Liquidity Risk and the Dynamics of Arbitrage Capital

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When non-financial firms/individuals trade, almost always some specialized agents like market makers, brokers, and hedge funds take the other side.

A view of financial markets: non-financial firms/individuals face various risks which financial firms are willing to partially absorb for compensation.

- main issue: no immediate new inflows of financial capital (even if returns are high)

We take this view in this paper and characterize the joint dynamics of asset prices and financial capital.

Main application: outcome generate “liquidity facts” in essentially frictionless economy.
Liquidity facts

- assets’ illiquidity: difficulty to sell
  - measures: price impact/ negative autocorrelation/ spread
- Liquidity varies over time and in correlated manner across assets (aggregate illiquidity)
- Liquidity can be priced risk factor: higher expected return for
  - ...stocks paying off when liquidity high?
  - ...stocks more liquid when liquidity high?
  - ...stocks more liquid when aggregate return high?

- Aggregate illiquidity depends on financial institutions’ level of capital (Market makers, arbitrageurs, speculators, hedge funds, trading desks of investment banks...)
A dynamic model of risk-sharing

- Continuous time, infinite horizon, $t \in [0, \infty)$.

**Hedgers.** (e.g. non-financial firms, farmers, individuals)

- Endowment $u^\top dD_t$ at $t + dt \Rightarrow$ hedging demand at $t$, where
  \[ dD_t = \bar{D} dt + \sigma^\top dB_t, \]
  and $B_t$ is $N$-dimensional Brownian motion. Payoff covariance matrix $\Sigma \equiv \sigma^\top \sigma$.

- Mean-variance preferences over change $dv_t$ in wealth between $t$ and $t + dt$
  \[ \frac{\mathbb{E}_t(dv_t)}{dt} - \frac{\alpha}{2} \frac{\text{Var}_t(dv_t)}{dt} \]
  $\Rightarrow$ Demand for insurance is constant over time.
Arbitrageurs. (e.g. dealers, brokers, hedge funds, insurance companies, etc.)

- CRRA preferences over intertemporal consumption

\[ \mathbb{E}_t \left( \int_t^{\infty} \frac{c^{1-\gamma}_s}{1-\gamma} e^{-\rho(s-t)} ds \right) \]

⇒ Supply for insurance is time-varying because of wealth effects.
Assets.

- (for now), $N$ short-lived risk-sharing contracts at each time $t$.
  - Payoff $dD_t$ at time $t + dt$
  - Price $\pi_t dt$ at time $t$.
  - Zero net supply.
  - (we’ll introduce long-lived financial assets in a bit)

- exogenous riskless rate $r$
Equilibrium Prices and Positions

- Arbitrageur positions:
  \[ y_t = \frac{\alpha}{\alpha + A(w_t)} u. \]

  Arbitrageurs hold fraction of portfolio \( u \) that hedgers want to sell.  
  Standard risk-sharing rule, but with effective risk aversion \( A(w_t) \).

- Asset prices:
  \[ \pi_t = D - \frac{\alpha A(w_t)}{\alpha + A(w_t)} \Sigma u. \]

  Portfolio \( u \) that hedgers want to sell is single pricing factor.
Effective Risk Aversion

\[ A(w_t) \equiv \frac{\gamma}{w_t} - \frac{q'(w_t)}{q(w_t)} \]

is effective risk aversion

- Logarithmic preferences (\( \gamma = 1 \)).
  - Consumption proportional to wealth
  - Effective risk aversion is
    \[ A(w_t) = \frac{1}{w_t}. \]
  - Static ARA. No intertemporal hedging.
• Risk-neutral preferences ($\gamma \to 0$) and riskless rate $r \to 0$.
  • Consumption is zero for $w_t \in (0, \bar{w})$ and at infinite rate for $w_t \in (\bar{w}, \infty)$.
  • Effective risk aversion is
    \[
    A(w_t) = \frac{\alpha}{1 + z} \left( \sqrt{z} \cot \left( \frac{\alpha w_t}{\sqrt{z}} \right) - 1 \right) \quad \text{for} \quad w_t \in (0, \bar{w}),
    \]
    where $z \equiv \frac{\alpha^2 u^T \Sigma u}{2 \rho}$ and $\cot \left( \frac{\alpha \bar{w}}{\sqrt{z}} \right) \equiv \frac{1}{\sqrt{z}}$.
  • Static ARA=0. Only intertemporal hedging.
Effective Risk Aversion

Liquidity Risk and the Dynamics of Arbitrage Capital

Kondor and Vayanos (2013)
Stationary Distribution

- Self-correcting dynamics: Arbitrageur Sharpe ratio (SR)
- Depends only on $z \equiv \frac{\alpha^2 u^\top \Sigma u}{2(\rho - r)}$.

<table>
<thead>
<tr>
<th>$0 &lt; z &lt; 1$</th>
<th>$1 &lt; z &lt; \tilde{z}$</th>
<th>$\tilde{z} &lt; z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_t$ converges to 0</td>
<td>decreasing pdf</td>
<td>bimodal pdf</td>
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- $\tilde{z} = \frac{27}{8}$ in logarithmic case, $\tilde{z} = 4$ in risk-neutral case decreases in wealth.
- $z$ pushes distribution to the left in a Monotone Likelihood Ratio-sense
- Bimodal distribution, sign of systemic risk?
Shape of Stationary Density

Liquidity Risk and the Dynamics of Arbitrage Capital

Kondor and Vayanos (2013)
Unconditional objects

Average residual risk, $(u - y)^\top \Sigma (u - y)$

Average Sharpe ratio

$z$ (with $\Sigma$ varying)
Long-Lived Assets

- $N$ risky assets.
  - Price $S_t$ at time $t$.
  - Payoff $dD_{t'}$ for times $t' > t$. (Infinite stream of short-lived assets’ payoffs.)
  - Zero net supply.

Comparison with short-lived assets:
- Same allocation of risk and market prices of risk.
- But an asset $\neq$ a claim on unit risk: return depends on (endogenous) price-dynamics $\rightarrow$ Liquidity risk.
  - Zero with short-lived assets.
- Time-varying volatilities and correlations.
  - Constant with short-lived assets.
- key: how $dD_t$ shocks are transformed to wealth shocks and how this affects expected returns.
Endogenous wealth shocks and expected returns

- Long-lived assets do not affect risk sharing.
- Risk holding proportional to $u$.
- $dD_t$ changes wealth proportionally to $u$.
- More compensation for assets covarying with $u$.
- Price negatively, expected return positively proportional to $\Sigma u$.
Equilibrium

- Asset prices proportional to $\Sigma u$ ($g(0) = 0, \frac{\alpha}{r} > g(w_t) > 0, g'(w_t) > 0$):

$$S(w_t) = \frac{\bar{D}}{r} - \left( \frac{\alpha}{r} - g(w_t) \right) \Sigma u.$$  

- risk-neutral price

- premium

- expected return also proportional to $\Sigma u$:

$$\mathbb{E}_t(dR_t) = \frac{\alpha A(w_t)}{\alpha + A(w_t)} \left[ \frac{\alpha g'(w_t) u^\top \Sigma u}{\alpha + A(w_t)} + 1 \right] \Sigma u.$$  

- scalar, hump-shaped in $w_t$

- Arbitrageurs’ holding in assets is also proportional to $u$ and grows from 0 to $u$
Closed-Form Solutions

- Compute \( g'(w_t) \) in logarithmic and risk-neutral cases when \( r \to 0 \).

- Volatilities are hump-shaped in arbitrageur wealth.
  - Price volatility = Volatility of arb. wealth \( \times \) Price sensitivity to wealth.
  - \( w_t \approx 0 \) ⇒ Volatility of arb. wealth \( \approx 0 \).
  - \( w_t \) large ⇒ Price sensitivity to wealth \( \approx 0 \).

- Expected returns are also hump-shaped
  - Price of risk still decreasing but volatility is hump-shaped

- Non-fundamental covariance because of wealth shocks: humped shape in wealth
  - effect all returns the most when their non-fundamental volatility is highest
Liquidity Risk and the Dynamics of Arbitrage Capital

Kondor and Vayanos (2013)
**Illiquidity of an asset: Kyle’s lambda**

- Define illiquidity $\lambda_{nt}$ of asset $n$ as price change per unit of quantity traded, following shock to $u_n$, hedgers’ willingness to hold asset $n$
  - Kyle’s lambda: $\lambda_{nt} \equiv \frac{\partial S_{nt}}{\partial u_n} \frac{\partial X_{nt}}{\partial u_n}$
  - Can also interpret $\lambda$ as concerning price difference between asset pair with different $u$’s.
- Illiquidity $\lambda_{nt}$ of asset $n$ is equal to

$$
\left(1 + \frac{A(w_t)}{\alpha} + g'(w_t)u^\top \Sigma u\right) \left(\frac{\alpha}{r} - g(w_t)\right) \Sigma_{nn}.
$$

- Depends on $n$ through variance $\Sigma_{nn}$ of asset’s payoff (consistent with Stoll (1978), Chen, Lesmond and Wei (2007) etc).
- Decreasing in arbitrageur wealth.
Illiquidity factor

- consider one of A-P illiquidity betas (covariance of asset’s return and aggregate illiquidity, \( \Lambda_t = -\sum \frac{\lambda_{nt}}{N} \)):

\[
\beta_n(w) \equiv \text{Cov}(d\Lambda_t, dR_{nt})
\]

- as \( \Lambda_t \) monotonic in wealth and wealth shocks affect prices proportional to \( \Sigma u \), we have \( \beta_n(w) = \beta^C(w)\Sigma u \) (while illiquidity proportional to \( \Sigma_{nn} \))

- as expected returns are also proportional to \( \Sigma u \) we can write

\[
\frac{\mathbb{E}_t(dR_{nt})}{dt} = \Pi(w)\beta_n(w)
\]

with \( \Pi(w) \) premium for illiquidity risk

- \( \Rightarrow \) Asset \( n \)'s expected return is explained by:
  - Covariance between asset’s return and aggregate liquidity.
  - Not by covariance between asset's liquidity and aggregate liquidity or return.
Our model can be extended in a number of directions. We sketch the main extensions in this section, and analyze them more thoroughly in Kondor and Vayanos (2014). One extension is to assume that the supply of long-lived assets is positive instead of zero. This assumption makes the model more directly applicable to stocks and bonds, and to the empirical findings on priced liquidity factors in those markets. Introducing positive supply preserves the basic structure of the model.

\( w_t \approx 0 \): Illiquidity is large and highly sensitive to \( w_t \Rightarrow \) Covariance is large and premium is small.
Intuition

- as arbitrageurs' take one side of each trade, their f.o.c determines prices: their portfolio is the pricing factor
- Assets covarying most with hedgers’ portfolio:
  - Have high expected returns.
  - And drop the most when arbitrageur wealth decreases.
- empirical measures of illiquidity factor are proxying this pricing factor
Infinitely lived CARA hedgers and positive supply

- add a positive supply vector of assets, and ...

- CARA preferences over intertemporal consumption:

\[ \mathbb{E}_t \left( \int_t^\infty - \exp(-\alpha \hat{c}_s) e^{-\hat{\rho}(s-t)} ds \right). \]

- Preserves no wealth effects for hedgers.
- Adds intertemporal hedging demand.
  - Hedge against changes in arbitrageur wealth: effective risk-aversion for hedgers depends on \( w_t \) also
- Solutions become numerical (ODE), but main results remain the same.
Infinitely lived CARA hedgers and positive supply

Liquidity Risk and the Dynamics of Arbitrage Capital
Kondor and Vayanos (2013)
Related Literature

Arbitrageur Capital, Asset Prices and Liquidity.

- Survey by Gromb-Vayanos (2010).

Dynamic Risk-Sharing with CRRA Agents.

Conclusion

- Continuous-time, multi-asset model of liquidity provision with wealth effects.

- Implications for: Expected asset returns, volatilities, correlations, arbitrageur positions, short-run and long-run dynamics.

- Pricing of illiquidity factors:
  - as arbitrageurs' take one side of each trade, their f.o.c determines prices: their portfolio is the pricing factor
  - Assets covarying most with hedgers' portfolio:
    - Have high expected returns.
    - And drop the most when arbitrageur wealth decreases.
  - empirical measures of illiquidity factor are proxying this pricing factor

- Extensions in online appendix (positive supply assets, long-horizon hedgers, stochastic $u_t$)