Optimal Stopping for American Type Options

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Outline of communication

- Multivariate Modulated Markov price processes and American type options
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Multivariate modulated Markov price processes and American type options

- A multivariate Markov log-price process modulated by a stochastic index
  $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), \ldots, Y_k^{(\varepsilon)}(t)), t \geq 0$ is a vector càdlàg log-price process and $X^{(\varepsilon)}(t), t \geq 0$ is a measurable index process such that $Z^{(\varepsilon)}(t) = (\vec{Y}^{(\varepsilon)}(t), X^{(\varepsilon)}(t))$ is a Markov process with a phase space $\mathbb{Z} = \mathbb{R}_k \times \mathbb{X}$, an initial distribution $P^{(\varepsilon)}(A)$, and transition probabilities $P^{(\varepsilon)}(t, z, t+u, A)$.
- A multivariate price process modulated by a stochastic index
  $\vec{S}^{(\varepsilon)}(t) = (S_1^{(\varepsilon)}(t), \ldots, S_k^{(\varepsilon)}(t)), t \geq 0$ is a vector price process, where $S_i^{(\varepsilon)}(t) = \exp\{Y_i^{(\varepsilon)}(t)\}, i = 1, \ldots, k, t \geq 0$.
- American type options

$$\Phi^{(\varepsilon)} = \sup_{0 \leq \tau^{(\varepsilon)} \leq T} \mathsf{E} g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})).$$

A: Not more than polynomial growth of partial derivatives of payoff function $g(t, \vec{s})$ $(\cdot \leq L_1 + L_2|\vec{s}|^{\gamma})$.

B: There exist measurable sets $\mathbb{Z}_t \subseteq \mathbb{Z}$, $t \in [0, T]$ such that: **(a)** $P^{(\varepsilon)}(t, z_\varepsilon, t+u, \cdot) \Rightarrow P^{(0)}(t, z, t+u, \cdot)$ as $\varepsilon \to 0$, for any $z_\varepsilon \to z \in \mathbb{Z}_t$ as $\varepsilon \to 0, 0 \leq t < t+u \leq T$; **(b)** $P^{(0)}(t, z, t+u, \mathbb{Z}_{t+u}) = 1$ for every $z \in \mathbb{Z}_t, 0 \leq t < t+u \leq T$.

C: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \sup_{0 \leq t \leq t' \leq t+c \leq T} \sup_{z \in \mathbb{Z}} \mathsf{E}_{z,t}(e^{\beta|\vec{Y}^{(\varepsilon)}(t') - \vec{Y}^{(\varepsilon)}(t)|} - 1) = 0$ for some $\beta > \gamma + 1$.

D: $Z^{(\varepsilon)}(0) = z_0 \in \mathbb{Z}_0$.

Theorem 1: **A** – **D** $\Rightarrow$

$$\Phi^{(\varepsilon)} \to \Phi^{(0)} \text{ as } \varepsilon \to 0.$$

Convergence of option rewards

- Time skeleton approximations

$$\Phi^{(\varepsilon)}(\mathcal{P}(n)) = \sup_{\tau^{(\varepsilon)} \in \mathcal{P}(n)} \mathsf{E} g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}))).$$

where $\mathcal{P}(n) = < 0 = t_{n,0} < \cdots < t_{n,n} = T >$ such that $d_n = \max_{1 \leq k \leq n}(t_{n,k} - t_{n,k-1}) \to 0$ as $n \to \infty$.

$$\sup_{\varepsilon \geq 0} |\Phi^{(\varepsilon)} - \Phi^{(\varepsilon)}(\mathcal{P}(n))| \leq \Delta(n) \to 0 \text{ as } n \to \infty.$$

- Convergence of option rewards for discrete time models

$$\Phi^{(\varepsilon)}(\mathcal{P}(n)) \to \Phi^{(0)}(\mathcal{P}(n)) \text{ as } \varepsilon \to 0.$$

Applications of convergence results

- Types of price processes

  – Price processes represented by exponential Markov and semi-Markov chains;

  – Gaussian Markov random walk type price processes;

  – Exponential ARMA type price processes;

  – General multivariate Markov price processes with Markov and semi-Markov modulation;

  – Exponential modulated Lévy type price processes;

  – Exponential multivariate diffusion price process.

- Approximation models

  – Space-time skeleton approximations;

  – Tree approximation models (binomial, trinomial, etc.);

  – Monte-Carlo type approximations.

- Types of options

- A reselling model

$$\begin{cases} d\ln S(t) = \mu dt + \sigma dW_1(t), \\ d\ln \sigma(t) = -\alpha(\ln \sigma(t) - \ln \sigma)dt + \nu dW_2(t), \\ t \in [0, T], \end{cases}$$

where (a) $\mu \in \mathbb{R}$; $\alpha, \nu, \sigma > 0$; (b) $S(0) = s_0 = const > 0$; $\sigma(0) = \sigma$; (c) $\vec{W}(t) = (W_1(t), W_2(t)), t \geq 0$ is a standard bivariate Brownian motion with $\mathsf{E}W_1(1)W_2(1) = \rho$.

- Formulation of the reselling problem

$$\Phi^{(0)} = \sup_{\tau \leq T} \mathsf{E}e^{-r\tau}C(\tau, S(\tau), \sigma(\tau)),$$

where

$$C(t, S, \sigma) = SF(d_t) - Ke^{-r(T-t)}F(d_t - \sigma\sqrt{T-t}),$$

$$d_t = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}, \ F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

Tree type approximations

- Approximation of the SDE by a stochastic difference equation

$$\begin{cases} \Delta \ln S(t_n) = \mu \Delta t_n + \sigma \Delta W_1(t_n), \\ \Delta \ln \sigma(t_n) = -\alpha(\ln \sigma(t_{n-1}) - \ln \sigma)\Delta t_n + \nu \Delta W_2(t_n), \\ n = 1, \ldots, N, \end{cases}$$

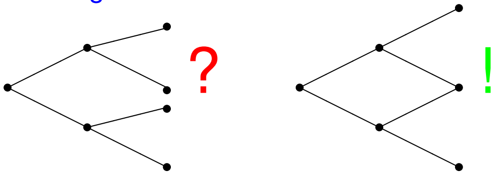
where $\Delta f(t_n) = f(t_n) - f(t_{n-1})$, $t_n = nT/N$, $n = 0, 1, \ldots, N$.

- Fitting of a bivariate binomial model

$$\Delta W_i(t_n) \sim Y_{n,i} \Rightarrow \begin{cases} \mathsf{E} Y_{in} = \mathsf{E} \Delta W_i(t_n), \\ \mathsf{E} Y_{in} Y_{jn} = \mathsf{E} \Delta W_i(t_n) \Delta W_j(t_n), \\ i, j = 1, 2, n = 1, \ldots, N. \end{cases}$$

- A recombining condition

$Y_{n,i}$:



Transformation of the reselling model

- A solution for the system of SDE for the reselling model

$$\begin{cases} S(t) = s_0 e^{\mu t + \sigma W_1(t)}, \\ \sigma(t) = \sigma e^{\nu e^{-\alpha t} \int_0^t e^{\alpha s} dW_2(s)}, \\ t \in [0, T]. \end{cases}$$

- Transformation of the reselling model

$$\begin{aligned} \Phi^{(0)} &= \sup_{\tau \le T} \mathsf{E} e^{-r\tau} C(\tau, s_0 e^{\mu\tau} \cdot S_1(\tau), \sigma \cdot S_2(\tau)^{e^{\alpha(T-\tau)}}) \\ &= \sup_{\tau \le T} \mathsf{E} g(\tau, (S_1(\tau), S_2(\tau))), \end{aligned}$$

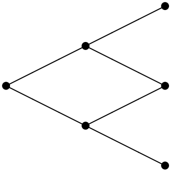
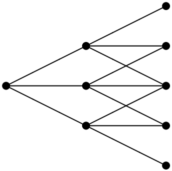
where

$$S_1(t) = e^{\sigma W_1(t)}, \ S_2(t) = e^{\nu e^{-\alpha T} \int_0^t e^{\alpha s} dW_2(s)}, t \ge 0,$$

$$g(t, (s_1, s_2)) = e^{-rt} C(t, s_0 e^{\mu t} \cdot s_1, \sigma \cdot s_2^{e^{\alpha(T-t)}}).$$

An approximation tree bivariate binomial-trinomial model

- A tree bivariate binomial-trinomial model

$Y_{n,1}$:  $Y_{n,2}$: 

$\varepsilon = T/N$

Number of nodes $= (N+1)(2N+1)$!

$\mathbf{E_k}$: $|\rho| < e^{-\alpha T/k}$. In the case $k = 1$:

$$u_{n,1}^{(\varepsilon)} = \sigma\sqrt{\varepsilon}, \quad u_{n,2}^{(\varepsilon)} = u\sqrt{\varepsilon}, \text{where } u \in [\nu, \nu|\rho|^{-1}e^{-\alpha T}]$$

$$p_{n,++}^{(\varepsilon)} = p_{n,--}^{(\varepsilon)} = \frac{\nu^2 e^{-2\alpha T}}{4u^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon} + \frac{\rho\nu e^{-\alpha T}}{4u} e^{\alpha n\varepsilon} \frac{1-e^{-\alpha\varepsilon}}{\alpha\varepsilon},$$

$$p_{n,+-}^{(\varepsilon)} = p_{n,-+}^{(\varepsilon)} = \frac{\nu^2 e^{-2\alpha T}}{4u^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon} - \frac{\rho\nu e^{-\alpha T}}{4u} e^{\alpha n\varepsilon} \frac{1-e^{-\alpha\varepsilon}}{\alpha\varepsilon},$$

$$p_{n,+\cdot}^{(\varepsilon)} = p_{n,-\cdot}^{(\varepsilon)} = \frac{1}{2} - \frac{\nu^2 e^{-2\alpha T}}{2u^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon},$$

$$n = 1, \ldots, N.$$

Convergence of tree approximations

- A recurrence backward algorithm

$$(1): \vec{y}_{n,l_1,l_2} = ((2l_1 - n)\sigma\sqrt{\varepsilon}, l_2 u \sqrt{\varepsilon})$$

$$l_1 = 0, 1, \ldots, n, \ l_2 = 0, \pm 1, \ldots, \pm n, n = 0, \ldots, N;$$

$$(2): w^{(\varepsilon)}(t_N, \vec{y}_{N,l_1,l_2}) = g(t_N, e^{\vec{y}_{N,l_1,l_2}}),$$

$$l_1 = 0, 1, \ldots, N, \ l_2 = 0, \pm 1, \ldots, \pm N;$$

$$(3): w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2}) = g(t_n, e^{\vec{y}_{n,l_1,l_2}}) \vee \Big(w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2+1}) p^{(\varepsilon)}_{n,++}$$

$$+ w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2}) p^{(\varepsilon)}_{n,+\cdot} + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2-1}) p^{(\varepsilon)}_{n,+-}$$

$$+ w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2+1}) p^{(\varepsilon)}_{n,-+} + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2}) p^{(\varepsilon)}_{n,-\cdot}$$

$$+ w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2-1}) p^{(\varepsilon)}_{n,--} \Big),$$

$$l_1 = 0, 1, \ldots, n, \ l_2 = 0, \pm 1, \ldots, \pm n, n = N-1, \ldots, 0.$$

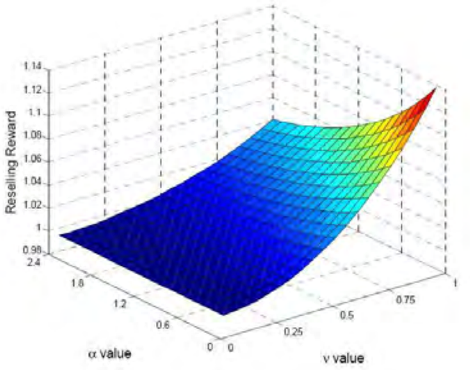
- Convergence of tree approximations

Theorem 2: $\mathbf{E_1} \Rightarrow$

$$w^{(\varepsilon)}(0, (0, 0)) \to \Phi^{(0)} \text{ as } \varepsilon \to 0.$$

Numerical examples

- The optimal expected reselling rewards



The optimal expected reselling rewards for the models with parameters $r = 0.04$; $S(0) = 10$, $\mu = 0.02$, $\sigma = 0.2$, $0.12 < \alpha < 2.4$, $0.05 < \nu < 1$, $\rho = 0.3$; and $K = 10$, $T = 0.5$.

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