *} \left(\mu_t^{\tau *} + \mu_t^\pi + \rho_\pi \sigma_t^{\tau *} \sigma^\pi + \rho_{\pi *} \sigma_{t*}^{\tau *} \sigma^\pi - r_t\right) d\tau \right] dt$$
$$- \left[\int_0^T x_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau + \int_0^T x_t^{\tau *} I_t \Lambda_t^{\tau *} \sigma_t^{\tau *} d\tau \right] dB_t - \left[\int_0^T x_t^{\tau *} I_t \Lambda_t^{\tau *} \sigma_{t*}^{\tau *} d\tau \right] dB_t^* - \left[\int_0^T x_t^{\tau *} I_t \Lambda_t^{\tau *} \sigma^\pi d\tau \right] dB_t^\pi.$$

Substituting it into (3) and differentiating with respect to x_t^τ and $x_t^{\tau *}$ for all τ, we obtain the first-order conditions that optimal demands have to satisfy. Finally, imposing market clearing in the nominal and real bond markets, i.e., setting $x_t^\tau = s_t^\tau$ and $x_t^{\tau *} = s_t^{\tau *} = 0$ for all $\tau \in (0, T]$, we obtain the following relationships that must hold in equilibrium:

$$\mu_t^\tau - r_t = \alpha \sigma_t^\tau \int_0^T s_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau$$

and

$$\mu_t^{\tau *} + \mu_t^\pi + \rho_\pi \sigma_t^{\tau *} \sigma^\pi + \rho_{\pi *} \sigma_{t*}^{\tau *} \sigma^\pi - r_t = \alpha \left(\sigma_t^{\tau *} + \rho \sigma_{t*}^{\tau *} + \rho_\pi \sigma^\pi\right) \int_0^T s_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau,$$

that is, the market prices of the nominal, real, and inflation risks are given by

$$\lambda_t = -\alpha \int_0^T s_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau, \; \lambda_{t*} = -\alpha \int_0^T s_t^{\tau *} I_t \Lambda_t^{\tau *} \sigma_{t*}^{\tau *} d\tau = 0, \text{ and } \lambda_{\pi t} = -\alpha \int_0^T s_t^{\tau *} I_t \Lambda_t^{\tau *} \sigma_\pi d\tau = 0,$$

and we have

$$\mu_t^\tau - r_t = \left(-\sigma_t^\tau\right) \lambda_t \text{ and } \mu_t^{\tau *} + \mu_t^\pi + \rho_\pi \sigma_t^{\tau *} \sigma^\pi + \rho_{\pi *} \sigma_{t*}^{\tau *} \sigma^\pi - r_t = -\left(\sigma_t^{\tau *} + \rho \sigma_{t*}^{\tau *} + \rho_\pi \sigma^\pi\right) \lambda_t. \quad \text{(A-88)}$$

The total amount of interest rate risk is the same as before due to

$$d\left[\int_0^T s_t^\tau \Lambda_t^\tau d\tau + \int_0^T s_t^{\tau *} I_t \Lambda_t^{\tau *} d\tau \right] = \left[\int_0^T s_t^\tau \Lambda_t^\tau \mu_t^\tau d\tau \right] dt - \left[\int_0^T s_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau \right] dB_t,$$

therefore the market price of (nominal) interest rate risk are all determined by its total quantity, and (5) holds. As our focus is on changes in interest rate risk induced by the interest rate risk inherent in MBS, using the same argument as in the main part of Section 1, we obtain that the market price of nominal interest rate risk is given by (10). Therefore, under the risk-neutral measure, the dynamics of the nominal interest rate, the duration factor, and the real interest rate are given by (12), (13), and, from (18), by

$$dr_t^* = \left[\kappa^* \left(\theta^* - r_t^*\right) + \alpha \sigma^* \sigma_y^{\bar{\tau}} D_t\right] dt + \sigma^* dB_t^{*\mathbb{Q}}, \quad \text{(A-89)}$$

respectively.

Finally, we look to determine equilibrium bond prices. Because the nominal short rate and the market price of risk are the same as in the baseline model, nominal yields are affine in the nominal short rate and duration: (15) holds with the functions $\mathcal{A}(\tau)$, $\mathcal{B}(\tau)$, and $\mathcal{C}(\tau)$

determined in the proof of Theorem 1. Notice that this implies the endogenous parameters $\sigma_y^{\bar{\tau}}$, δ_r, δ_D, and δ_0 are all as given in (15).

Next, we conjecture that equilibrium yields on real bonds in the model defined by (12), (13) and (A-89) are in the form

$$y_t^{\tau *} = \mathcal{E}(\tau) + \mathcal{F}(\tau) r_t^* + \mathcal{G}(\tau) D_t + \mathcal{H}(\tau) r_t, \tag{A-90}$$

i.e., prices of real bonds are

$$\Lambda_t^{\tau *} = e^{-[\tau\mathcal{E}(\tau)+\tau\mathcal{F}(\tau)r_t^*+\tau\mathcal{G}(\tau)D_t+\tau\mathcal{H}(\tau)r_t]}. \tag{A-91}$$

Applying Itô's Lemma to (A-91), substituting in (12), (13), and (A-89) and imposing condition (A-87) on the relationship between the drift and volatility of real bonds under $\mathbb{Q}$, we obtain an equation affine in the factors r_t^*, D_t, and r_t. Collecting the r_t^*, D_t, r_t, and constant terms, respectively, we get the following set of ODEs:

$$1 = \tau\mathcal{F}'(\tau) + \mathcal{F}(\tau) + \kappa^*\tau\mathcal{F}(\tau), \tag{A-92}$$

$$0 = \tau\mathcal{G}'(\tau) + \mathcal{G}(\tau) + \delta_D^{\mathbb{Q}}\tau\mathcal{G}(\tau) - \alpha\sigma^*\sigma_y^{\bar{\tau}}\tau\mathcal{F}(\tau) - \alpha\sigma\sigma_y^{\bar{\tau}}\tau\mathcal{H}(\tau), \tag{A-93}$$

$$0 = \tau\mathcal{H}'(\tau) + \mathcal{H}(\tau) + \kappa\tau\mathcal{H}(\tau) + \delta_r\tau\mathcal{G}(\tau), \tag{A-94}$$

and

$$\begin{aligned} 0 = {} & \tau\mathcal{E}'(\tau) + \mathcal{E}(\tau) - \tau\mathcal{F}(\tau)\kappa^*\theta^* - \tau\mathcal{G}(\tau)\delta_0 - \tau\mathcal{H}(\tau)\kappa\theta + \frac{1}{2}\tau^2\mathcal{F}^2(\tau)(\sigma^*)^2 \\ & + \frac{1}{2}\tau^2\mathcal{G}^2(\tau)\left(\eta_y\sigma_y^{\bar{\tau}}\right)^2 + \frac{1}{2}\tau^2\mathcal{H}^2(\tau)\sigma^2 + \tau^2\mathcal{F}(\tau)\mathcal{G}(\tau)\rho\sigma^*\eta_y\sigma_y^{\bar{\tau}} + \tau^2\mathcal{F}(\tau)\mathcal{H}(\tau)\rho\sigma^*\sigma \\ & + \tau^2\mathcal{G}(\tau)\mathcal{H}(\tau)\sigma\eta_y\sigma_y^{\bar{\tau}}, \end{aligned} \tag{A-95}$$

with terminal conditions $\mathcal{E}(0) = \mathcal{G}(0) = \mathcal{H}(0) = 0$ and $\mathcal{F}(0) = 1$. From (A-92) it is imminent that

$$\mathcal{F}(\tau) = \frac{1 - e^{-\kappa^*\tau}}{\kappa^*\tau}. \tag{A-96}$$

Combining (A-93) and (A-94), we write the following second-order ODE for $\mathcal{G}$:

$$\begin{aligned} 0 = {} & \tau\mathcal{G}''(\tau) + 2\mathcal{G}'(\tau) + \left(\kappa + \delta_D^{\mathbb{Q}}\right)\tau\mathcal{G}'(\tau) + \left(\kappa + \delta_D^{\mathbb{Q}}\right)\mathcal{G}(\tau) + \kappa\delta_D\tau\mathcal{G}(\tau) \\ & - \alpha\sigma^*\sigma_y^{\bar{\tau}}\left[\tau\mathcal{F}'(\tau) + \mathcal{F}(\tau) + \kappa\tau\mathcal{F}(\tau)\right]. \end{aligned} \tag{A-97}$$

Solving (A-97) for $\mathcal{G}$, then from (A-94) deriving $\mathcal{H}$, and applying the terminal conditions, yields the following solution:

$$\mathcal{G}(\tau) = \frac{\rho\sigma^*}{\sigma}\mathcal{C}(\tau) + (\kappa^* - \kappa)\frac{\rho\alpha\sigma^*\sigma_y^{\bar{\tau}}\left[\mathcal{F}(\tau) - \frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} + \frac{\kappa^*-(\kappa_n+\varepsilon)}{\alpha\sigma\sigma_y^{\bar{\tau}}}\mathcal{C}(\tau)\right]}{\kappa\left(\delta_D^{\mathbb{Q}} - \delta_D\right) + (\kappa^* - \kappa)\left(\delta_D^{\mathbb{Q}} - \kappa^*\right)} \tag{A-98}$$

and

$$\mathcal{H}(\tau) = \frac{\rho\sigma^*}{\sigma}\left[\mathcal{B}(\tau) - \mathcal{F}(\tau)\right] + (\kappa^* - \kappa)\frac{\rho\sigma^*\left(\delta_D^{\mathbb{Q}} - \kappa^*\right)\left[\mathcal{F}(\tau) - \frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} + \frac{\varepsilon[\kappa^*-(\kappa_n+\varepsilon)]}{\alpha\sigma\sigma_y^{\tau}\left(\delta_D^{\mathbb{Q}}-\kappa^*\right)}C(\tau)\right]}{\sigma\left[\kappa\left(\delta_D^{\mathbb{Q}} - \delta_D\right) + (\kappa^* - \kappa)\left(\delta_D^{\mathbb{Q}} - \kappa^*\right)\right]}. \tag{A-99}$$

Finally, (A-95) together with (A-96)-(A-99) would yield $\mathcal{E}(\tau)$, but we omit the exact solution here as it is not essential for the results that follow.

Notice that in the special case when the speed of mean reversion for the nominal and real short rates are very close to each other, $\kappa \approx \kappa^*$, (A-98) and (A-99) simplify to

$$\mathcal{G}(\tau) \approx \frac{\rho\sigma^*}{\sigma}\mathcal{C}(\tau) \text{ and } \mathcal{H}(\tau) \approx \frac{\rho\sigma^*}{\sigma}\left[\mathcal{B}(\tau) - \mathcal{F}(\tau)\right]. \tag{A-100}$$

Proof of Proposition 5. The excess return over horizon $(t, t+h)$ on a maturity-τ real bond is

$$rx_{t,t+h}^{\tau*} = \log\Lambda_{t+h}^{\tau-h*} - \log\Lambda_t^{\tau*} + \log\Lambda_t^{h*}. \tag{A-101}$$

To express the RHS, we follow the same steps as in the nominal bond case. Applying Itô's lemma to $\log\Lambda_t^{\tau*}$ and using (A-81) we obtain

$$d\log\Lambda_t^{\tau*} = \left[\mu_t^{\tau*} - \frac{1}{2}\left(\sigma_t^{\tau*}\right)^2 - \frac{1}{2}\left(\sigma_{t*}^{\tau*}\right)^2 - \rho\sigma_t^{\tau*}\sigma_{t*}^{\tau*}\right]dt - \sigma_t^{\tau*}dB_t - \sigma_{t*}^{\tau*}dB_t^*. \tag{A-102}$$

Next we consider the change in the log price of a real bond, which at time t has time to maturity τ, over a horizon of h. The price of this bond at time $s \in [t, t+h]$ is given by $\Lambda_s^{t+\tau-s*}$, hence integrating (A-102) implies

$$\log\Lambda_{t+h}^{\tau-h*} - \log\Lambda_t^{\tau*} = \int_t^{t+h} d\log\Lambda_s^{\tau-(s-t)*} = \int_t^{t+h}\mu_s^{\tau-(s-t)*}ds - \int_t^{t+h}\rho\sigma_s^{\tau-(s-t)*}\sigma_{s*}^{\tau-(s-t)*}ds$$
$$-\int_t^{t+h}\left[\frac{1}{2}\left(\sigma_{s*}^{\tau-(s-t)*}\right)^2 + \frac{1}{2}\left(\sigma_s^{\tau-(s-t)*}\right)^2\right]ds - \int_t^{t+h}\sigma_s^{\tau-(s-t)*}dB_s - \int_t^{t+h}\sigma_{s*}^{\tau-(s-t)*}dB_s^*$$

and

$$-\log\Lambda_t^{h*} = \int_t^{t+h}\left[\mu_s^{h-(s-t)*} - \frac{1}{2}\left(\sigma_s^{h-(s-t)*}\right)^2 - \frac{1}{2}\left(\sigma_{s*}^{h-(s-t)*}\right)^2 - \rho\sigma_s^{h-(s-t)*}\sigma_{s*}^{h-(s-t)*}\right]ds$$
$$-\int_t^{t+h}\sigma_s^{h-(s-t)*}dB_s - \int_t^{t+h}\sigma_{s*}^{h-(s-t)*}dB_s^*,$$

where we used $\log \Lambda_t^{0*} = 0$. Substituting the last two expressions into (A-101), the excess return on the bond over a horizon h becomes

$$rx_{t,t+h}^{\tau *} = \int_t^{t+h} \left(\mu_s^{\tau-(s-t)*} - \mu_s^{h-(s-t)*}\right) ds + \int_t^{t+h} \left[\frac{1}{2}\left(\sigma_s^{h-(s-t)*}\right)^2 - \frac{1}{2}\left(\sigma_s^{\tau-(s-t)*}\right)^2\right] ds \quad \text{(A-103)}$$

$$+ \int_t^{t+h} \left[\frac{1}{2}\left(\sigma_{s*}^{h-(s-t)*}\right)^2 - \frac{1}{2}\left(\sigma_{s*}^{\tau-(s-t)*}\right)^2 + \rho\sigma_s^{h-(s-t)*}\sigma_{s*}^{h-(s-t)*} - \rho\sigma_s^{\tau-(s-t)*}\sigma_{s*}^{\tau-(s-t)*}\right] ds$$

$$- \int_t^{t+h} \sigma_s^{\tau-(s-t)*} dB_s + \int_t^{t+h} \sigma_s^{h-(s-t)*} dB_s - \int_t^{t+h} \sigma_{s*}^{\tau-(s-t)*} dB_s^* + \int_t^{t+h} \sigma_{s*}^{h-(s-t)*} dB_s^*.$$

We want to examine the regression coefficients when we regress $rx_{t,t+h}^{\tau}$ on state variables D_t, r_t and r_t^*. As the volatility expressions do not depend on any of these state variables, and the Brownian increments dB_s, $s \in [t, t+h]$, are independent of state variables at time t, we can ignore all the non-μ terms in (A-103). On the other hand, using (10) and (A-88) we have

$$\mu_t^{\tau *} = r_t^* + \left(\sigma_t^{\tau *} + \rho\sigma_{t*}^{\tau *} + \rho_\pi\sigma^\pi\right)\alpha\sigma_t^{\bar{\tau}} D_t - \rho_\pi\sigma_t^{\tau *}\sigma^\pi - \rho_{\pi *}\sigma_{t*}^{\tau *}\sigma^\pi. \quad \text{(A-104)}$$

Applying (A-104) to $\tau - (s-t)$ and $h - (s-t)$ implies

$$\int_t^{t+h} \left(\mu_s^{\tau-(s-t)*} - \mu_s^{h-(s-t)*}\right) ds = \alpha\sigma_t^{\bar{\tau}} \int_t^{t+h} \Delta\sigma_s^{s-t*} D_s ds + const, \quad \text{(A-105)}$$

where the last term collects the volatility adjustments that depend on maturities but not on the real rate or duration, and $\Delta\sigma_s^{s-t*} \equiv \sigma_t^{\tau-(s-t)*} + \rho\sigma_{t*}^{\tau-(s-t)*} - \sigma_t^{h-(s-t)*} - \rho\sigma_{t*}^{h-(s-t)*}$.

We consider three regressions in this theoretical model. First, when we regress $rx_{t,t+h}^{\tau *}$ on D_t and r_t in the form

$$rx_{t,t+h}^{\tau *} = \beta_{0,D,r}^{\tau,h*} + \left(\beta_1^{\tau,h*}, \beta_2^{\tau,h*}\right)(D_t, r_t)^\top + \epsilon_{t+h},$$

the vector of coefficients is $\left(\beta_1^{\tau,h*}, \beta_2^{\tau,h*}\right)^\top = V^{-1} Cov\left[rx_{t,t+h}^{\tau *}, (D_t, r_t)^\top\right]$, where V is given by (A-17). Combining (A-17), (A-22), (A-24) (A-103), (A-105), and the discussion in between, after some algebra we obtain

$$\beta_1^{\tau,h*} = \alpha\sigma_y^{\bar{\tau}} \int_t^{t+h} \Delta\sigma_s^{s-t*} e^{-\delta_D(s-t)} ds \text{ and} \quad \text{(A-106)}$$

$$\beta_2^{\tau,h*} = -\frac{\delta_r}{\delta_D - \kappa}\alpha\sigma_y^{\bar{\tau}} \int_t^{t+h} \Delta\sigma_s^{s-t*}\left(e^{-\kappa(s-t)} - e^{-\delta_D(s-t)}\right) ds, \quad \text{(A-107)}$$

analogously to (A-62)-(A-63) since $Cov\left[D_s, (D_t, r_t)^\top\right]$ is the same as in the basic model.

Second, we run a multivariate regression of $rx^{\tau*}_{t,t+h}$ on D_t, r_t, and r^*_t in the form

$$rx^{\tau*}_{t,t+h} = \beta^{\tau,h*}_{0,D,r,r^*} + \left(\gamma^{\tau,h*}_1, \gamma^{\tau,h*}_2, \gamma^{\tau,h*}_3\right)(D_t, r_t, r^*_t)^\top + \epsilon_{t+h},$$

which yields $\left(\gamma^{\tau,h*}_1, \gamma^{\tau,h*}_2, \gamma^{\tau,h*}_3\right)^\top = (V^*)^{-1}\, Cov\left[rx^{\tau*}_{t,t+h}, (D_t, r_t, r^*_t)^\top\right]$, where V^* is given by (A-25). Following the same steps as in the bivariate regression case, after some algebra we obtain the following relationship between the regression coefficients: $\gamma^{\tau,h*}_1 = \beta^{\tau,h*}_1$, $\gamma^{\tau,h*}_2 = \beta^{\tau,h*}_2$, and $\gamma^{\tau,h*}_3 = 0$.

Finally, we run the univariate regression on duration only:

$$rx^{\tau*}_{t,t+h} = \beta^{\tau,h*}_{0,D} + \beta^{\tau,h*} D_t + \epsilon_{t+h},$$

where the regression coefficient is $\beta^{\tau,h*} = Cov\left[rx^{\tau*}_{t,t+h}, D_t\right]/Var\left[D_t\right]$. From (A-103), after some algebra, this coefficient simplifies to

$$\beta^{\tau,h*} = \alpha\sigma^{\bar{\tau}}_y \int_t^{t+h} \Delta\sigma^{s-t*}_s \frac{\delta_D e^{-\delta_D(s-t)} - \kappa e^{-\kappa(s-t)}}{\delta_D - \kappa} ds. \tag{A-108}$$

Proving that in general $\beta^{\tau,h*}_1$ and $\beta^{\tau,h*}$ are positive and increasing across maturity τ, while possible, would be much more cumbersome than in the case of nominal excess returns. Hence, to save space, and motivated by the calibrated parameters, we only provide the proof for the special case when the speed of mean reversion of the real and nominal short rates are approximately close to each other, i.e., $\kappa \approx \kappa^*$. For this, we first apply Itô's Lemma to (A-91) and contrast it with (A-81) to obtain the general formulas for real bond return volatilities

$$\sigma^{\tau*}_t = \tau\mathcal{G}(\tau)\,\eta_y\sigma^{\bar{\tau}}_y + \tau\mathcal{H}(\tau)\,\sigma \text{ and } \sigma^{\tau*}_{t*} = \tau\mathcal{F}(\tau)\,\sigma^*.$$

Substituting (A-100) into these formulas, after some algebra we get that in the case $\kappa \approx \kappa^*$

$$\sigma^{\tau*}_t + \rho\sigma^{\tau*}_{t*} = \tau\mathcal{G}(\tau)\,\eta_y\sigma^{\bar{\tau}}_y + \tau\mathcal{H}(\tau)\,\sigma + \rho\tau\mathcal{F}(\tau)\,\sigma^* \approx \frac{\rho\sigma^*}{\sigma}\sigma^\tau_t.$$

Thus, $\Delta\sigma^{s-t*}_s$ simplifies to $\frac{\rho\sigma^*}{\sigma}\left(\sigma^{\tau-(s-t)}_t - \sigma^{h-(s-t)}_t\right)$, which in turn leads to

$$\beta^{\tau,h*}_1 \approx \frac{\rho\sigma^*}{\sigma}\beta^{\tau,h}_1,\ \beta^{\tau,h*}_2 \approx \frac{\rho\sigma^*}{\sigma}\beta^{\tau,h}_2, \text{ and } \beta^{\tau,h*} \approx \frac{\rho\sigma^*}{\sigma}\beta^{\tau,h}.$$

Since $\beta^{\tau,h}_1$ and $\beta^{\tau,h}$ are positive and increasing across maturities, $\beta^{\tau,h*}_1$ and $\beta^{\tau,h*}$ must also be positive and increasing across maturities; moreover, the corresponding real and nominal regression coefficients are proportional to each other by a multiplier of $\frac{\rho\sigma^*}{\sigma}$. This concludes the proof of Proposition 5 and the statements thereafter. □

Appendix C.4 Time-varying convexity

Here we present a tractable way to relax the assumption of constant MBS convexity and capture the non-linearities inherent to the prepayment option. This version of the model allows for an additional degree of freedom and provides a better statistical description of MBS duration and convexity series. However, the qualitative implications of the model are identical to the ones outlined in Section 1. More precisely, we allow the sensitivity of outstanding MBS to the short rate to be quadratic:

$$\frac{dMBS_t}{dr_t} = z_t + \phi z_t^2, \quad \text{and} \tag{A-109}$$

$$dz_t = -\kappa_z^{\mathbb{Q}} z_t dt + \sigma_z dB_t^{\mathbb{Q}}. \tag{A-110}$$

In the data, when interest rates and MBS duration decrease, the negative convexity of MBS increases. In other words MBS duration has negative skewness. The skewness of the monthly series of MBS duration in our sample is equal to -1.32 compared to the 10 year yield which displays only a moderate skewness of -0.06. The parameter ϕ can be calibrated to match this feature of the data. From an economic point of view negative skewness corresponds to the asymmetry in MBS duration response to changes in interest rates: it reacts more to falling than to rising interest rates.

In the model described by (1), (9), and (A-109)-(A-110) yields are given by

$$y_t^\tau = \mathcal{A}_z(\tau) + \mathcal{B}_z(\tau) r_t + \mathcal{C}_z(\tau) z_t + \mathcal{D}_z(\tau) z_t^2,$$

The key to quadratic closed form solution is that while quadratic terms appear under $\mathbb{Q}$ in the dynamics of r_t, z_t is still affine under $\mathbb{Q}$ (and therefore not affine under $\mathbb{P}$); see also Cheng and Scaillet (2007). Bond prices are given by

$$\Lambda_t^\tau = e^{-\left[A_z(\tau)+B_z(\tau)r_t+C_z(\tau)z_t+D_z(\tau)z_t^2\right]},$$

where $A_z(\tau) \equiv \tau\mathcal{A}_z(\tau)$, $B_z(\tau) \equiv \tau\mathcal{B}_z(\tau)$, $C_z(\tau) \equiv \tau\mathcal{C}_z(\tau)$, and $D_z(\tau) \equiv \tau\mathcal{D}_z(\tau)$. No-arbitrage pricing of bonds results in the following system of ODEs (where we remove the time-dependence to simplify the notation):

$$\begin{aligned}
0 &= A_z' - \kappa\theta B_z + \frac{1}{2}\sigma^2 B_z^2 + \frac{1}{2}\sigma_z^2\left(C_z^2 - 2D_z\right) + \sigma\sigma_z B_z C_z, \\
1 &= B_z' + \kappa B_z, \\
0 &= C_z' + \left(\kappa_z^{\mathbb{Q}} + 2\sigma_z^2 D_z\right) C_z - \alpha\sigma^2 B_z + 2\sigma\sigma_z B_z D_z, \text{ and} \\
0 &= D_z' + 2\kappa_z^{\mathbb{Q}} D_z + 2\sigma_z^2 D_z^2 - \alpha\sigma^2\phi B_z,
\end{aligned}$$

together with the boundary conditions $A_z(0) = B_z(0) = C_z(0) = D_z(0) = 0$. The solution to the system above can be written in terms of J- and Y-type Bessel functions. To simplify, we can also solve for $A_z(\tau)$, $B_z(\tau)$, $C_z(\tau)$, and $D_z(\tau)$ recursively using a discrete time approximation of the dynamics of the state variables.

By Itô's lemma the second order dollar sensitivity of outstanding MBS to short rate shocks ($\frac{d^2MBS_t}{dr_t^2} \equiv -\gamma$) is equal to:

$$\sigma_z + 2\phi\sigma_z z_t,$$

implying time-varying convexity. The instantaneous volatility of maturity-τ yield is given by

$$\mathcal{B}_z(\tau)\sigma + \mathcal{C}_z(\tau)\sigma_z + 2\mathcal{D}_z(\tau)\sigma z_t.$$

The code that calculates $A_z(\tau)$, $B_z(\tau)$, $C_z(\tau)$, and $D_z(\tau)$ and allows to verify Propositions 1 and 4 in the context of stochastic convexity is available upon request.

Appendix D Robustness results

Other measures: Since our MBS duration and convexity measures are provided by Barclays, one might suspect that our results could be driven by model misspecification. To address this issue, we re-run our benchmark regressions using the Bank of America MBS indices. The results in Table 9 are qualitatively in line with those reported earlier using the Barclays indices. MBS dollar duration is highly significant in particular for longer maturities. Similarly, for the bond yield volatility regressions, we find MBS dollar convexity to be significant.

Real bonds: In the main analysis of the paper, we use real yields data which has been adjusted for liquidity because a large literature documents a significant liquidity premium in the TIPS market which (i) varies a lot over time and (ii) is found to be particularly high during the 2008 financial crisis (see, e.g., D'Amico, Kim, and Wei (2014) and Pflueger and Viceira (2015), among others). For example, Pflueger and Viceira (2015) emphasize that in order to understand the predictability of real bond returns, it is important to disentangle the predictability due to liquidity and the predictability due to cash flow risks.

To illustrate this point, the upper panel of Figure 7 plots the 10-year TIPS yield (bold line) and two real yields adjusted for liquidity from Pflueger and Viceira (2015) and D'Amico, Kim, and Wei (2014), respectively.[31] In the lower panel, we plot the difference between the TIPS yield and the liquidity adjusted yields and note the large spike at the beginning of 2008, indicating a large liquidity premium in the TIPS market.

It is therefore a natural question to ask how and whether this affects our results. To address these concerns we present in Table 10 real bond return predictability results using different real yields. The first column reveals that regressing the actual 10-year TIPS on dollar duration in the 1999-2013 sample produces a slope coefficient that is positive but not significant. At the same time the second and third columns show that using either Pflueger and Viceira (2015) or D'Amico, Kim, and Wei (2014) real yields leads to positive and significant coefficients of comparable magnitude. In order to gauge the effect of the liquidity dry-up during the 2008-2009 financial crisis, we can run regressions from annual bond returns from TIPS and control for a crisis dummy. We note that the estimated coefficient on duration (fourth column) is quantitatively in line with the ones for liquidity-adjusted yields and statistically significant.

Interest rate swaps: Interest rate risk is primarily hedged in either the Treasury or interest rate swap market and the main focus in the previous section has been on Treasury data. The reason for this is twofold. First, interest rate swap data contain a considerable credit risk component (see Feldhütter and Lando (2008)) which is outside the scope of our paper to explain. Second, after the Lehman default in 2008, prices of interest rate swaps (especially at longer maturities) got possibly distorted due to a decline in arbitrage capital (see Krishnamurthy (2010)). In particular, our data sample also covers the time period where the swap spread, defined as the difference between the fixed rate on a fixed-for-floating 10-year swap and 10-year Treasury rate, turned negative. Nevertheless, for robustness reasons, we also run bond risk premia regressions using swap rather than Treasury data and we report estimated coefficients in Table 11.[32] We note that the size and significance of the estimated coefficients are almost identical to

[31] Pflueger and Viceira (2015) estimate the liquidity premium using an approach consisting of regressing breakeven inflation onto bond market liquidity proxies while controlling for inflation expectations. D'Amico, Kim, and Wei (2014) on the other hand use a no-arbitrage term structure model.

[32] We bootstrap a zero-coupon curve from swap rates and calculate excess returns that are directly comparable to the Treasury excess returns we use in the benchmark results.

Table 9

Bond risk premia and volatility regressions: Alternative measures

The upper panel reports estimated coefficients from regressing annual bond excess returns onto BoA MBS dollar duration and the level of interest rates. The lower panel presents coefficients from regressing yield volatilities onto BoA MBS dollar convexity. t-Statistics are calculated either using Newey and West (1987) (in parentheses) or Hansen and Hodrick (1980) (in brackets). Data is monthly and runs from December 1989 to December 2012 (bond excess returns) and January 1997 to December 2012 (bond yield volatilities).

Panel A: Bond excess returns

	2y	3y	4y	5y	6y	7y	8y	9y	10y
constant	-0.0028	-0.1112	-0.3609	-0.7291	-1.1951	-1.7381	-2.3380	-2.9776	-3.6439
	(-0.01)	(-0.17)	(-0.36)	(-0.56)	(-0.74)	(-0.91)	(-1.07)	(-1.22)	(-1.37)
	[-0.01]	[-0.17]	[-0.37]	[-0.56]	[-0.74]	[-0.91]	[-1.06]	[-1.20]	[-1.33]
duration	0.0534	0.1166	0.1864	0.2594	0.3332	0.4062	0.4776	0.5468	0.6137
	(2.63)	(3.32)	(3.94)	(4.34)	(4.50)	(4.52)	(4.47)	(4.40)	(4.34)
	[3.44]	[4.54]	[5.48]	[5.75]	[5.47]	[5.03]	[4.66]	[4.38]	[4.17]
Adj. R^2	6.09%	8.35%	10.97%	13.53%	15.84%	17.81%	19.46%	20.83%	22.00%
constant	-0.0454	-0.0708	-0.1764	-0.3721	-0.6547	-1.0140	-1.4369	-1.9108	-2.4253
	(-0.14)	(-0.11)	(-0.19)	(-0.30)	(-0.43)	(-0.57)	(-0.70)	(-0.83)	(-0.96)
	[-0.16]	[-0.12]	[-0.19]	[-0.30]	[-0.42]	[-0.54]	[-0.65]	[-0.77]	[-0.88]
duration	0.0467	0.1230	0.2156	0.3158	0.4185	0.5206	0.6200	0.7153	0.8062
	(2.02)	(2.99)	(4.04)	(4.94)	(5.49)	(5.69)	(5.71)	(5.65)	(5.57)
	[2.19]	[3.48]	[5.24]	[6.94]	[7.58]	[7.27]	[6.73]	[6.28]	[5.95]
level	0.0471	-0.0447	-0.2042	-0.3950	-0.5981	-0.8014	-0.9973	-1.1806	-1.3485
	(0.46)	(-0.25)	(-0.85)	(-1.36)	(-1.75)	(-2.04)	(-2.24)	(-2.37)	(-2.46)
	[0.56]	[-0.32]	[-1.18]	[-1.96]	[-2.54]	[-2.89]	[-3.07]	[-3.15]	[-3.17]
Adj. R^2	6.70%	8.54%	12.64%	17.46%	22.20%	26.43%	29.98%	32.87%	35.16%

Panel B: Bond yield volatilities

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
constant	-3.9255	-2.0995	0.0209	1.3622	2.1274	2.5320	2.7141	2.7640	2.7413	2.6835
	(-1.44)	(-0.85)	(0.01)	(0.77)	(1.40)	(1.88)	(2.23)	(2.48)	(2.66)	(2.79)
	[-1.29]	[-0.77]	[0.01]	[0.70]	[1.27]	[1.71]	[2.04]	[2.27]	[2.43]	[2.55]
convexity	0.1038	0.1017	0.0798	0.0616	0.0491	0.0406	0.0349	0.0310	0.0282	0.0262
	(4.95)	(6.82)	(7.43)	(7.41)	(7.02)	(6.48)	(5.96)	(5.54)	(5.22)	(5.00)
	[4.43]	[6.21]	[6.99]	[7.15]	[6.86]	[6.33]	[5.78]	[5.32]	[4.98]	[4.74]
Adj. R^2	28.95%	27.55%	24.84%	22.43%	20.16%	18.20%	16.62%	15.42%	14.55%	13.94%

those reported for Treasuries. Adding explanatory factors such as the level of yields does not deteriorate the significance of MBS dollar duration.

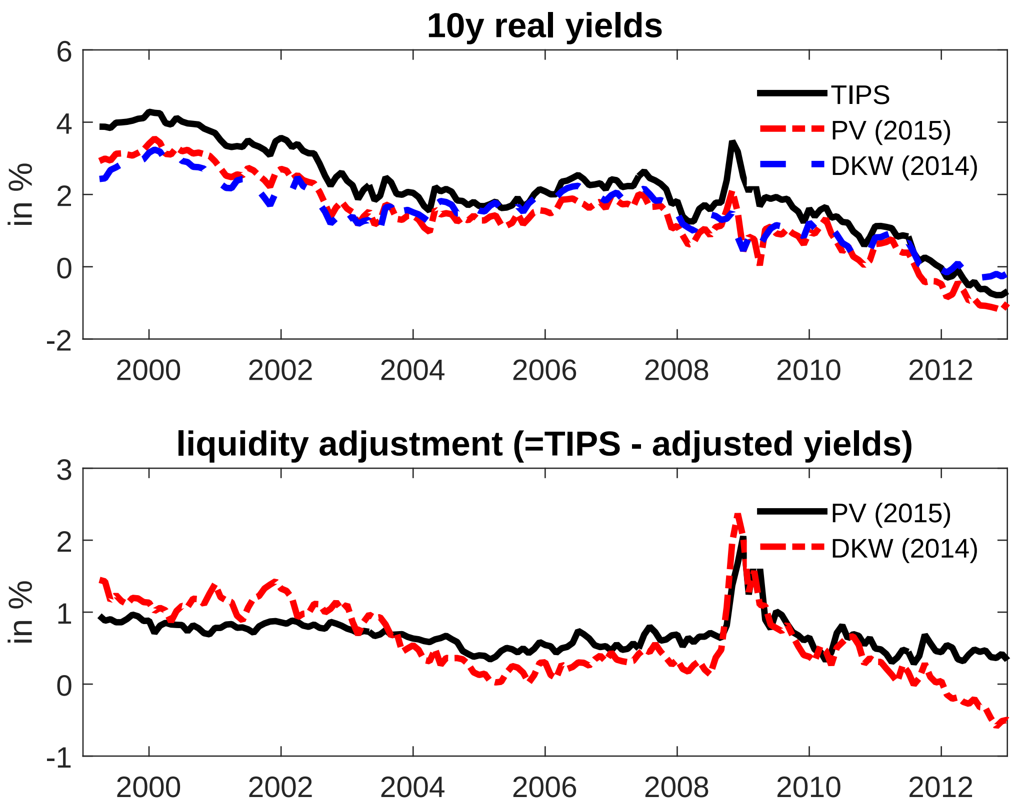

Figure 7. Real yields, 1999 - 2013

The upper panel plots the 10-year TIPS yield and liquidity adjusted yields from D'Amico, Kim, and Wei (2014) and Pflueger and Viceira (2015). The lower panel plots the difference between the 10-year TIPS yield and the two liquidity adjusted yields.

Table 10
Real bond risk premia regressions

This table reports estimated coefficients from regressing annual real bond excess returns, $rx^{10y*}_{t,t+1y}$, onto MBS dollar duration:

$$rx^{10y*}_{t,t+1y} = \beta^{10y}_0 + \beta^{10y}_1 \text{duration}_t + \beta^{10y}_2 \text{duration}_t \times d^{2008}_t + \epsilon^{10y}_{t+1y},$$

where $rx^{10y*}_{t,t+1y}$ is the annual bond return calculated from TIPS (first and fourth column), liquidity adjusted yields from Pflueger and Viceira (2015) (PV) and D'Amico, Kim, and Wei (2014) (DKW). d^{2008}_t is a dummy which takes a value of one during 2008 and 2009 and zero otherwise. t-Statistics are calculated using Newey and West (1987). Data is monthly and runs from March 1999 through December 2012.

	TIPS	PV	DKW	TIPS
constant	-4.918	-11.245	-8.241	-10.43
	(-0.69)	(-1.88)	(-2.58)	(-1.21)
duration	0.022	0.034	0.028	0.030
	(1.37)	(2.40)	(3.45)	(1.78)
interaction	no	no	no	yes
Adj. R^2	4.88%	12.72%	16.13%	11.00%

Table 11
Bond risk premia regressions: swaps

This table reports estimated coefficients from regressing annual bond excess returns constructed from interest rate swaps, $rxsw^{\tau}_{t+1y}$, onto a set of variables:

$$rxsw^{\tau}_{t+1y} = \beta^{\tau}_0 + \beta^{\tau}_1 \text{duration}_t + \beta^{\tau}_2 \text{level}_t + \epsilon^{\tau}_{t+1y},$$

where duration_t is MBS dollar duration and level_t is the one-year yield. t-Statistics are calculated either using Newey and West (1987) (in parentheses) or Hansen and Hodrick (1980) (in brackets). Data is monthly and runs from December 1989 to December 2012

	2y	3y	4y	5y	6y	7y	8y	9y	10y
constant	-1.8231	-3.6439	-6.0636	-9.0721	-12.4627	-16.0119	-19.7017	-23.4686	-27.1796
	(-1.76)	(-1.79)	(-2.15)	(-2.62)	(-3.11)	(-3.56)	(-3.94)	(-4.26)	(-4.56)
	[-1.54]	[-1.53]	[-1.82]	[-2.22]	[-2.65]	[-3.04]	[-3.38]	[-3.68]	[-3.95]
duration	0.0058	0.0118	0.0186	0.0264	0.0347	0.0433	0.0521	0.0610	0.0696
	(2.38)	(2.61)	(3.04)	(3.59)	(4.15)	(4.66)	(5.10)	(5.45)	(5.77)
	[2.04]	[2.20]	[2.55]	[3.02]	[3.49]	[3.94]	[4.33]	[4.66]	[4.96]
Adj. R^2	6.82%	7.62%	10.00%	13.03%	16.00%	18.80%	21.44%	23.93%	26.18%
constant	-1.5139	-3.3676	-5.9083	-9.0704	-12.6109	-16.2879	-20.0893	-23.9528	-27.7451
	(-1.56)	(-1.67)	(-2.08)	(-2.58)	(-3.07)	(-3.54)	(-3.93)	(-4.26)	(-4.56)
	[-1.39]	[-1.46]	[-1.81]	[-2.25]	[-2.70]	[-3.12]	[-3.48]	[-3.80]	[-4.09]
duration	0.0038	0.0100	0.0175	0.0263	0.0357	0.0451	0.0546	0.0642	0.0733
	(1.75)	(2.18)	(2.67)	(3.21)	(3.70)	(4.17)	(4.59)	(4.95)	(5.25)
	[1.62]	[1.98]	[2.41]	[2.89]	[3.35]	[3.79]	[4.19]	[4.54]	[4.85]
level	0.1739	0.1554	0.0874	0.0009	-0.0833	-0.1553	-0.2180	-0.2724	-0.3181
	(2.17)	(0.97)	(0.39)	(0.00)	(-0.26)	(-0.43)	(-0.55)	(-0.64)	(-0.70)
	[2.11]	[0.92]	[0.37]	[0.00]	[-0.25]	[-0.42]	[-0.55]	[-0.64]	[-0.72]
Adj. R^2	14.77%	9.37%	10.32%	13.07%	16.18%	19.18%	22.00%	24.63%	26.98%