Irving Fisher Committee on
Central Bank Statistics

IFC-Banco de Portugal Conference: "Communication on central bank statistics: unlocking the next level"

19-20 September 2022

Construction of bivariate fan chart from joint distribution[1]

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[1] This contribution was prepared for the workshop. The views expressed are those of the authors and do not necessarily reflect the views of the Banco de Portugal, the BIS, the IFC or the other central banks and institutions represented at the event.

Communicating Statistics: Construction of Bivariate Fan Chart from Joint Distribution

RESERVE BANK OF INDIA

Communication on Central Bank Statistics: Unlocking the Next Level

organised jointly by the Irving Fisher Committee on Central Bank Statistics of the Bank for International Settlements and Banco de Portugal

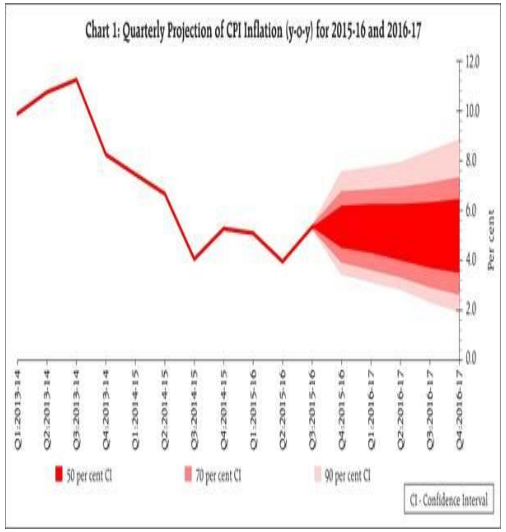
Lisbon, September 19-20, 2022

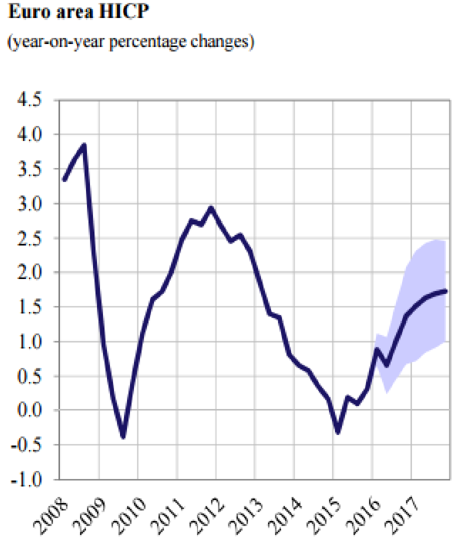
Statistical Communication

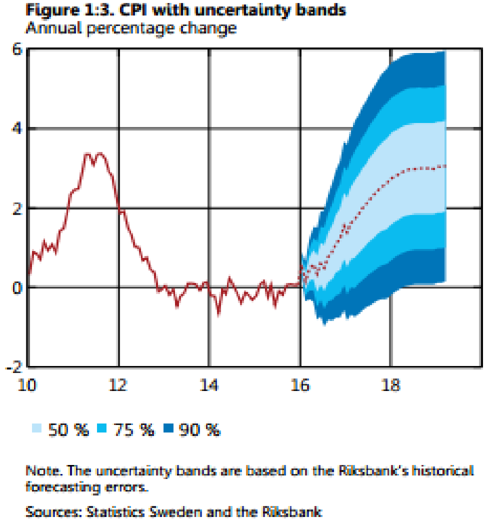
- Central Banks communicate forecasts of macroeconomic variables
  - ✓Risks to forecasts based on which policy stance taken – need to be minimized
  - ✓Forecast errors – required to be estimated
  - ✓Risk quantification – Confidence interval around forecasts

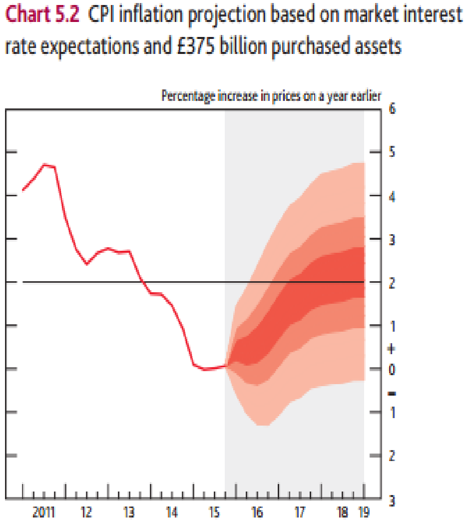
- Central Banks publish fan charts to
  - ✓communicate risks/ uncertainty associated to forecasts of inflation and growth
  - ✓Fan chart – communication of quantified uncertainty around forecast using confidence intervals

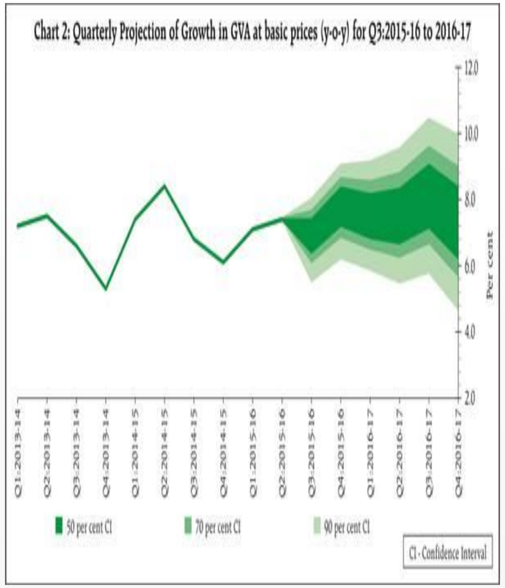
Fan-Charts for Economy’s Growth and Inflation: India, Europe, Sweden and UK

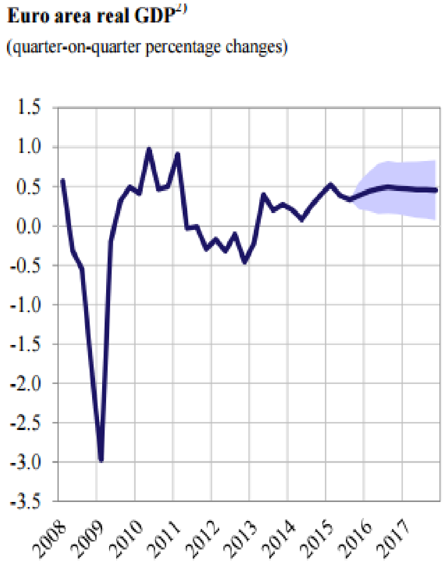


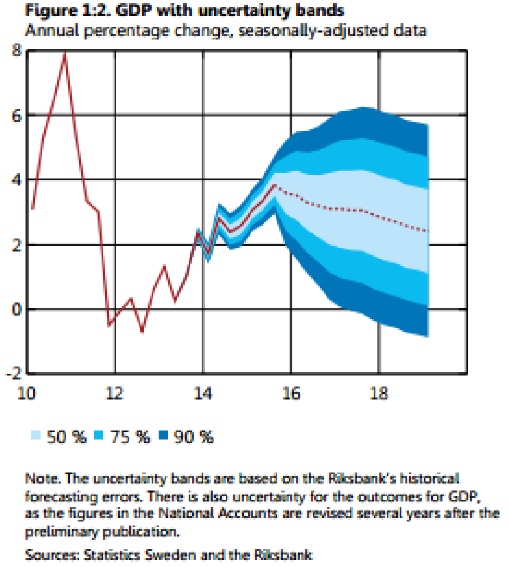


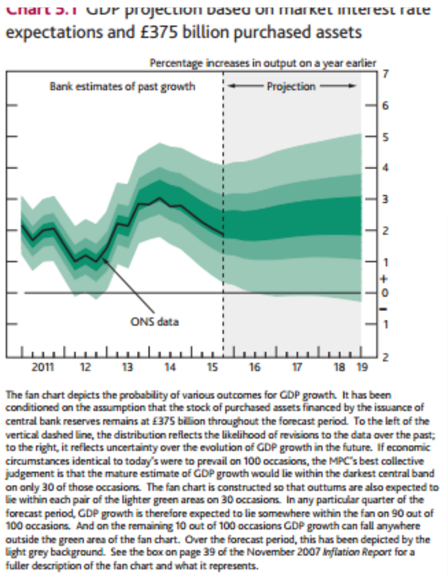












Fan Chart

Fan chart depicts:

- ✓a measure of central tendency (central view)
- ✓a view on degree of uncertainty
- ✓a view on balance of risk (often non-symmetric)

Distribution in fan chart is asymmetric if risk assessment is either skewed upside or downside

Literatures

- Theory on incorporating asymmetric risks to forecasts using two-piece Normal distribution
  - ✓by M. Blix and P. Sellin in 1998 – application in Indian context discussed by N. Banerjee and A. Das in 2011
- Methodology for fan chart for government deficit and debt ratios over medium-term
- Bayesian fan charts
- Quantification of skewness in fan charts by assessing probability of future dip using probit model

The Forecasting Distribution

Join halves of two Normal distributions with same mode but different standard deviations

$$f(x) = \begin{cases} C\ exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2\right], if\ -\infty < x \leq \mu \\ C\ exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right], if\ x > \mu \end{cases}$$

$$with\ \ C = \frac{\sqrt{2}}{\sqrt{\pi}(\sigma_1+\sigma_2)}, -\infty < \mu < \infty, \sigma_1, \sigma_2 > 0$$

$$Balance\ of\ Risk\ \ p = P[X \leq \mu] = \frac{\sigma_1}{\sigma_1+\sigma_2}$$

Confidence Interval for fan chart

Quantile k of the distribution:

$$k = \begin{cases} \mu + \sigma_1 \Phi^{-1}\left(\frac{\alpha}{C\sqrt{2\pi}\sigma_1}\right), & , if\ \alpha \leq p \\ \mu + \sigma_2 \Phi^{-1}\left(\frac{\alpha + C\sqrt{2\pi}\sigma_2 - 1}{C\sqrt{2\pi}\sigma_2}\right), & if\ \alpha > p \end{cases}$$

$$0 \leq \alpha \leq 1$$

Construction

μ_{xt}

- Forecast for time t

p

- Balance of risk based on judgement/ data-driven

γ

- Skewness parameter $\gamma = \frac{2p-1}{1-2p+2p^2}$

σ

- $\sigma = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T-1}(x_t - \mu_{xt})^2}$, $\quad x_t = actual\ at\ time\ t \quad \mu_{xt} = forecast\ for\ time\ t$

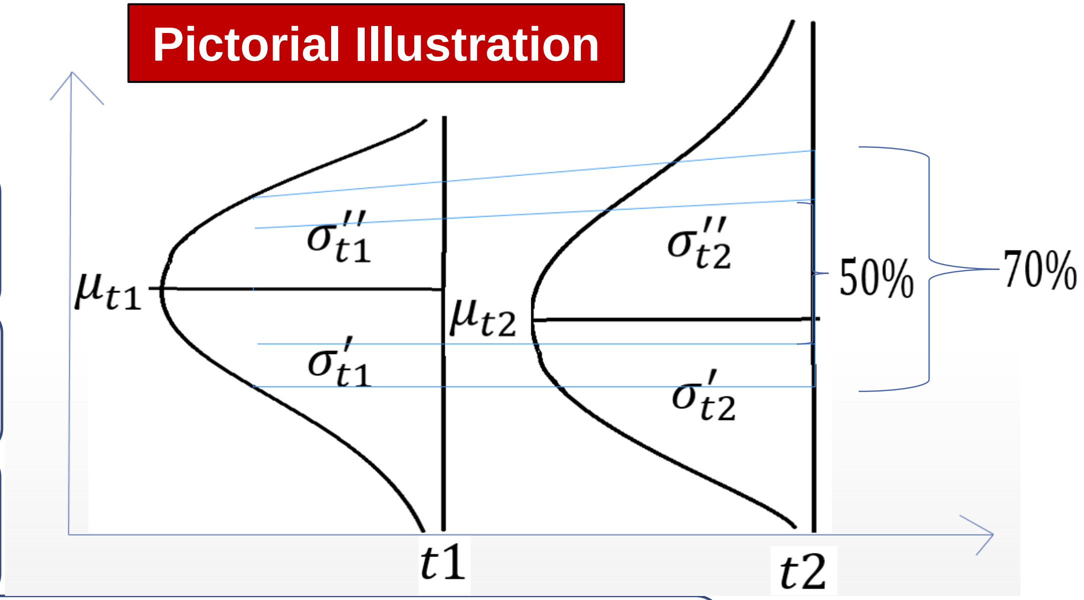
σ_1, σ_2

- $\sigma_1 = \frac{\sigma}{\sqrt{1-\gamma}}, \quad \sigma_2 = \frac{\sigma}{\sqrt{1+\gamma}}$

k

- $k = \begin{cases} \mu + \sigma_1 \Phi^{-1}\left(\frac{\alpha}{C\sqrt{2\pi}\sigma_1}\right), & if\ \alpha \le p \\ \mu + \sigma_2 \Phi^{-1}\left(\frac{\alpha + C\sqrt{2\pi}\sigma_2 - 1}{C\sqrt{2\pi}\sigma_2}\right), & if\ \alpha > p \end{cases}, \quad 0 \le \alpha \le 1$

Pictorial Illustration



Way Forward – Bivariate Fan Chart

- Macroeconomic variables of interest are inter-dependent – literature contains limited work on dimensional extension of fan charts

- M. Blix and P. Sellin in 2000 derived bivariate fan chart of inflation and output
  - ✓opined that bivariate framework allows for arriving at forecast of a variable conditional on information on other
  - ✓estimating confidence bands from econometric forecasting model itself, is not possible as multiple models are used, specific information related to particular forecast period and important subjective judgments cannot be incorporated
  - ✓assumed that error distribution of inflation and output forecasts are separately two-piece Normal and linked the standardized form of these variables into a standard Bivariate Normal (BN) distribution through an estimated correlation coefficient; then, derived conditional distribution of one variable given another

- **Alternative:** Initial assumption of joint distribution of two variables

Initial Thoughts

What is a Bivariate Fan Chart? – Joint error distribution around two-dimensional forecast coordinate

Why a Bivariate Fan Chart?

- impact of movement of one variable on future path of other
- known information on forecast error of one variable
  - ✓may improve forecast of other variable – useful when information on variables are published at different lags

Framework

- $(X, Y) \sim$ Bivariate Normal
- Join halves of 2 different Bivariate Normal distributions
- Each halve has its own set of variance matrix
- Forecasts of X and Y for a period are μ_x and μ_y, respectively – forecast coordinate (μ_x, μ_y)

Bivariate Fan Chart – Methodology

Handling Asymmetry

- Asymmetry in risks associated to forecast coordinates (μ_x, μ_y)
  - ✓different in each of the four quadrants of (X, Y) plane
- Join sections of two Bivariate Normal distributions

$$BN(\mu_x, \mu_y, \sigma_{1x}, \sigma_{1y}, \rho) \qquad \text{and} \qquad BN(\mu_x, \mu_y, \sigma_{2x}, \sigma_{2y}, \rho)$$

- ✓same modal coordinate (μ_x, μ_y), $\quad -\infty < \mu_x, \mu_y < \infty$
- ✓same correlation coefficient ρ, $\quad -1 \leq \rho \leq 1$
- ✓$\sigma_{1x}(> 0)$, $\sigma_{1y}(> 0)$ in first distribution
- ✓$\sigma_{2x}(> 0)$, $\sigma_{2y}(> 0)$ in second distribution

Bivariate Fan Chart – Methodology......contd.

The Cut Line Equation – Equation of line to cut distribution into two pieces

- x-axis, y-axis with $x, y \in \mathbb{R}$ and z-axis for probability with $0 \leq z \leq 1$
- Which half of distribution is to be taken?
- Cut line may either be aligned to x-axis, y-axis or at any other angle
- A general cut line $\left(y - \mu_y\right) = m(x - \mu_x),\ x, y \in \mathbb{R}$ may be used
  - ✓slope of cut line is $m \in \mathbb{R}$
- Join pieces of $BN\left(\mu_x, \mu_y, \sigma_{1x}, \sigma_{1y}, \rho\right)$ and $BN\left(\mu_x, \mu_y, \sigma_{2x}, \sigma_{2y}, \rho\right)$

Bivariate Fan Chart – Methodology……contd.

Probability Distribution Function

$$f(x,y) = \begin{cases} C\ exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{1x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{1x}}\right)\left(\frac{y-\mu_y}{\sigma_{1y}}\right) + \left(\frac{y-\mu_y}{\sigma_{1y}}\right)^2\right\}\right], \\ \qquad if\ (y-\mu_y) \le m(x-\mu_x), \qquad x,y \in \mathbb{R} \\ C\ exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_{2x}}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_{2x}}\right)\left(\frac{y-\mu_y}{\sigma_{2y}}\right) + \left(\frac{y-\mu_y}{\sigma_{2y}}\right)^2\right\}\right], \\ \qquad if\ (y-\mu_y) > m(x-\mu_x), \qquad x,y \in \mathbb{R} \end{cases}$$

C is a constant such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$

Balance of Risks (BOR) – the Highest risk to forecast coordinate in quadrant with the highest BOR

$$\begin{aligned} &BOR\ in\ quadrant - 1, \qquad p(1) = P\{X > \mu_x, Y > \mu_y + m(x-\mu_x)\} \\ &BOR\ in\ quadrant - 2, \qquad p(2) = P\{X \le \mu_x, Y > \mu_y + m(x-\mu_x)\} \\ &BOR\ in\ quadrant - 3, \qquad p(3) = P\{X \le \mu_x, Y \le \mu_y + m(x-\mu_x)\} \\ &BOR\ in\ quadrant - 4, \qquad p(4) = P\{X > \mu_x, Y \le \mu_y + m(x-\mu_x)\} \end{aligned}$$

Bivariate Fan Chart – Methodology……contd.

Alternate Parametrization $f(x,y) = \begin{cases} BN\left(\mu_x, \mu_y, \frac{\sigma_x}{\sqrt{(1-\gamma_x)}}, \frac{\sigma_y}{\sqrt{(1-\gamma_y)}}, \rho\right), if\ (y-\mu_y) \le m(x-\mu_x)\ x, y \in \mathbb{R} \\ BN\left(\mu_x, \mu_y, \frac{\sigma_x}{\sqrt{(1+\gamma_x)}}, \frac{\sigma_y}{\sqrt{(1+\gamma_y)}}, \rho\right), if\ (y-\mu_y) > m(x-\mu_x)\ x, y \in \mathbb{R} \end{cases}$

γ_x and γ_y are inverse skewness indicators, σ_x and σ_y are uncertainty parameters.

Solving for parameters

$$\sigma_{1x} = \frac{\sigma_x}{\sqrt{1-\gamma_x}}, \qquad \sigma_{1y} = \frac{\sigma_y}{\sqrt{1-\gamma_y}}, \qquad \sigma_{2x} = \frac{\sigma_x}{\sqrt{1+\gamma_x}}, \qquad \sigma_{2y} = \frac{\sigma_y}{\sqrt{1+\gamma_y}}$$

$$p_x = \frac{\sigma_{1x}}{\sigma_{1x}+\sigma_{2x}} \Rightarrow \gamma_x = \frac{2p_x - 1}{1 - 2p_x + 2{p_x}^2}, \qquad p_y = \frac{\sigma_{1y}}{\sigma_{1y}+\sigma_{2y}} \Rightarrow \gamma_y = \frac{2p_y - 1}{1 - 2p_y + 2{p_y}^2}$$

$$\sigma_x \ proxied\ by\ {\sigma_x}^* = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T-1}(x_t - \mu_{xt})^2}$$

$$\sigma_y \ proxiedby\ {\sigma_y}^* = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T-1}\left(y_t - \mu_{yt}\right)^2}$$

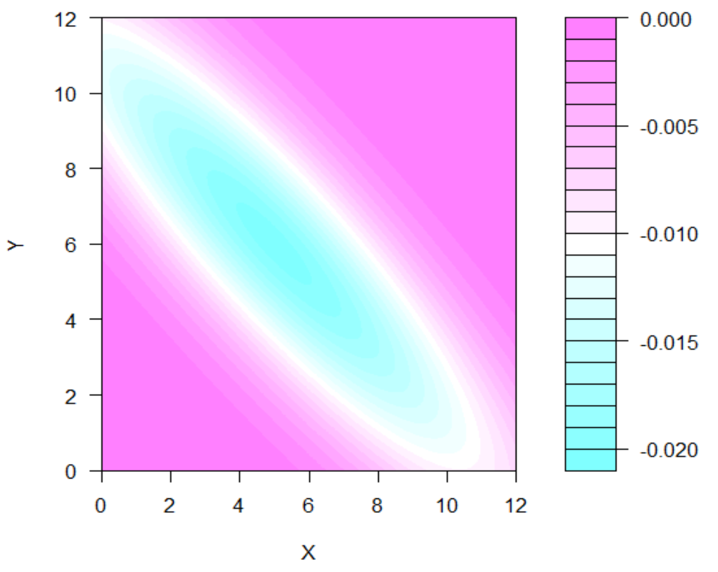
μ_{xt} and μ_{yt}	=	forecasts of X and Y, respectively for t
x_t and y_t	=	realized values of X and Y, respectively at t
T	=	number of periods considered from history

$$\rho\ proxied\ by\ \rho^* = \frac{\sum_{t=1}^{T-1}(x_t-\bar{x})(y_t-\bar{y})}{\sqrt{\sum_{t=1}^{T-1}(x_t-\bar{x})^2}\sqrt{\sum_{t=1}^{T-1}(y_t-\bar{y})^2}}$$

Bivariate Fan Chart – Methodology……contd.

- Confidence bands for bivariate distribution is a set of equi-probability contours
  - ✓ an equi-probability contour joins all coordinates with same probability
- Consider all coordinates (x, y) for which density is same, say δ
  - ✓ solving this equation, equi-probability contour can be derived
- Thus, for constructing bivariate fan chart, values of seven parameters required
  - ✓ baseline forecasts (μ_x, μ_y), uncertainties σ_x, σ_y, balance of risks p_x, p_y and correlation coefficient ρ
- Conditional Distribution
  - ✓ Given known information on X, distribution of Y derived as $f_Y(y|x) = \frac{f(x,y)}{g(x)}$
  - ✓ mean of conditional (revised forecast of Y) expected to be better than original forecast μ_y
- Quantile from Conditional Distribution → Conditional fan chart for Y

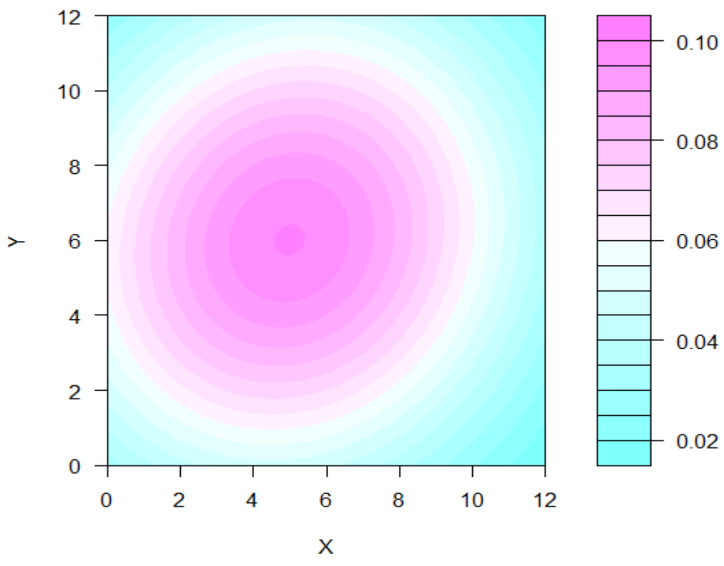
Bivariate Fan Chart – Contours for Demonstration



$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \rho = -0.9$

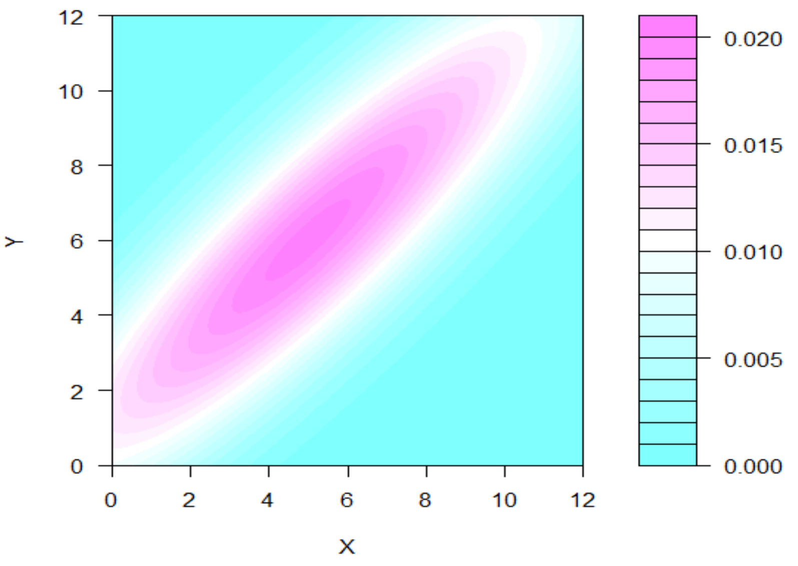
Bias in 2nd and 4th quadrants

Opposite risks to forecasts



$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \rho = 0.1$

Risk symmetric in all quadrants

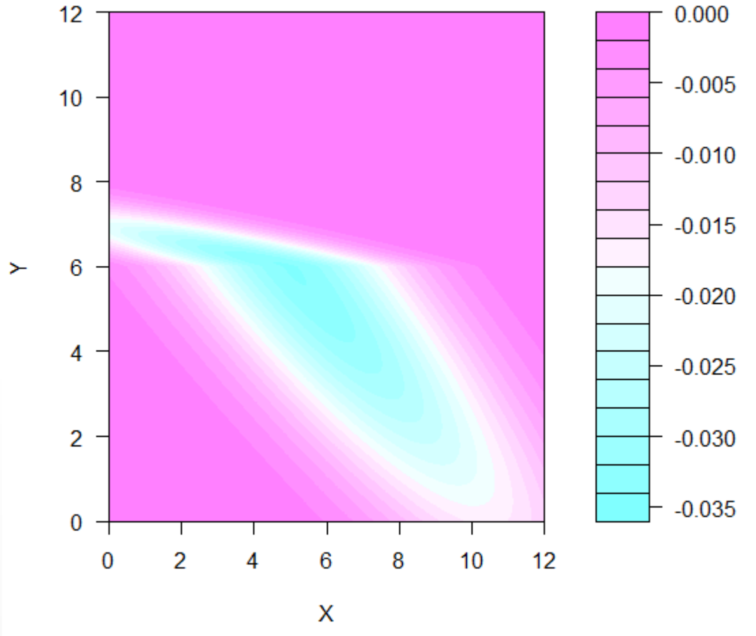


$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \rho = 0.9$

Bias in 1st and 3rd quadrants

Unidirectional risks to forecasts

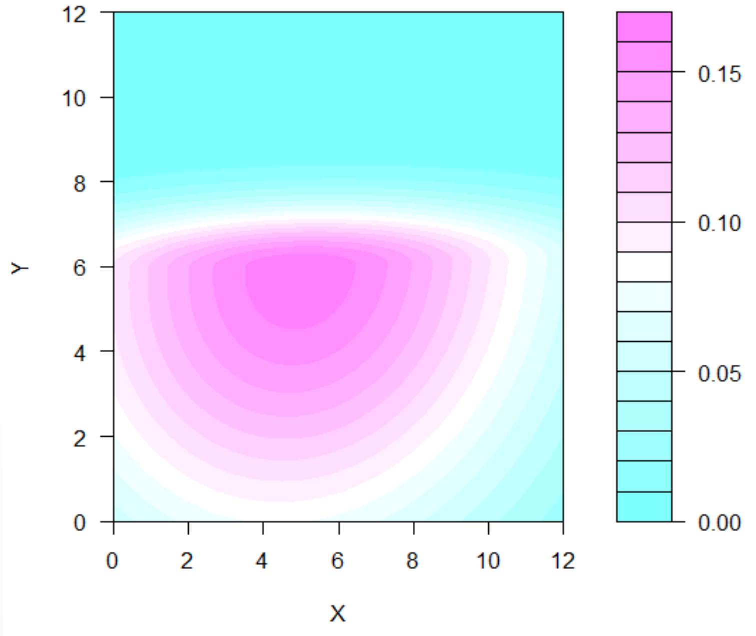
Bivariate Fan Chart – Contours for Demonstration……contd.



$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} > \sigma_{2y}, \rho = -0.9$

Bias in 4th quadrant

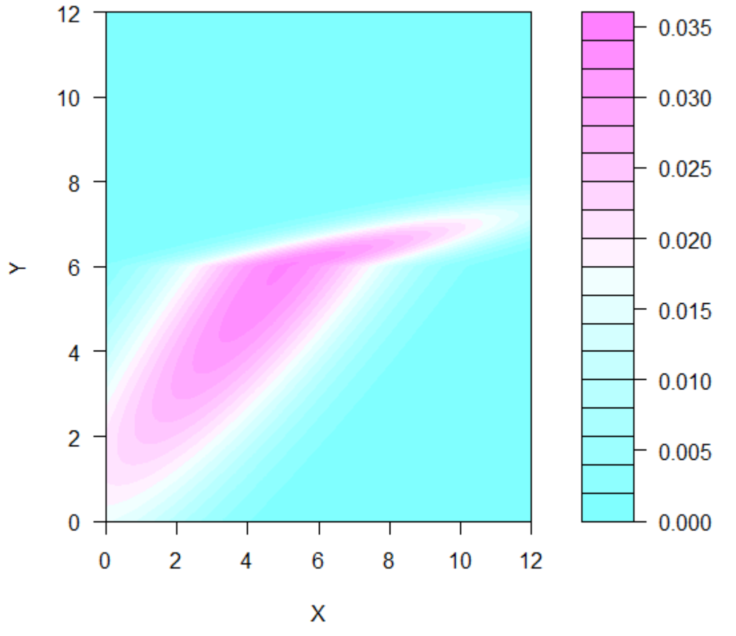
Upside risk to X and downside risk to Y



$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} > \sigma_{2y}, \rho = 0.1$

Bias in 3rd and 4th quadrants

Balanced risk to X and downside risk to Y

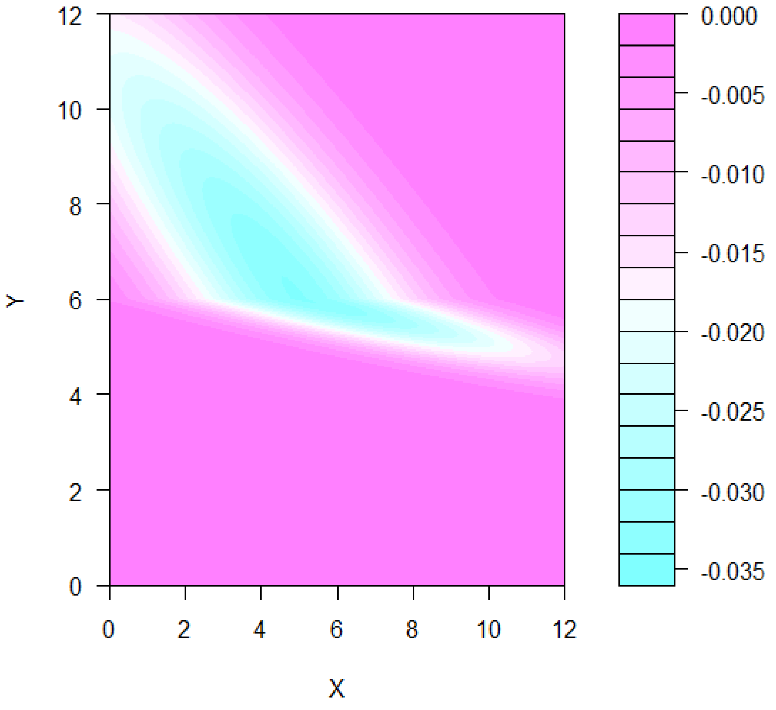


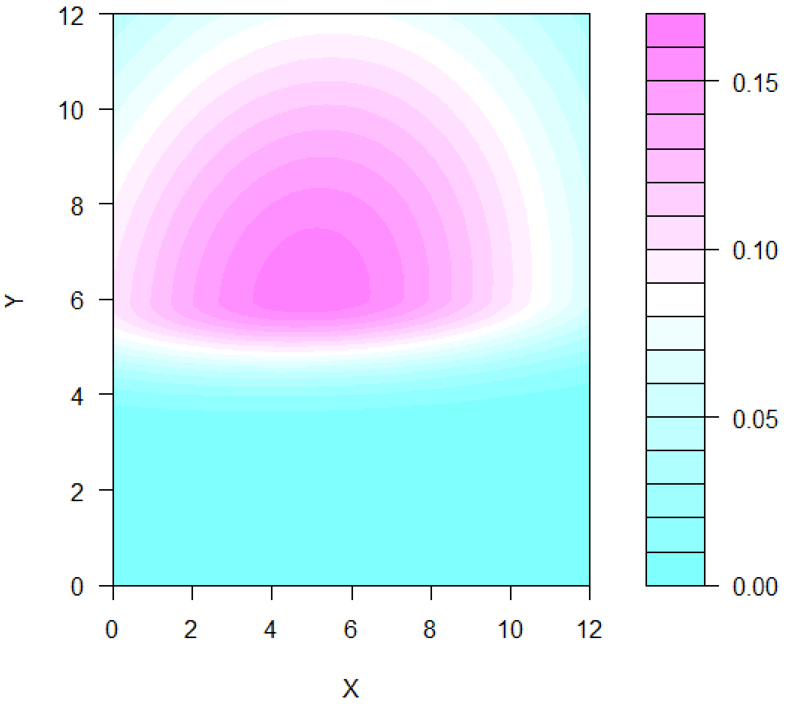
$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} > \sigma_{2y}, \rho = 0.9$

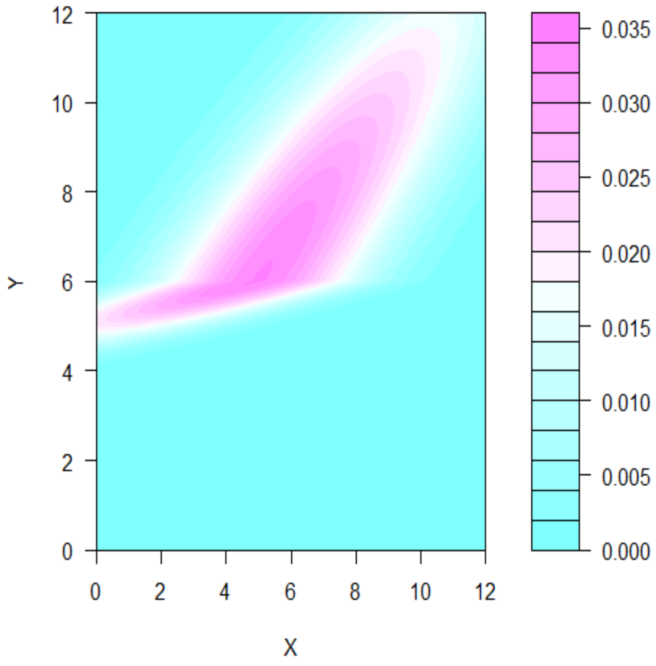
Bias in 3rd quadrant

Downside risk to both X and Y

Bivariate Fan Chart – Contours for Demonstration……contd.







$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} < \sigma_{2y}, \rho = -0.9$

Bias in 2nd quadrant

Downside risk to X and upside risk to Y

$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} < \sigma_{2y}, \rho = 0.1$

Bias in 1st and 2nd quadrants

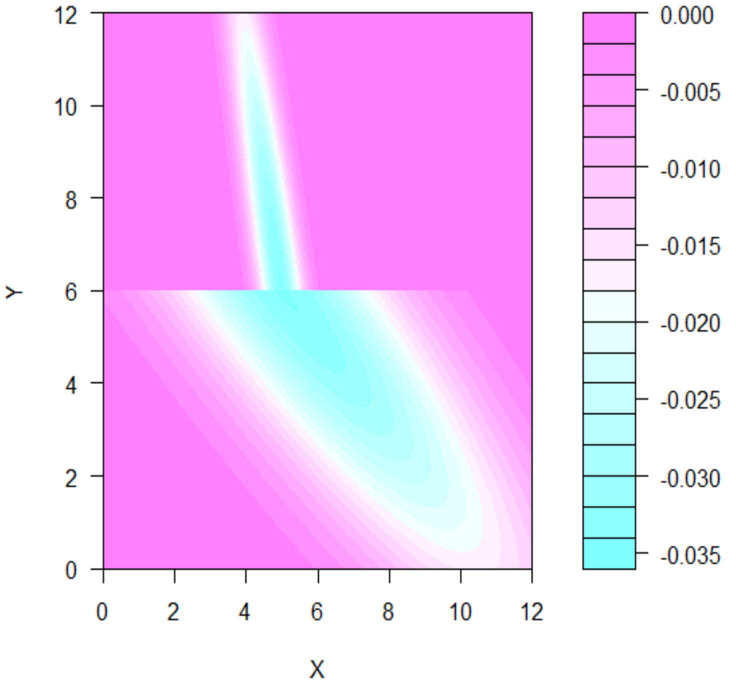
Balanced risk to X and upside risk to Y

$\sigma_{1x} = \sigma_{2x}, \sigma_{1y} < \sigma_{2y}, \rho = 0.9$

Bias in 1st quadrant

Upside risk to both X and Y

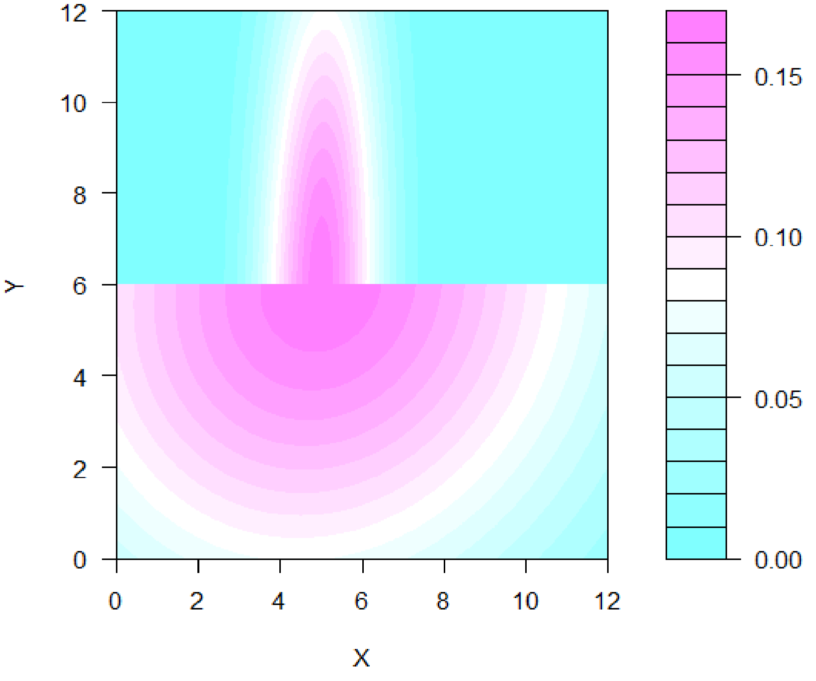
Bivariate Fan Chart – Contours for Demonstration……contd.



$\sigma_{1x} > \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \rho = -0.9$

Bias in 4th quadrant

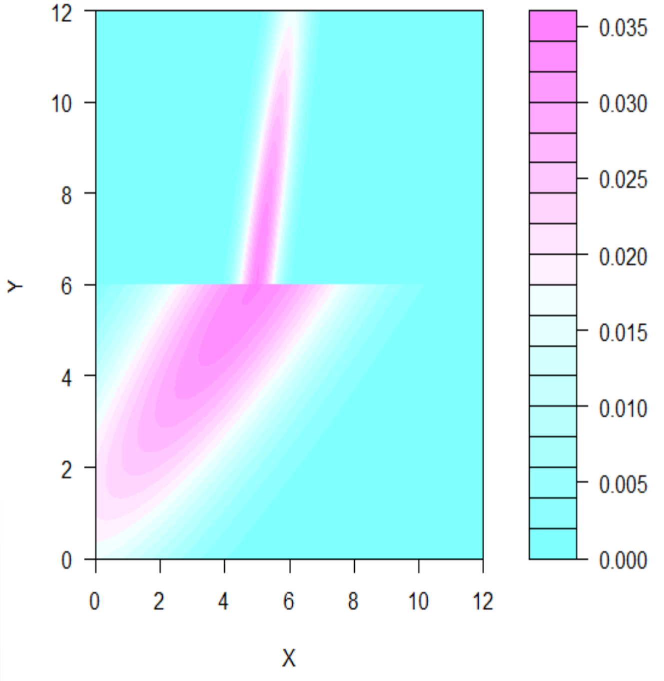
Upside risk to X and downside risk to Y



$\sigma_{1x} > \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \rho = 0.1$

Bias in 3rd and 4th quadrants

Balanced risk to X and downside risk to Y



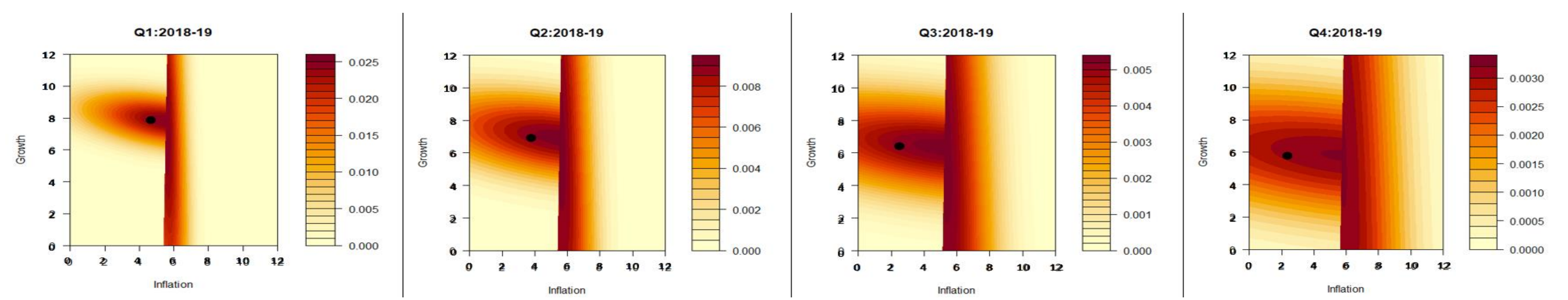
$\sigma_{1x} > \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \rho = 0.9$

Bias in 3rd quadrant

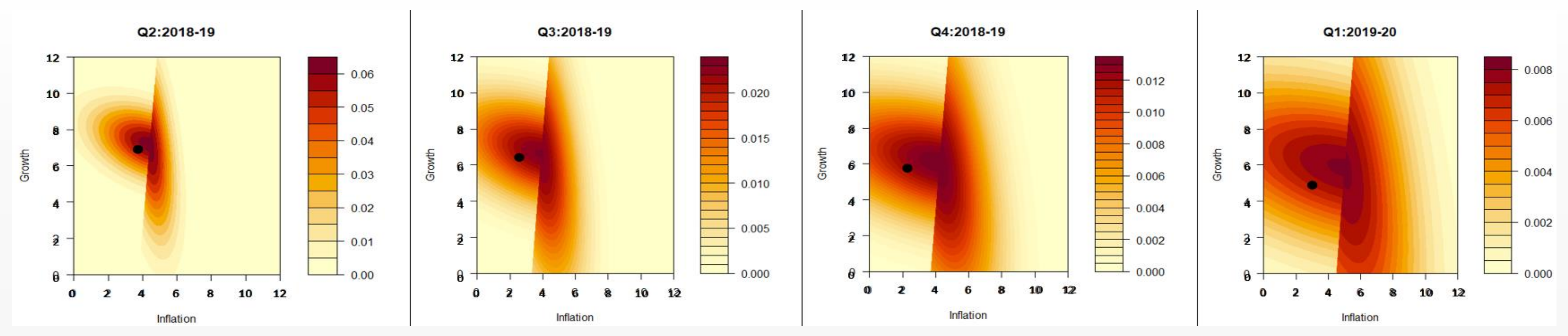
Downside risk to both X and Y

Numerical Illustration

Q1:2018-19 to Q4:2018-19 $p_x = 0.2,\ p_y = 0.9,\ \theta = 89^o$ ● **indicates actual inflation and growth coordinate** (x, y)

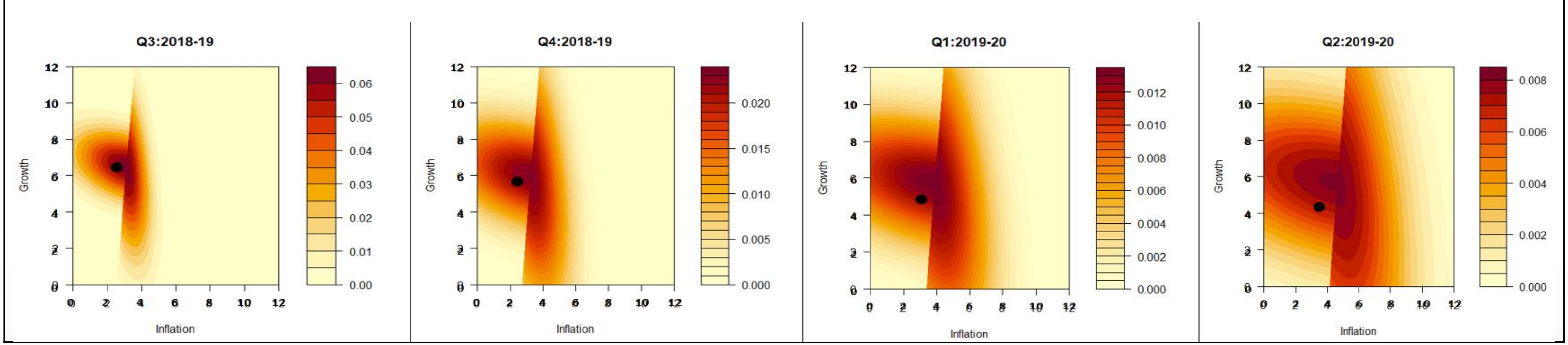


Q2:2018-19 to Q1:2019-20 $p_x = 0.3,\ p_y = 0.7,\ \theta = 85^o$



Numerical Illustration……contd.

Q3:2018-19 to Q2:2019-20 $p_x = 0.3,\ p_y = 0.7,\ \theta = 85^o$ ● **indicates actual inflation and growth coordinate** (x, y)



Numerical Illustration……contd.

Performance of Conditional Fan Charts										
Quarter	θ	p_x	p_y	Average absolute deviation (basis points) from actual growth		Average absolute deviation (basis points) of actual growth from central path		Average Width (basis points) of confidence band		
				Univariate fan	Conditional fan	Univariate fan	Conditional fan	Band (per cent)	Univariate fan	Conditional fan
Q1:2018-19 to Q4:2018-19	89^o	0.2	0.9	14.3	6.1	1341.2	7.9	50	1791.9	278.9
								70	2689.9	428.6
								90	4074.7	680.8
Q2:2018-19 to Q1:2019-20	85^o	0.3	0.7	30.5	12.7	205.3	34.8	50	573.9	312.1
								70	869.1	487.6
								90	1359.9	980
Q3:2018-19 to Q2:2019-20	85^o	0.3	0.7	53.3	21.5	183.5	45.5	50	566.5	321.5
								70	857.9	506.9
								90	1342.3	1078.8

Conclusion

- In the current scenario of uncertainty, Central Bank's communication plays a crucial role
- Univariate fan chart is a traditional way of communicating uncertainty around forecasts
  - ✓Central Banks publish fan charts to communicate the risks/ uncertainty associated to forecasts of macroeconomic variables – inflation and growth
- Bivariate fan chart can be an improvement in the manner of communication
  - ✓can be used for representing error bands – for forecasts of two linearly related variables
  - ✓it incorporates the asymmetric risks to forecast coordinates
  - ✓suitable when information of two variables released at different lags
  - ✓conditional fan chart can be derived for one variable given information on other
  - ✓additional information on one variable refines the forecast of the other

THANK YOU