Alternative tools of trade for central banks and other financial institutions: foreign exchange liquidity options[1]

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1. Introduction

We propose a class of exotic derivative securities, called the foreign exchange liquidity options (FELO), to be offered by a central bank to the domestic banks to allow them to borrow foreign currency denominated short-term loans from the central bank reserves in the event of sudden up moves in a foreign currency to the domestic currency exchange rate. Since the FELO provide the option holder banks with foreign exchange liquidity when the spot price of the contract foreign currency crosses a predetermined contract level, they can be viewed as an insurance the central bank sells to the domestic banks against potential liquidity crises. The most distinguishing feature of the FELO is that they give their holders not the right to buy, but the right to borrow foreign currency from the central bank reserves, for a predetermined term at the then current foreign currency country default-free interest rates on the exercise date of the option, if the option gets exercised. Since the option holder is not default-free, the price of this insurance is the market price of the default risk of the option holder. Further, since the FELO do not involve transactions in which the domestic currency is exchanged for a foreign currency, they allow the central banks to provide foreign currency liquidity to the domestic market at times of liquidity crises without affecting the domestic money supply.

In terms of their effect on the central bank balance sheets, the FELO are similar to the gold loans by which the central banks earn interest on their gold reserves that otherwise earn none. Gold Fields Minerals Service Gold Survey (2005) estimates that the annual volume of gold loans by central banks nears 5,000 tones. The same survey reports also that the total annual mine production is roughly about 2,500 tones whereas the annual central bank gold sales are about 656 tones. These figures indicate how popular the gold loans are and how much liquidity they add to the gold market. One reason for this popularity is that by lending the gold they own, the central banks obtain interest on their idle gold without removing the rented gold from their reserves. As Takeda (2006) points out, the central banks are not required, only recommended by the IMF, to exclude the rented gold from their reported reserve totals.

As has been witnessed time and again over the past few decades, although the long run equilibrium of foreign exchange markets depends on the macroeconomic fundamentals, the global capital flows, international liquidity conditions of the global financial markets and expectations of the market participants may create short-term departures of the exchange rate from the equilibrium levels. The central banks in "dollarized" economies where the exchange rate pass-through to prices is high have been particularly sensitive to such

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exchange rate fluctuations, because in such economies the exchange rate stability is a prerequisite for price, as well as financial, stability. More often than not, the response of the central banks to such crises has been to intervene in the currency market either to “correct” such short run departures or to reduce the potential and/or observed exchange rate volatility.

The conventional tools the central banks use for foreign exchange management are the direct intervention, selling/purchasing auctions and foreign exchange warehouse utilities. In the case of the first two, the domestic currency is exchanged for a foreign currency whereas in the case of the warehouse utility, a foreign currency is rented out in exchange for a foreign currency denominated collateral. Since the FELO can be viewed as exchange rate contingent short-term reverse repurchase agreements into which the central bank enters with the domestic banks (in which the domestic banks borrow foreign currency from the central bank reserves using their foreign currency denominated debt as “collateral”), the FELO resemble the warehouse utility the most. The main difference is that the launch of a warehouse utility is totally under the discretion of the central bank and it is not used unless a severe crisis hits the economy. Under the FELO agreements, on the other hand, the domestic banks are assured to have access to foreign currency reserves whenever the contract conditions are met. Therefore, unlike the FELO, the warehouse utility is not an insurance against foreign currency liquidity crises and gets employed only after the fact.

The aforementioned conventional tools are not the only tools available to the central banks and several derivatives based alternatives have also been employed. Neely (2001) reports the following derivatives based central bank interventions: the 1993 sale of put options by Bank of Spain so as to induce the market participants to support the spot value of peseta; the 1996–2001 sale of put options on the US dollar by Banco de Mexico for reserve accumulation purposes; the 1997 forward market purchase of bath by Bank of Thailand; and the ongoing efforts of the Reserve Bank of Australia to sterilize some of its spot interventions via swap transactions in the foreign exchange market. In addition, Breuer (1999) reports the utilization of the Hong Kong dollar put/the US dollar call options by the Bank of China in order to support the currency peg in 1998 whereas Mandeng (2003) reports the option-based foreign exchange market interventions of the Central Bank of Colombia between 1999 and 2003 in order to accumulate/decumulate reserves and reduce exchange rate volatility, to name but a few. Although the central banks started to use the derivative products for foreign exchange management in the early 1990s, many of the theory papers proposing various derivatives instruments predate their practical use in central bank currency interventions by at least a decade, and the literature on such proposals is growing. We give a brief review of the literature below. To our knowledge, all of the proposed derivative products in the literature involve the sale of a foreign currency and, as we document in our literature review, such derivatives based interventions suffer from several drawbacks. Since in the case of the FELO the foreign currency is not sold but rented, the FELO are free from many of the drawbacks mentioned above.

The use of options to borrow is not new to central banks. For example, in 1999 the Federal Reserve Bank of New York auctioned the so-called millennium date change (Y2K) options which gave the option holder bond dealers to execute overnight repos with the Federal Reserve Bank of New York at 150 basis points above the prevailing target federal funds rate. Through Y2K options, the Fed assured the availability of a large amount of liquidity to the bond dealers around the millennium change. Sundaresan and Wang (2006) investigated the impact of Y2K options on the liquidity premium of the US Treasury Securities, and based on the implied volatilities of Y2K options and the aggressiveness of demand for the instruments, concluded that Y2K options indeed eased the fears of bond dealers, contributing to a drop in the liquidity premium. The similarities between the FELO and Y2K options are obvious. However, the FELO differ from Y2K options in two aspects: Firstly, in the case of Y2K options, the central bank assured the injection of domestic currency to the domestic market whereas in the case of FELO, the central bank assures the injection foreign currency to the domestic market. More importantly, in the case of Y2K options, the time at which a liquidity crisis could occur was known in advance whereas in the case of the FELO, the time of the onset of such a crisis is uncertain.

It should be noted that the FELO are similar also to the state-contingent government bonds proposed by Holmstrom and Tirole (1996). The difference is that in our case the "government", that is, the central bank, does not sell state-contingent "government" bonds, but sells state-contingent "foreign government" bonds from the central bank reserves. Based on their theory, Holmstrom and Tirole suggested that governments should issue state-contingent bonds that pay off only when the private sector experiences a shortage of liquidity. Later, Holmstrom and Tirole (1998) argued that state-contingent bonds were not used in reality because there was *"no aggregate, measurable state that unequivocally identifies times when firms should be provided more liquidity."* Their view is that *"rather than the use of bonds that are contingent on a few foreseeable and verifiable variables, a discretionary policy may be more effective."*

One problem with the FELO, as well as with all other derivatives based strategies, is that there is *"no aggregate, measurable state that unequivocally identifies times when"* the domestic banks should be provided with foreign exchange liquidity, either. Specifically, there are two issues with providing foreign exchange liquidity to the domestic banks through the FELO. The first issue is the timing: since the time a liquidity crisis begins is not foreseeable, when should the central bank offer the FELO? The second issue is the choice of the exchange rate barriers associated with the FELO: above (below) what exchange rate level does a liquidity crisis begin (end)? Neither of these questions has an answer.

Therefore, we suggest that central banks should sign the contracts whenever a domestic bank wishes to do so and give the timing discretion to the domestic banks so that the banks get insured whenever they perceive that a liquidity crisis is imminent. Since the insurance fee the banks are charged is the market price of their default risk, they would be indifferent between borrowing from the central bank and private lenders when they do not perceive an imminent liquidity crisis. Further, since the loans are short-term and the short-term information asymmetry between the central bank and the domestic banks is negligible, signing such contracts during normal times would allow the central bank to earn higher interest on its reserves for a small default risk. Furthermore, since the FELO are private contracts between the central bank and domestic banks, if a high risk bank approaches the central bank singularly, then the central bank can always discourage the bank by asking a large insurance fee.

As for the choice of the exchange rate barriers associated with the FELO, we suggest that the central bank should leave this decision also to the domestic banks, provided that the chosen barriers satisfy the simple conditions we spell out below. Since the barriers are among the variables that determine the contract price, the central bank is appropriately compensated for any choice. Further, if the central bank chooses the barriers, this could be interpreted as a signal by the domestic banks that the central bank is exchange rate targeting. In a floating exchange rate regime, the central bank neither can send such signals nor can target the exchange rate level.

The rest of the paper is organized as follows. In the next section, we specify the FELO in detail, review the related literature and discuss the monetary policy implications of the FELO. In Section 3, we price the FELO in a general setting and then specialize our result to the Republic of Turkey in a five-factor affine jump diffusion model. In Section 4, we discuss our estimation procedure, estimate the model and present our estimation results. In Section 5, we discuss the computation of the FELO price through Monte Carlo simulations first, and then present and discuss our results for a set of model parameters. Finally, In Section 6, we conclude.

2. The foreign exchange liquidity options

2.1 Review of the related literature

As Eaton and Turnovsky (1984) noted, proposals of derivatives based strategies as additional means to intervene in the foreign exchange market can be traced back to Keynes. Keynes (1930) wrote:[4] *"I conceive of (Central Banks) as fixing week by week not only their official rate of discount, but also the terms on which they are prepared to buy or sell forward exchange on one or two leading foreign exchange centres and the terms on which they are prepared to buy or sell gold points."* Of course, at the time Keynes made his proposal, the gold standard was still in place. Analyzing the role of forward exchange market in stabilizing domestic income against stochastic exchange rate disturbances after the demise of the gold standard and Bretton Woods, Eaton and Turnovsky (1984) found the following: *"Forward market intervention does not provide monetary authorities additional leverage in stabilizing income beyond unsterilized spot market intervention. Intervention rules based on reactions to both the forward and spot exchange rates, however, can outperform intervention policies responding to the spot rate alone, regardless of the market in which the intervention occurs."* It is with this, and similar conclusions of other authors from their analyses of other types of derivatives based strategies, in mind that we propose our foreign exchange liquidity options to central banks as additional tools for foreign exchange management.

With the rapid expansion of derivatives transactions since the latter half of the 1980s, many central banks started to investigate the potential use of derivatives products in their foreign exchange management efforts in the late 1980s and early 1990s. A comprehensive survey of the literature after 1982 until 1993 can be found in Hali(1993). In 1995, two years after Bank of Spain sold put options to induce the market participants to support the spot value of peseta, Taylor(1995) suggested that central banks should buy put options written on the domestic currency in order to accumulate reserves at reasonable prices when there is a depreciating trend of the currency due to a speculative attack. In his proposal, he argued that central banks may then reinject the reserves accumulated through the put options to the market to mitigate the downward pressure on the domestic currency.

Breuer (1999) demonstrated that a strategy of central banks buying currency options is flawed because of the destabilizing impact on the exchange rate of the dynamic hedging by market makers, as well as because of a moral hazard incentive for the central bank to influence the spot exchange rate. He then suggested an alternative strategy whereby the central bank sells options, either individually or packages in a "strangle". He argued that under certain conditions this strategy could lower exchange rate volatility, may boost the credibility of an exchange rate target zone and could have lower expected costs than spot market interventions. He pointed out, however, that selling options exposes the central bank to an unlimited loss potential.

In the same year, Wiseman (1999) suggested that central banks should commit themselves to frequent and regular auctions of short-dated physically-delivered foreign exchange options. His suggestion is very similar to that of Breuer(1999), since both mechanisms are based on the exchange rate volatility reducing delta hedges of the option buyers. Stremme (1999) analyzed the proposal of Breuer(1999) and Wiseman(1999) in scenarios in which market participants, rather than simply hedge their option positions, strategically exploit the leverage provided by such options. He concluded that options issued by central banks create an incentive for speculators to manipulate exchange rates which would otherwise not exist. Put differently, Stremme argued that rather than protecting the domestic currency against speculative attacks, options in fact create additional tools for such attacks. The FELO we

[4] "A Treatise on Money", p. 327.

propose are immune from such attacks for a number of reasons: Firstly, similar to Y2K options the Federal Reserve Bank of New York auctioned to the bond dealers only, we suggest that the FELO should be offered by the central bank to the domestic banks only. Secondly, we suggest that the FELO should not be auctioned by the central bank, but should be signed by the central bank and each domestic bank as over-the-counter contracts. Since over-the-counter contracts are private contracts between the central bank and each domestic bank, the terms of the contracts are private information which need not be disclosed publicly.

Later in 2000, Blejer and Schumacher (2000) questioned the rationale for central banks to use derivative products and contingent liabilities, and suggested a portfolio approach to analyze the whole spectrum of central bank contingent liabilities. They identified five essential reasons for central banks to use derivative products in foreign exchange management:

1. to support the development of the derivatives market;
2. to smooth the volatility in the spot market;
3. to attain a desired level or band of the exchange rates;
4. to sustain financial stability while defending a fixed exchange rate regime; and, finally,
5. to add to the existing tools an additional tool for foreign exchange liquidity management.

The FELO we propose are essentially for the second and fifth reasons Blejer and Schumacher identifed, not to mention that we are concerned also with the financial stability of the domestic financial sector irrespective of whether the exchange rate regime is fixed or floating.

In closing our by no means comprehensive review of a broad literature, we mention also the recent notable work of Suh and Zapatero (2007). In this recent work, Suh and Zapatero introduce a class of new financial instruments that they call quadratic options, which are designed to mitigate the exchange rate volatility without the unlimited loss potential Breuer(1999) identified. Their literature review brings the interested reader up to date and we refer the reader to their review for additional coverage. Based on their quadratic options, they compare the spot only, option only, spot and option and no intervention strategies, and find that the best alternative for a central bank is to combine their options with spot intervention. Unfortunately, the stabilization potential of the contracts Suh and Zapatero propose rely heavily on the hedging activities of the issuer investment banks, as well.

2.2 Specifcation of the FELO

We consider three types of foreign exchange liquidity options. The first and simplest of these is a numerator currency loan commitment of the seller to the buyer such that if the numerator currency to the denominator currency exchange rate crosses a certain barrier above the current level, then the buyer has the right to borrow from the seller a prespecifed amount and maturity date zero-coupon loan in the numerator currency at the then current numerator currency country default-free interest rates on the exercise date of the option. We call this option the plain vanilla foreign exchange liquidity option or the PFELO.

The second type extends the first type in such a way that if, after crossing the first barrier prior to the option exercise date, the exchange rate crosses a second barrier above the first after the option exercise date on or before the maturity date of the loan, then the buyer has the additional right to extend the maturity of the loan to a second prespecified date. We call this option the extendible foreign exchange liquidity option or the EFELO.

Finally, the third type extends the first type in such a way that if, after crossing the first barrier prior to the option exercise date, the exchange rate crosses a second barrier below the first after the option exercise date on or before a second prespecified date prior to the maturity date of the loan, then the seller has the right to retract the maturity of the loan to the second date. We call this option the retractable foreign exchange liquidity option or the RFELO.

In this paper, we deal with the retractable foreign exchange liquidity options only since any plain vanilla foreign exchange liquidity option is a special case of a retractable foreign exchange liquidity option with the zero second exchange rate barrier whereas an extendible foreign exchange liquidity option is simply a long position in two plain vanilla foreign exchange liquidity options and a short position in a retractable foreign exchange liquidity option, provided the second barrier is chosen above the first. These will become clear after we write down the pricing formula for the retractable foreign exchange liquidity option.

To set the specific terms of the RFELO, let $t \in \mathbb{R}_+$ denote the time, where $\mathbb{R}_+$ is the field of non-negative real numbers, and consider the following four dates: $t = 0$ is the date on which the option contract is signed; $t = T$, such that $T > 0$, is the date on which the option contact expires; $t = L$, such that $L > T$, is the original maturity date of the loan, if the loan comes to life; and $t = S$, such that $T < S < L$, is the retraction date of the loan, if it gets retracted. Let $x(t)$ denote the natural logarithm of the exchange rate and consider two exchange rate barriers $x_T, x_S \in \mathbb{R}$ such that $x_T > x(0)$ and $x_S < x_T$. Therefore, if $x(t)$ crosses the first barrier x_T on or before the exercise date, that is, if there is some $t \in (0, T]$ such that $x(t) > x_T$, then the option holder exercises the RFELO and borrows an *L*-maturity zero-coupon loan from the central bank *at the L-maturity zero-coupon rate on date* $t = T$ *of the foreign currency country*. If, after the exercise of the option, $x(t)$ crosses the second barrier x_S on or before the retraction date S, that is, if there is some $t \in (T, S]$ such that $x_S > x(t)$, then the central bank retracts the loan to the retraction date S.

It is evident that the retractable foreign exchange liquidity option can be viewed as a portfolio of two retractable bonds, one defaultable, the other default-free: The option buyer has a short position in the defaultable retractable zero-coupon bond with a set maturity date and a long position in the default-free equivalent of this bond, each of which comes to life on the date the exchange rate crosses the first barrier before or on the option expiry date, if that happens. Clearly, the option seller has the opposite positions in these bonds. Provided that they come to life, both bonds retract to a predetermined retraction date before the original maturity date if the exchange rate crosses the second barrier between the option exercise date and the retraction date, if that happens. Of course, the defaultable bond of this portfolio is to be priced at the credit spread of the option buyer to the then current numerator currency country default-free interest rates on the date the contract is signed. This is the replicating portfolio of the option and in what follows we will price this portfolio.

2.3 Monetary policy implications

As the emerging market currency and several associated banking crises of the past three decades attest, with the ongoing liberalization of their financial markets since the late 1970s the emerging market economies are now vulnerable to undesirable side effects of the capital flows. The positive interest rate differentials of the emerging market economies not only attract large capital inflows, but also create incentives for the domestic banks to borrow in hard currencies to extend loans in their domestic currency. Especially during periods of excess global liquidity, large and continuous capital inflows suppress the hard currency exchange rates downward, creating an illusion of forever-cheap hard currencies in some agents of the domestic economy. Although not necessarily all the agents suffer from this illusion, some of those who do not may suffer from the alternative illusion that they can get out before a crisis.

As long as hard currencies remain cheap, the described carry trade is highly profitable to the domestic financial sector, deepening the illusions. However, since the resulting currency mismatch (not to mention the maturity mismatch resulting from the usual short-term borrowing and long-term lending) between assets and liabilities of the financial sector balance sheets is a source of substantial risk, the outcome of these illusions is a fragile domestic financial sector of the emerging market economies. This risk has to be removed by entering into offsetting off-balance sheet agreements with counter-parties abroad.

The problem is that most, if not all, of such agreements into which the financial sectors of the emerging market economies can enter with counter-parties abroad are short-term contracts. Covering short foreign exchange positions with short-term contracts, usually with maturities not longer than a month, is problematic because at times of severe exchange rate distortions rolling these contracts over becomes difficult, if not almost impossible. As a result, especially when the turmoil is long and deep, despite their robust net on and off balance sheet foreign exchange positions, the domestic banks begin to bid the spot prices up in the foreign exchange market, leading to an upward spiral.

As Blejer and Schumacher (2000) argue, state-contingent liabilities issued by central banks may release some of this pressure the market players put on the spot exchange rates at times of heavy speculation. The first implication of the FELO is then that they may scale the upward push in the spot foreign exchange market down by meeting the aforementioned artificial demand coming from the domestic banks at times of uncertainty. By assuming the counter-party role for a short period and guaranteeing foreign exchange liquidity to the domestic banks at times of uncertainty, the central bank may ease the fears of the domestic banks regarding rolling their short-term obligations over.

To have such an additional gadget to relax the foreign exchange market is important for highly "dollarized" economies such as the Turkish economy where the exchange rate pass-through to prices is high. For example, in the recent May 2006 foreign exchange turmoil in Turkey, although the high and volatile foreign exchange market lasted for only three months, together with the deteriorating expectations about the economy the inflation deviated from the target for an expected period of more than two years. The central bank operational borrowing rate was raised by 400 basis points in order to drive the inflation back into its decreasing trend prior to the turmoil, the obvious and unwanted side effect of which was increased incentives for continued carry trade by the domestic banks.

In addition, since under the FELO contracts a portion of the foreign reserves are rented rather than sold, the central bank may ease the fears of the domestic banks without depleting its net reserves during a crisis. The FELO transactions change the composition of foreign reserves held on the credit side of the balance sheet only: that is, they replace a fraction of the previously liquid assets with less liquid and riskier loans for a short period, a price any central bank would be willing to pay for the sake of domestic financial stability. Further, since the loan terms are short, the risk is minimal.

As we suggested in the introduction, the central banks should allow the domestic banks to choose the exchange rate barriers associated with the FELO contracts. For example, the central banks may offer a broad menu of exchange rate barriers and let the domestic banks decide which barriers meet their needs the best. This way, not only the central bank can avoid sending signals that may be misinterpreted, but also can collect valuable information regarding what exchange rate levels the domestic banks consider unsafe for their operations. This valuable information may help the central bank in overseeing the soundness of the financial system.

A potential moral hazard with allowing the domestic banks to choose the exchange rate barriers is that some banks may choose barriers that are higher than they believe they need to signal better than actual financial positions to the third parties. This potential moral hazard can be avoided if the barriers are kept private information between the central bank and each of the option holder domestic banks. For example, the central bank may enforce the public nondisclosure of the barriers through appropriate clauses in the contract. Since in this case the option holder domestic banks cannot disclose the barriers as signals of soundness to the third parties, they would have no incentive to choose barriers that are higher than they believe they need. Even then, since the central bank may refuse to let or, by asking a large insurance fee, discourage high risk banks sign such contracts, being able to enter into FELO contracts with the central bank is still a positive signal to the third parties for a domestic bank and may improve the overall position of the domestic banks in the foreign exchange market.

Lastly, recall that we proposed three flavors of the foreign exchange liquidity options, namely, the plain vanilla, extendible and retractable flavors. Although the plain vanilla foreign exchange liquidity option may be sufficient for the central bank to provide hard currency liquidity to the domestic banks, since not only when a liquidity crisis begins, but also when it ends is not foreseeable, the extendible and retractable flavors offer additional insurance to the domestic banks. The extendible liquidity option provides additional insurance against the possible underestimation while the retractable foreign exchange liquidity option provides additional insurance against the possible overestimation of the length and severity of the potential crisis. We suggest that the central bank should offer all of the flavors in its broad menu of contracts and let the domestic banks choose the flavor depending on their perceived needs, as well. Not only this would allow the central bank to handle a potential crisis more flexibly, but also would help the central bank collect valuable information about the market sentiment.

3. The RFELO price

3.1 Mathematical preliminaries

We work in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where, with $\mathcal{F}_\infty = \mathcal{F}$, the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfies the usual conditions and $\mathbb{P}$ is the historical measure of the data. Further, the filtration $\mathcal{F}$ is generated by an N-dimensional standard Brownian motion $W^{\mathbb{P}}(t)$ and a K-dimensional pure jump process $J(t)$ with jumps $\Delta J(t) \in \mathbb{R}_+^K$ where $\lambda^{\mathbb{P}}(t)$ is the arrival intensity of the jumps and $\nu^{\mathbb{P}}(z)$, such that $z \in \mathbb{R}_+^K$; is the jump amplitude distribution. The distributions of $W^{\mathbb{P}}(t)$, $J(t)$ and $\Delta J(t)$ are assumed independent.

Let $P^n(t, U)$ be the price at time t of the non-defaultable zero-coupon bond in the foreign currency country, maturing at time U where $U > t$, and consider an equivalent defaultable zero-coupon bond, whose price at time t is $P^d(t, U)$. Let $\mathbb{Q}$ be an equivalent risk-neutral measure to the historical measure $\mathbb{P}$, which need not be unique in the presence of jumps. We will fix this risk-neutral measure when we write down our interest rate, spread and exchange rate models. Prior to this, however, we suppose that the market price of the jump risk is constant (see, Dai and Singleton, 2003, p. 650) and, for our RFELO pricing purposes, conveniently set it to 0. Consequently, we have $\lambda^{\mathbb{Q}}(t) = \lambda^{\mathbb{P}}(t)$ and $\nu^{\mathbb{Q}}(z) = \nu^{\mathbb{P}}(z)$ almost everywhere so that we stop distinguishing between them and drop the superscripts.

In the risk-neutral probability measure $\mathbb{Q}$, the price of the non-defaultable zero coupon bond is given by

$$P^n(t, U) = E^{\mathbb{Q}} \left[\exp\{-\int_t^U r(u) du\} | \mathcal{F}_t \right] \tag{1}$$

where $r(t)$ is the default-risk free short rate in the foreign currency country, whereas for the defaultable zero coupon bond, following Berndt (2004), we adopt the recovery of market value convention of Duffie and Singleton(1999) and suppose that

$$P^d(t,U) = E^{\mathbb{Q}}\left[\exp\{-\int_t^U (r(u)+s(u))\,du\}|\mathcal{F}_t\right] \qquad (2)$$

where $s(t)$ is the short credit spread. We refer the reader to Duffie and Singleton (1999) and Brendt (2004), as well as to Duffie, Pedersen and Singleton (2003), for discussions on the implications of this convention.

In closing this subsection, we define the non-defaultable and defaultable money market accounts $M^n(t) = \exp\{\int_0^t r(u)du\}$ and $M^d(t) = \exp\{\int_0^t (r(u)+s(u))\,du\}$, respectively, for later use.

3.2 The RFELO price in a general setting

Taking up from where we left in the opening of the previous section, let us denote by *RFELO(t)* the price at time $t \in [0,T]$ of our retractable foreign exchange liquidity option. As we explained, this option may be viewed as a portfolio of two retractable bonds as viewed by the buyer: a long position in a non-defaultable and a short position in a defaultable bond. We denote the prices of these non-defaultable and defaultable bonds by $R^n(t)$ and $R^d(t)$, respectively. Consequently, the price of our retractable foreign exchange liquidity option, as viewed by the buyer, is

$$RFELO(t) = R^n(t) - R^d(t) \qquad (3)$$

Let us now define the stopping times

$$\tau_T = \inf\{t \in (0,T] | x(t) \geq x_T\} \qquad (4)$$

$$\tau_S = \inf\{t \in (T,S] | x(t) \leq x_S,\ \tau_T < \infty\} \qquad (5)$$

where, with ϕ as the empty set, we used the usual convention that inf $\phi = \infty$. It is clear that our retractable bonds come to life on the exercise date T with maturity date L if $\tau_T \in (0,T]$, and if $\tau_S \in (T,S]$ then they retract to S. Therefore, the prices of these bonds in the risk-neutral measure $\mathbb{Q}$ are given by

$$\begin{aligned} R^n(t) &= E^{\mathbb{Q}}\left[\exp\{-\int_t^S r(u)du\}1(0<\tau_T \leq T)1(T<\tau_S\leq S)|\mathcal{F}_t\right] \\ &\quad + E^{\mathbb{Q}}\left[\exp\{-\int_t^L r(u)du\}1(0<\tau_T\leq T)1(\tau_S > S)|\mathcal{F}_t\right] \end{aligned} \qquad (6)$$

$$\begin{aligned} R^d(t) &= E^{\mathbb{Q}}\left[\exp\{-\int_t^S (r(u)+s(u))du\}1(0<\tau_T \leq T)1(T<\tau_S\leq S)|\mathcal{F}_t\right] \\ &\quad + E^{\mathbb{Q}}\left[\exp\{-\int_t^L (r(u)+s(u))du\}1(0<\tau_T\leq T)1(\tau_S > S)|\mathcal{F}_t\right] \end{aligned} \qquad (7)$$

Although the above pricing formulae in the risk-neutral probability measure $\mathbb{Q}$ can be computed numerically, an easier to compute and interpret alternative is to re-express them in the "so-called" forward measures in the manner of Jamsidian (1997), as we introduce below. To this end, let us define the non-defaultable forward measure $\mathbb{Q}_U^n$ and the defaultable forward measure $\mathbb{Q}_U^d$ through the Radon-Nikodým derivatives

$$\frac{d\mathbb{Q}_U^i}{d\mathbb{Q}} = \frac{P^i(U,U)}{P^i(0,U)M^i(U)} = \frac{1}{P^i(0,U)M^i(U)}, \qquad i = n,d \qquad (8)$$

where last of the above equalities follows from the fact that $P^i(U,U)=1, i=n,d$. It is evident from the above that these forward measures depend on the maturity date of $P^i(t,U), i=n,d$ and, hence, the subscript U.

It then follows that the above pricing formulae can be rewritten as

$$R^i(t)=p^i_S(t)P^i(t,S)+p^i_L(t)P^i(t,L), \quad i=n,d \tag{9}$$

where

$$p^i_S(t)=E^{Q^i_S}[1(0<\tau_T\leq T)1(T<\tau_S\leq S)|\mathcal{F}_t], \quad i=n,d \tag{10}$$

$$p^i_L(t)=E^{Q^i_L}[1(0<\tau_T\leq T)1(\tau_S>S)|\mathcal{F}_t], \quad i=n,d \tag{11}$$

It is evident also that $p^i_S(t)$ and $p^i_L(t),\ i=n,d$ are the $\mathcal{F}_t$ conditional probabilities of the i^{th} bonds with maturity dates S and L coming to life in the corresponding forward measures, respectively. This indicates that the pricing of our foreign exchange liquidity options boils down to an assessment of these probabilities, since the prices at time t of the S and L maturity date, zero coupon, non-defaultable and defaultable bonds can be read from the then current market conditions.

3.3 Specialization of the FELO price to the Republic of Turkey

As we promised earlier, with YTL as the domestic and USD as the foreign currency, we specialize our general pricing formulae to Central Bank of the Republic of Turkey in a five-factor affine jump-diffusion model. This requires an estimation of a US term structure model, a USD/YTL exchange rate model and a country credit spread model for the sovereign Eurodollar Bonds issued by Treasury of the Republic of Turkey, as well as its extension to individual domestic banks and foreign banks with domestic operations in the country.[5] In doing this, we rely on the works of Duffie, Pedersen and Singleton (2003) and Brendt (2004), as well as on Brandt and Santa-Clara (2003), heavily.

The US term structure part of our five-factor affine jump-diffusion model is identical to that of Duffie, Pedersen and Singleton (2003). Therefore, in the manner of Brendt (2004), we could have taken their parameter estimates as the “true” parameter values. The reason for why we do not do this is that Duffie, Pedersen and Singleton (2003) estimated their model based on pre-1999 data whereas we are interested in post-2001 data since in February 2001, a structural change occurred in Turkey: In February 2001, Turkey had to let go off of its crawling-peg to the US dollar after a political crisis and switched to a floating exchange regime under the guidance of the International Monetary Fund. A scrutiny of the exchange rate data suggests that this structural change had taken place roughly between February 2001 and June 2001, after which the new regime appears to have settled down. Since in our model the correlations between the US short rate, Turkish sovereign spread and USD/YTL exchange rate play a significant role, we re-estimate the model of Duffie, Pedersen and Singleton (2003), together with the rest of our model, for the post-June 2001 data.

5 We are thankful to Central Bank of the Republic of Turkey for providing us with bank level loan data.

3.3.1 The model

To the already introduced short rate $r(t)$ and natural logarithm of the exchange rate $x(t)$, we now add the short rate volatility $v(t)$ and exchange rate volatility $w(t)$, and, with $s(t)$ representing the short sovereign Eurodollar spread of the Republic of Turkey, suppose that

$$s(t) = u(t) + \pi_v v(t) + \pi_w w(t) + \pi_r r(t) + \pi_x x(t) \tag{12}$$

where the country specific factor $u(t)$ follows the square-root process

$$du(t) = K_u^{\mathbb{P}}(\theta_u^{\mathbb{P}} - u(t))dt + \Sigma_u \sqrt{u(t)} dW_u^{\mathbb{P}}(t) \tag{13}$$

Although with this assumption we force the country specific factor $u(t)$ to be positive, as is evident, this does not mean that the sovereign spread $s(t)$ is forced to be positive also. The logical consistency of the possibility of negative sovereign spreads with their theoretical model, an extension of which we adopt here, has been discussed in Duffie, Pedersen and Singleton (2003) at length, and we refer the reader to their paper for this discussion.[6]

To complete the model, let us introduce the vectors $y(t) = (v(t), w(t), r(t), x(t))^{\mathsf{T}}$, $W_y^{\mathbb{P}}(t) = (W_v^{\mathbb{P}}(t), W_w^{\mathbb{P}}(t), W_r^{\mathbb{P}}(t), W_x^{\mathbb{P}}(t))^{\mathsf{T}}$ and $J(t) = (0, 0, 0, J_x(t))^{\mathsf{T}}$, and suppose that

$$dy(t) = K_y^{\mathbb{P}}(\theta_y^{\mathbb{P}} - y(t))dt + \Sigma_y \sqrt{S(t)} dW_y^{\mathbb{P}}(t) + dJ(t) \tag{14}$$

where

$$S(t) = \begin{bmatrix} v(t) & 0 & 0 & 0 \\ 0 & w(t) & 0 & 0 \\ 0 & 0 & v(t) & 0 \\ 0 & 0 & 0 & w(t) \end{bmatrix}$$

whereas $\theta_y^{\mathbb{P}}(t) = (\theta_v^{\mathbb{P}}(t), \theta_w^{\mathbb{P}}(t), \theta_r^{\mathbb{P}}(t), \theta_x^{\mathbb{P}}(t))^{\mathsf{T}}$,

$$K_y^{\mathbb{P}} = \begin{bmatrix} K_{vv}^{\mathbb{P}} & 0 & 0 & 0 \\ K_{wv}^{\mathbb{P}} & K_{ww}^{\mathbb{P}} & 0 & 0 \\ K_{rv}^{\mathbb{P}} & 0 & K_{rr}^{\mathbb{P}} & 0 \\ K_{xv}^{\mathbb{P}} & K_{xw}^{\mathbb{P}} & K_{xr}^{\mathbb{P}} & K_{xx}^{\mathbb{P}} \end{bmatrix} \quad \text{and} \quad \Sigma_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \Sigma_{wv} & \Sigma_{ww} & 0 & 0 \\ \Sigma_{rv} & 0 & \Sigma_{rr} & 0 \\ \Sigma_{xv} & \Sigma_{xw} & \Sigma_{xr} & \Sigma_{xx} \end{bmatrix}$$

With the above, the formulation of our model is now complete. It is clear that the model we postulated is not the most general five-factor affine jump-diffusion model we could have postulated[7], but even in the above restricted form there are far too many parameters in our model for it to be considered parsimonious. Further, we will have to add more parameters as we move along. We choose to sacrifice generality for the sake of manageability, which is not very easy even in the above restricted form. It should be noted also that, in the manner of Bates (1996), we allow jumps only in the exchange rate. We could have allowed jumps also in the US short rate, but, as can be seen from the pricing formula (9), it is the distributional properties of the exchange rate which play the significant role in the price of the RFELO

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6 See, Duffie, Pedersen and Singleton (2003), p. 133.

7 See, Dai and Singleton (2000).

through their influence on the $\mathcal{F}_t$ conditional probabilities $p_S^i(t)$ and $p_L^i(t),\ i = n, c$. Consequently, we do not do that.

Finally, we fix our risk-neutral measure $\mathbb{Q}$ along the lines of Dai and Singleton (2000)[8] by specifying the market price of the risks associated with $W_u^{\mathbb{P}}(t)$ and $W_y^{\mathbb{P}}(t)$ respectively as $\sqrt{u(t)}\Lambda_u$, and $\sqrt{S(t)}\Lambda_y$, where $\Lambda_y = (\Lambda_v, \Lambda_w, \Lambda_r, \Lambda_x)^{\mathrm{T}}$. With $W^{\mathbb{Q}}(t)$ as the corresponding standard Brownian motion in the risk-neutral measure $\mathbb{Q}$, itthen follows that

$$dW_u^{\mathbb{Q}}(t) = dW_u^{\mathbb{P}}(t) + \Sigma_u u(t)\Lambda_u dt \tag{15}$$

$$dW_y^{\mathbb{Q}}(t) = dW_y^{\mathbb{P}}(t) + \Sigma_y S(t)\Lambda_y dt \tag{16}$$

and, therefore, in the risk-neutral measure $\mathbb{Q}$, we have

$$du(t) = K_u^{\mathbb{Q}}(\theta_u^{\mathbb{Q}} - u(t))dt + \Sigma_u\sqrt{u(t)}dW_u^{\mathbb{P}}(t) \tag{17}$$

$$dy(t) = K_y^{\mathbb{Q}}(\theta_y^{\mathbb{Q}} - y(t))dt + \Sigma_y\sqrt{S(t)}dW_y^{\mathbb{Q}}(t) + dJ(t) \tag{18}$$

where

$$K_u^{\mathbb{Q}}(\theta_u^{\mathbb{Q}} - u(t)) = K_u^{\mathbb{P}}(\theta_u^{\mathbb{P}} - u(t)) - \Sigma_u u(t)\Lambda_u \tag{19}$$

and

$$K_y^{\mathbb{Q}}(\theta_y^{\mathbb{Q}} - y(t)) = K_y^{\mathbb{P}}(\theta_y^{\mathbb{P}} - y(t)) - \Sigma_y S(t)\Lambda_y, \tag{20}$$

from which $K^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}}$ are solved.

3.3.2 Determination of the non-defaultable yield curve

We know from Duffie and Kan (1996) and Dai and Singleton (2000) for the model we postulated that the non-defaultable pure discount bond price (1) takes the form

$$P^n(t, U) = \exp\{A_r(t, U) + B_r(t, U)^{\mathrm{T}} y_r(t)\} \tag{21}$$

where $y_r(t) = (v(t), r(t))^{\mathrm{T}}$ and $B_r(t, U) = (B_{rv}(t, U), B_{rr}(t, U))^{\mathrm{T}}$. Since $P^n(U, U) = 1$, we must have the terminal condition

$$A_r(U, U) = 0, \quad B_r(U, U) = 0 \tag{22}$$

A straight forward application of the Feyman-Kac Theorem in view of (1) and (21) gives the ordinary differential equations

$$\partial_t A_r(t, U) + (K_{y_r}^{\mathbb{Q}}\theta_{y_r}^{\mathbb{Q}})^{\mathrm{T}} B_r(t, U) = 0 \tag{23}$$

$$\begin{gathered}\partial_t B_{rv}(t, U) - K_{y_r v}^{\mathbb{Q}} B_r(t, U) \\ +\frac{1}{2}\left[\Sigma_{y_r}^{\mathrm{T}} B_r(t, U) B_r(t, U)^{\mathrm{T}} \Sigma_{y_r}\right]_{vv} + \frac{1}{2}\left[\Sigma_{y_r}^{\mathrm{T}} B_r(t, U) B_r(t, U)^{\mathrm{T}} \Sigma_{y_r}\right]_{rr} = 0\end{gathered} \tag{24}$$

$$\partial_t B_{rr}(t, U) - K_{rr}^{\mathbb{Q}} B_{rr}(t, U) - 1 = 0 \tag{25}$$

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8 See, p.642 of their paper.

where $\theta_{y_r}^{\mathbb{Q}} = (\theta_v^{\mathbb{Q}}, \theta_r^{\mathbb{Q}})^{\mathrm{T}}$, $K_{y_r v}^{\mathbb{Q}} = (K_{vv}^{\mathbb{Q}}, K_{rv}^{\mathbb{Q}})$ whereas

$$K_{y_r}^{\mathbb{Q}} = \begin{bmatrix} K_{vv}^{\mathbb{Q}} & 0 \\ K_{rv}^{\mathbb{Q}} & K_{rr}^{\mathbb{Q}} \end{bmatrix} \quad \text{and} \quad \Sigma_{y_r} = \begin{bmatrix} 1 & 0 \\ \Sigma_{rv} & \Sigma_{rr} \end{bmatrix}$$

Although subject to the terminal condition (22) the equation (25) is easily solved to give

$$B_{rr}(t,U) = \frac{e^{-K_{rr}^{\mathbb{Q}}(U-t)} - 1}{K_{rr}^{\mathbb{Q}}}, \tag{26}$$

the equations (23) and (24) subject to the terminal condition (22) have no closed form solutions. Because of this, all researchers in the field use numerical solution methods such as various forms of the Runge-Kutta method to solve problems similar to the above. Here, we take a different approach: we obtain a power series solution to the above system of Riccati Equations. The advantage of doing this is that since the above system of Riccati Equations is a constant coe¢cients system, if found, its solution is analytic and, hence, its power series solution converges uniformly in $[0, U]$. Further, it is well known for uniformly convergent power series that their main diagonal Padé approximants converge rapidly.[9] Consequently, these power series solutions can be summed with the use of main diagonal Padé approximants for economy in calculations, avoiding the often time consuming numerical integrations.[10]

To recall briefly, the $[m/n$-Padé approximant to any formal power series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k \tag{27}$$

is the quotient $P_m(x)/Q_n(x)$ such that

$$P_m(x) = \sum_{k=0}^{m} p_k x^k, \qquad Q_n(x) = \sum_{k=0}^{n} q_k x^k \tag{28}$$

and

$$Q_n(x) f(x) - P_m(x) = O(x^{m+n+1}), \quad x \to 0 \tag{29}$$

where $O(\cdot)$ is the big-O symbol with the usual significance. With the usual normalization $q_0 = 1$, one obtains from the above the following system of $m+n+1$ linear equations in the remaining $m+n+1$ unknowns:

$$\sum_{i=1}^{k \wedge n} f_{k-i} q_k = -f_k, \quad m+1 \le k \le m+n \tag{30}$$

$$p_k = \sum_{i=1}^{k \wedge n} f_{k-i} q_k, \quad 0 \le k \le m \tag{31}$$

where, as usual, $k \wedge n = \min(k, n)$. From now on, we will use the [10/10]-Padé approximants as described above to sum our uniformly convergent power series solutions.

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9 See, for example, Lubinsky (1995) and the references therein.

10 As will be clear in the next section, another important advantage of our approach is avoidance of the often cumbersome to compute Fourier Transforms; see, Duffe, Pan and Singleton (2000).

To obtain the power series solution of the system (23) through (25) subject to the terminal condition (22), let us set

$$A_r(t,U) = \sum_{k=0}^{\infty} a_{rk}(U-t)^k, \;\; B_r(t,U) = \sum_{k=0}^{\infty} b_{rk}(U-t)^k \tag{32}$$

where $b_{rk} = (b_{rvk}, b_{rrk})^{\mathrm{T}}$. Inserting these into (22) through (25) we then get

$$a_{r0} = 0, \quad b_{r0} = 0 \tag{33}$$

$$a_{r1} = b_{rv1} = 0, \quad b_{rr1} = -1 \tag{34}$$

$$(k+1)a_{r,k+1} = (K^{\mathrm{Q}}_{y_r}\theta^{\mathrm{Q}}_{y_r})^{\mathrm{T}} b_{rk}, \;\; k = 1,2,\ldots \tag{35}$$

$$(k+1)b_{rv,k+1} = -K^{\mathrm{Q}}_{y_r v} b_{rk} + \frac{1}{2}\sum_{\substack{i_1+i_2=k \\ i_1,i_2\geq 0}} \left\{ \left[\Sigma^{\mathrm{T}}_{y_r} b_{ri_1} b^{\mathrm{T}}_{ri_2} \Sigma_{y_r}\right]_{vv} + \left[\Sigma^{\mathrm{T}}_{y_r} b_{ri_1} b^{\mathrm{T}}_{ri_2} \Sigma_{y_r}\right]_{rr} \right\}, \;\; k = 1,2,\ldots \tag{36}$$

$$(k+1)b_{rr,k+1} = -K^{\mathrm{Q}}_{rr} b_{rrk}, \;\; k = 1,2,\ldots \tag{37}$$

Determination of the non-defaultable yield curve as implied by our model is now complete. In the next section, we turn our attention to the defaultable yield curve.

3.3.3 Determination of the defaultable yield curve

To make progress, we need to specify the arrival intensity $\lambda(t)$ of the jumps and the jump amplitude distribution $\nu(z)$, first. Since we are working in an affine jump-diffusion framework, we set $\lambda(t) = \lambda_0 + \lambda_u u(t) + \lambda_y^{\mathrm{T}} y(t)$, where $\lambda_y = (\lambda_v, \lambda_w, \lambda_r, \lambda_x)^{\mathrm{T}}$, and suppose that $\ln z \sim N(\mu, \sigma^2)$ so that the jump amplitude distribution $\nu(z)$ is log-normal. Finally, we define the moments

$$M_k = \int_{\mathbb{R}_+} z^k dv(z) = e^{k\mu + \frac{1}{2}k^2\sigma^2} \tag{38}$$

of our jump amplitude distribution for later use in what follows.

Similar to the non-defaultable pure discount bond $P^n(t,U)$, we look for solutions of the form

$$P^d(t,U) = \exp\{A_s(t,U) + B_{su}(t,U)u(t) + B_s(t,U)^{\mathrm{T}} y(t)\} \tag{39}$$

where $B_s(t,U) = (B_{sv}(t,U), B_{sw}(t,U), B_{sr}(t,U), B_{sx}(t,U))^{\mathrm{T}}$ for the defaultable pure discount bond $P^d(t,U)$ given by (2). Since $P^d(U,U) = 1$, we again have the terminal condition

$$A_s(U,U) = B_{su}(U,U) = 0, \quad B_s(U,U) = 0 \tag{40}$$

A straight forward application of the Feyman-Kac Theorem in view of (2) and (39) then gives the following system of integro-ordinary differential equations:

$$\partial_t A_s(t,U) + K^{\mathrm{Q}}_u \theta^{\mathrm{Q}}_u B_{su}(t,U) + (K^{\mathrm{Q}}_y \theta^{\mathrm{Q}}_y)^{\mathrm{T}} B_s(t,U) + \lambda_0 \int_{\mathbb{R}_+} \left(e^{B_{sx}(t,U)z} - 1\right) dv(z) = 0 \tag{41}$$

$$\partial_t B_{su}(t,U) - K^{\mathrm{Q}}_u B_{su}(t,U) + \frac{1}{2}\Sigma^2_u B_{su}(t,U)^2 + \lambda_u \int_{\mathbb{R}_+} \left(e^{B_{sx}(t,U)z} - 1\right) dv(z) - 1 = 0 \tag{42}$$

$$\partial_t B_{sv}(t,U) - K^{\mathbb{Q}}_{yv} B_s(t,U) + \frac{1}{2}\left[\Sigma_y^{\mathrm{T}} B_s(t,U) B_s(t,U)^{\mathrm{T}} \Sigma_y\right]_{vv}$$
$$+\frac{1}{2}\left[\Sigma_y^{\mathrm{T}} B_s(t,U) B_s(t,U)^{\mathrm{T}} \Sigma_y\right]_{rr} + \lambda_v \int\limits_{\mathbb{R}_+} \left(e^{B_{sx}(t,U)z} - 1\right) dv(z) - \pi_v = 0 \tag{43}$$

$$\partial_t B_{sw}(t,U) - K^{\mathbb{Q}}_{yw} B_s(t,U) + \frac{1}{2}\left[\Sigma_y^{\mathrm{T}} B_s(t,U) B_s(t,U)^{\mathrm{T}} \Sigma_y\right]_{ww}$$
$$+\frac{1}{2}\left[\Sigma_y^{\mathrm{T}} B_s(t,U) B_s(t,U)^{\mathrm{T}} \Sigma_y\right]_{xx} + \lambda_w \int\limits_{\mathbb{R}_+} \left(e^{B_{sx}(t,U)z} - 1\right) dv(z) - \pi_w = 0 \tag{44}$$

$$\partial_t B_{sr}(t,U) - K^{\mathbb{Q}}_{rr} B_{sr}(t,U) + \lambda_r \int\limits_{\mathbb{R}_+} \left(e^{B_{sx}(t,U)z} - 1\right) dv(z) - 1 - \pi_r = 0 \tag{45}$$

$$\partial_t B_{sx}(t,U) - K^{\mathbb{Q}}_{xx} B_{sx}(t,U) + \lambda_x \int\limits_{\mathbb{R}_+} \left(e^{B_{sx}(t,U)z} - 1\right) dv(z) - \pi_x = 0 \tag{46}$$

where $K^{\mathbb{Q}}_{yv} = (K^{\mathbb{Q}}_{vv}, K^{\mathbb{Q}}_{wv}, K^{\mathbb{Q}}_{rv}, K^{\mathbb{Q}}_{xv})$ and $K^{\mathbb{Q}}_{yv} = (0, K^{\mathbb{Q}}_{ww}, 0, K^{\mathbb{Q}}_{xw})$. We note that had the intensity parameter λ_u been 0, then the equation (42) would have been the usual CIR equation whose solution subject to the terminal condition (40) is:[11]

$$B_{su}(t,U) = \frac{2\left(1 - e^{\Omega(U-t)}\right)}{2\Omega - (\Omega + K^{\mathbb{Q}}_u)\left(1 - e^{\Omega(U-t)}\right)} \tag{47}$$

where $\Omega = (K^{\mathbb{Q}2}_u + \Sigma_u^2)^{\frac{1}{2}}$. In Fig. 1, we compare this closed form solution, as well as the closed form solution (26) we obtained for $B_{rr}(t,U)$ in the previous section, with their corresponding [10/10]-Padé approximations for a set of model parameters $K^{\mathbb{Q}}_u$, Σ_u and $K^{\mathbb{Q}}_{rr}$, graphically.

Proceeding as before, we now set

$$A_s(t,U) = \sum_{k=0}^{\infty} a_{sk}(U-t)^k,\ B_{su}(t,U) = \sum_{k=0}^{\infty} b_{suk}(U-t)^k,\ B_s(t,U) = \sum_{k=0}^{\infty} b_{sk}(U-t)^k \tag{48}$$

where $b_{sk} = (b_{svk}, b_{swk}, b_{srk}, b_{sxk})^{\mathrm{T}}$, insert these into the equations (40) through (46), Taylor expand $e^{B_{sx}(t,U)z}$ about 0, integrate the resulting Taylor series, equate the coefficients of the like terms and get:

$$a_{s0} = b_{su0} = 0,\ b_{s0} = 0 \tag{49}$$

$$b_{su1} = -1,\ b_{sv1} = -\pi_v,\ b_{sw1} = -\pi_w,\ b_{sr1} = -1 - \pi_r,\ b_{sx1} = -\pi_x \tag{50}$$

$$(k+1)a_{s,k+1} = K^{\mathbb{Q}}_u \theta^{\mathbb{Q}}_u b_{suk} + (K^{\mathbb{Q}}_y \theta^{\mathbb{Q}}_y)^{\mathrm{T}} b_{sk}$$
$$+\lambda_0 \sum_{j=1}^{k} \sum_{\substack{i_1+i_2+\ldots+i_j=k \\ i_1,i_2,\ldots,i_j \geq 1}} b_{sxi_1} b_{sxi_2} \ldots b_{sxi_j} \frac{M_j}{j!},\ k = 0, 1, \ldots \tag{51}$$

[11] See, Cox, Ingersoll and Ross (1985), p. 393.

$$(k+1)b_{su,k+1} = -K_u^{\mathbb{Q}} b_{suk} + \frac{1}{2}\sum_{\substack{i_1+i_2=k\\ i_1,i_2\geq 1}} \Sigma_u^2 b_{sui_1} b_{sui_2}$$

$$+\lambda_u \sum_{j=1}^{k} \sum_{\substack{i_1+i_2+...+i_j=k\\ i_1,i_2,...,i_j\geq 1}} b_{sxi_1} b_{sxi_2} ... b_{sxi_j} \frac{M_j}{j!},\ \ k=1,2,... \tag{52}$$

$$(k+1)b_{sv,k+1} = -K_{yv}^{\mathbb{Q}} b_{sk} + \frac{1}{2}\sum_{\substack{i_1+i_2=k\\ i_1,i_2\geq 1}} \{[\Sigma_y^{\mathrm{T}} b_{sui_1} b_{sui_2} \Sigma_y]_{vv} + [\Sigma_y^{\mathrm{T}} b_{sui_1} b_{sui_2} \Sigma_y]_{rr}\}$$

$$+\lambda_v \sum_{j=1}^{k} \sum_{\substack{i_1+i_2+...+i_j=k\\ i_1,i_2,...,i_j\geq 1}} b_{sxi_1} b_{sxi_2} ... b_{sxi_j} \frac{M_j}{j!},\ \ k=1,2,... \tag{53}$$

$$(k+1)b_{sw,k+1} = -K_{yw}^{\mathbb{Q}} b_{sk} + \frac{1}{2}\sum_{\substack{i_1+i_2=k\\ i_1,i_2\geq 1}} \{[\Sigma_y^{\mathrm{T}} b_{sui_1} b_{sui_2} \Sigma_y]_{ww} + [\Sigma_y^{\mathrm{T}} b_{sui_1} b_{sui_2} \Sigma_y]_{xx}\}$$

$$+\lambda_w \sum_{j=1}^{k} \sum_{\substack{i_1+i_2+...+i_j=k\\ i_1,i_2,...,i_j\geq 1}} b_{sxi_1} b_{sxi_2} ... b_{sxi_j} \frac{M_j}{j!},\ \ k=1,2,... \tag{54}$$

$$(k+1)b_{sr,k+1} = -K_{rr}^{\mathbb{Q}} b_{srk} + \lambda_r \sum_{j=1}^{k} \sum_{\substack{i_1+i_2+...+i_j=k\\ i_1,i_2,...,i_j\geq 1}} b_{sxi_1} b_{sxi_2} ... b_{sxi_j} \frac{M_j}{j!},\ \ k=1,2,... \tag{55}$$

$$(k+1)b_{sx,k+1} = -K_{xx}^{\mathbb{Q}} b_{sxk} + \lambda_x \sum_{j=1}^{k} \sum_{\substack{i_1+i_2+...+i_j=k\\ i_1,i_2,...,i_j\geq 1}} b_{sxi_1} b_{sxi_2} ... b_{sxi_j} \frac{M_j}{j!},\ \ k=1,2,... \tag{56}$$

With the above, determination of the defaultable yield curve is now complete.