Optimal Unconditional Monetary Policy, Trend Inflation and the Zero Lower Bound

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Abstract

As the real rate of interest shows a decreasing path during the last 10 years or so, I address how optimal monetary policy must conform to this new instance. For that, I first identify the way monetary policy influences the probability of the nominal interest rate hitting and remaining and the ZLB, by means of the expectations channel. Next, I derive the time-consistent (unconditionally) optimal monetary policy under commitment to be adopted between ZLB episodes, when the constraint is occasionally binding, in a standard New-Keynesian model, and the central bank internalize its role in determining ZLB episodes. My approach allows for directly retaining precautionary policy behavior even under the log-linearized version of the model. So, it is easily incorporated into standard business cycle models. Finally, I verify how optimal policy must be implemented as the natural real rate of interest decreases towards zero. Results suggest that optimal policy resembles price level targeting at low real interest rates and low levels of inflation targets (trend inflation). As the inflation target is increased, a more entangled policy must be implemented, due to the following policy tradeoff. Strong responses to negative demand shocks help output, but increase the probability of hitting the ZLB. Therefore, more attenuated responses are indicated. Finally, the effects of increasing the inflation target are not the same as the ones obtained under higher levels of natural real interest rates.

Keywords: Trend inflation, Optimal Policy, Zero Lower Bound
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1 Introduction

As the long-run real rate of interest shows a decreasing path during the last 10 years, hitting estimated levels as low as 1% or even smaller (see e.g. Laubach and Williams (2015), Bauer and Rudebusch (2016) and Yi and Zhang (2016)), I address how optimal monetary policy must conform to this new instance, in frameworks in which shocks (demand and technology) have continuos distributions. Here, I contribute for the discussion by assessing the consequences of monetary policy controlling future probabilities of hitting the zero lower bound (ZLB) in a economy in which the ZLB constraint is occasionally binding. Moreover, I follow Damjanovic et al. (2008) strategy for deriving optimal policies under unconditionally commitment, which is unconditionally time-consistent.

I first identify the way monetary policy influences the probability of the nominal interest rate hitting and remaining and the ZLB, by means of the expectations channel. Next, I derive the time-consistent (unconditionally) optimal monetary policy under commitment to be adopted between ZLB episodes, when the constraint is occasionally binding, and the central bank internalize its role in determining ZLB episodes.

An important result is that my results directly internalize precautionary behavior even under the first-order approximation of the model. This is an important difference from my result to similar ones in the literature, e.g. Eggertsson and Woodford (2003a,b) and Nakov (2008).

I find that optimal policy resembles price level targeting at low real interest rates and low levels of inflation targets (trend inflation). Under low levels of nominal interest rates, the optimal targeting rule becomes even more history dependent and more dependent on past values of the nominal interest rate. That is, the central bank consciously and directly adopts precautionary behavior in normal times, not decreasing the rate as much on spot in response to negative demand shocks in order to create more room for future effective monetary policy changes. It also takes even longer than what standard optimal policy (e.g. Nakov (2008)) prescribe to increase the rates after the shocks have dissipated. This precautionary behavior becomes stronger as the steady state level of nominal interest rate is reduced.

I also find that, as the inflation target is increased, a more entangled policy must be implemented, as the effects of increasing the inflation target are not the same as the ones obtained under higher levels of natural real interest rates. As a matter of fact, higher levels of trend inflation also brings distortions to the economy.

When the monetary authority does not directly internalize its role in affecting the probability of hitting the ZLB, as in e.g. Nakov (2008), the functional form of optimal targeting rules during normal times are the same as the ones obtained in economies where the ZLB is never hit. In this standard approach, policy precautionary behavior arises only indirectly in general equilibrium, and so the central bank does not benefit as much.

When comparing welfare-based performances of both types of optimal policies under occasionally binding ZLB constraint, I find that the precautionary optimal policy dominates the standard optimal policy for every level of real interest rates and trend inflation (inflation target).

The remainder of the paper is organized as follows. The model is described in Section 2. Key results on the probability of hitting the ZLB and the design of the precautionary optimal policy are derived in Section 3. The effect of declining real interest rates at different levels of trend inflation on welfare is discussed in Section 5.1, while Section 5.2 assesses how optimal policies perform after negative demand shocks. Section 6 summarizes the paper's conclusions.

2 The model

For simplicity, I follow Woodford (2003, chap. 4) to describe the standard new-Keynesian model with Calvo (1983) price setting and flexible wages. The economy consists of a representative infinite-lived household that consumes an aggregate bundle and supplies differentiated labor to a continuum of differentiated firms indexed by $z \in (0,1)$, which produce and sell goods in a monopolistic competition environment.

2.1 Households

Household's workers supply $h_t(z)$ hours of labor to each firm z, at nominal wage $W_t(z) = P_t w_t(z)$, where P_t is the consumption price index and $w_t(z)$ is the real wage. Disutility over hours worked in each firm is $\upsilon_t(z) \equiv \chi h_t(z)^{1+\nu}/(1+\nu)$, where ν^{-1} is the Frisch elasticity of labor supply. The household's aggregate disutility function is $\upsilon_t \equiv \int_0^1 \upsilon_t(z)\,dz$. Consumption $c_t(z)$ over all differentiated goods is aggregated into a bundle C_t, as in Dixit and Stiglitz (1977), and provides utility $u_t \equiv \epsilon_t C_t^{1-\sigma}/(1-\sigma)$, where σ^{-1} is the intertemporal elasticity of substitution and ϵ_t is a preference shock. Aggregation and expenditure minimization relations are described by:

$$\begin{array}{ll} C_t^{\frac{\theta-1}{\theta}} = \int_0^1 c_t(z)^{\frac{\theta-1}{\theta}}\,dz & ;\; P_t^{1-\theta} = \int_0^1 p_t(z)^{1-\theta}\,dz \\ c_t(z) = C_t\left(\frac{p_t(z)}{P_t}\right)^{-\theta} & ;\; P_t C_t = \int_0^1 p_t(z)\,c_t(z)\,dz \end{array} \tag{1}$$

where $\theta > 1$ is the elasticity of substitution between goods.

Financial markets are complete and the budget constraint is $P_t C_t + E_t q_{t+1} B_{t+1} \leq B_t + P_t \int_0^1 w_t(z)\,h_t(z)\,dz + d_t$, where B_t is the state-contingent value of the portfolio of financial securities held at the beginning of period t, d_t denotes nominal dividend income, and q_{t+1} is the stochastic discount factor from $(t+1)$ to t. The household chooses the sequence of C_t, $h_t(z)$ and B_{t+1} to maximize its welfare measure $\mathcal{W}_t \equiv \max E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t}(u_\tau - \upsilon_\tau)$, subject to the budget constraint and a standard no-Ponzi condition, where β denotes the subject discount factor. In equilibrium,[1] optimal labor supply satisfies $w_t(z) = \upsilon_t'(z)/u_t'$, where $u_t' \equiv \partial u_t/\partial C_t$ is the marginal utility to consumption and $\upsilon_t'(z) \equiv \partial \upsilon_t(z)/\partial h_t(z)$ is the marginal disutility to hours. The optimal consumption plan and dynamics of the stochastic discount factor are described as follows:

$$1 = \beta E_t\left(\frac{u_{t+1}'}{u_t'}\frac{I_t}{\Pi_{t+1}}\right) \;\; ; \; q_t = \beta \frac{u_t'}{u_{t-1}'}\frac{1}{\Pi_t} \tag{2}$$

where $\Pi_t = 1 + \pi_t$ and $I_t = 1 + i_t$ are the gross inflation and interest rates at period t, which satisfies $I_t = 1/E_t q_{t+1}$, and i_t is the riskless one-period nominal interest rate.

2.2 Firms

Firm $z \in (0,1)$ produces differentiated goods using the technology $y_t(z) = \mathcal{A}_t h_t(z)^{\varepsilon}$, where $\mathcal{A}_t$ is the aggregate technology shock and $\varepsilon \in (0,1)$. The aggregate output Y_t

[1] Equilibrium is defined as the equations describing the first order conditions, a transversality condition $\lim_{T\to\infty} E_T q_{t,T} B_T = 0$, where $q_{t,T} \equiv \Pi_{\tau=t+1}^{T} q_\tau$, and the market clearing conditions.

is implicitly defined by $P_tY_t = \int_0^1 p_t(z)\, y_t(z)\, dz$. Using the market clearing condition $y_t(z) = c_t(z)$, $\forall z$, the definition implies that the firm's demand function is $y_t(z) = Y_t(p_t(z)/P_t)^{-\theta}$, where $Y_t = C_t$.

With probability $(1-\alpha)$, the firm optimally readjusts its price to $p_t(z) = p_t^*$. With probability α, the firm sets its price according to $p_t(z) = p_{t-1}(z)\,\Pi_t^{ind}$, where $\Pi_t^{ind} \equiv \Pi_{t-1}^{\gamma_\pi}$ and $\gamma_\pi \in (0,1)$. When optimally readjusting at period t, the price p_t^* maximizes the expected discounted flow of nominal profits $\mathcal{P}_t(z) = p_t(z)\, y_t(z) - P_t w_t(z)\, h_t(z) + E_t q_{t+1} \mathcal{P}_{t+1}(z)$, given the demand function and the price setting structure. At this moment, the firm's real marginal cost is $mc_t^* = (1/\mu)\, X_t^{(\omega+\sigma)} (p_t^*/P_t)^{-\theta\omega}$, where $\omega \equiv (1+\nu)/\varepsilon - 1$ is a composite parameter, $\mu \equiv \theta/(\theta-1) > 1$ is the static markup parameter, $X_t \equiv Y_t/Y_t^n$ is the gross output gap, and Y_t^n is the natural (flexible prices) output, which evolves according to

$$Y_t^{n^{(\omega+\sigma)}} = \frac{\varepsilon}{\chi\mu}\epsilon_t \mathcal{A}_t^{(1+\omega)} \tag{3}$$

Following e.g. Ascari and Sbordone (2013, Section 3) and Ascari (2004, online Appendix), the firm's first order condition can be conveniently written, in equilibrium, as follows:

$$\left(\frac{p_t^*}{P_t}\right)^{1+\theta\omega} = \frac{N_t}{D_t} \tag{4}$$

The numerator N_t and the denominator D_t functions can be written in recursive forms, avoiding infinite sums:

$$\begin{aligned} N_t &= (X_t)^{(\omega+\sigma)} + E_t n_{t+1} N_{t+1} \quad &; n_t = \alpha q_t \mathcal{G}_t \Pi_t \left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta(1+\omega)} \\ D_t &= 1 + E_t d_{t+1} D_{t+1} \quad &; d_t = \alpha q_t \mathcal{G}_t \Pi_t \left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{(\theta-1)} \end{aligned} \tag{5}$$

where $\mathcal{G}_t \equiv Y_t/Y_{t-1}$ denotes the gross output growth rate. The price setting structure implies the following dynamics:

$$1 = (1-\alpha)\left(\frac{p_t^*}{P_t}\right)^{-(\theta-1)} + \alpha \left(\frac{\Pi_t}{\Pi_t^{ind}}\right)^{(\theta-1)} \tag{6}$$

2.3 Aggregates

Following, I present a set of equations describing the evolution of the aggregate disutility $\upsilon_t \equiv \int_0^1 \upsilon_t(z)\, dz$ to labor and the aggregate hours worked $h_t \equiv \int_0^1 h_t(z)\, dz$. For that, let $\mathcal{P}_t^{-\theta(1+\omega)} \equiv \int_0^1 (p_t(z)/P_t)^{-\theta(1+\omega)} dz$ and $\mathcal{P}_{ht}^{-\theta(1+\tilde{\omega})} \equiv \int_0^1 (p_t(z)/P_t)^{-\theta(1+\tilde{\omega})} dz$ denote two distinct measures of aggregate relative prices, where $\tilde{\omega} \equiv \frac{1}{\varepsilon} - 1$. Using the Calvo (1983) price setting structure, I am able to derive the laws of motion of $\mathcal{P}_t$ and $\mathcal{P}_{ht}$.[2] The result is general and independent of any level of trend inflation. The following system describes

[2] The way I derive the law of motion of $\mathcal{P}_t$ and $\mathcal{P}_{ht}$ is very similar to how e.g. Alves (2014), Schmitt-Grohe and Uribe (2007) and Yun (2005) derive relevant price dispersion variables for aggregate output, employment, resource constraints and aggregate disutility in their models.

the evolution of υ_t, h_t, $\mathcal{P}_t$ and $\mathcal{P}_{ht}$:

$$\upsilon_t = \tfrac{\chi}{1+\nu}\left(\tfrac{Y_t}{\mathcal{A}_t}\right)^{(1+\omega)}\mathcal{P}_t^{-\theta(1+\omega)} \quad ; \; h_t = \left(\tfrac{Y_t}{\mathcal{A}_t}\right)^{(1+\tilde{\omega})}\mathcal{P}_{ht}^{-\theta(1+\tilde{\omega})}$$
$$\mathcal{P}_t^{-\theta(1+\omega)} = (1-\alpha)\left(\tfrac{p_t^*}{P_t}\right)^{-\theta(1+\omega)} + \alpha\left(\tfrac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta(1+\omega)}\mathcal{P}_{t-1}^{-\theta(1+\omega)}$$
$$\mathcal{P}_{ht}^{-\theta(1+\tilde{\omega})} = (1-\alpha)\left(\tfrac{p_t^*}{P_t}\right)^{-\theta(1+\tilde{\omega})} + \alpha\left(\tfrac{\Pi_t}{\Pi_t^{ind}}\right)^{\theta(1+\tilde{\omega})}\mathcal{P}_{ht-1}^{-\theta(1+\tilde{\omega})}$$

where $\wp_t^* \equiv p_t^*/P_t$ is the optimal resetting relative price.

2.4 The log-linearized model

For any variable $\mathfrak{W}_t$, $\widehat{\mathfrak{w}}_t \equiv \log\left(\mathfrak{W}_t/\overline{\mathfrak{W}}\right)$ represents its log-deviation from its steady state level $\overline{\mathfrak{W}}$ with non-zero trend inflation (Trend StSt). All steady state levels and parameter definitions are shown in Appendix A.

Under flexible prices ($\alpha = 0$), the (log-deviation) real interest rate and (log-deviation) output $\hat{y}_t^n$ evolve according to the following equations:

$$\hat{r}_t^n = E_t\left[\sigma\left(\hat{y}_{t+1}^n - \hat{y}_t^n\right) - \left(\hat{\epsilon}_{t+1} - \hat{\epsilon}_t\right)\right] \quad ; \hat{y}_t^n = \tfrac{1}{(\omega+\sigma)}\left[(1+\omega)\widehat{\mathcal{A}}_t + \hat{\epsilon}_t\right] \tag{7}$$

Under sticky prices ($\alpha > 0$),[3] the (log-deviation) output gap $\hat{x}_t$ is defined as follows:

$$\hat{x}_t = \hat{y}_t - \hat{y}_t^n \tag{8}$$

The log-linearized IS curve is:

$$\hat{x}_t = E_t\hat{x}_{t+1} - \frac{1}{\sigma}E_t\left(\hat{\imath}_t - \hat{\pi}_{t+1} - \hat{r}_t^n\right) \tag{9}$$

The Generalized New Keynesian Phillips Curve (GNKPC) under trend inflation, as coined by Ascari and Sbordone (2013), is obtained by log-linearizing the firm's first order system $(4) - (5)$ and the price setting structure (6) about the Trend StSt. As in Alves (2014), I describe the GNKPC system in terms of the output gap as the only demand variable:

$$\begin{aligned}
\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) &= \beta E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + \bar{\kappa}\hat{x}_t + \left(\bar{\vartheta} - 1\right)\bar{\kappa}_{\varpi}\beta E_t\hat{\varpi}_{t+1} + \widehat{\mathfrak{u}}_t \\
\hat{\varpi}_t &= \bar{\alpha}\bar{\vartheta}\beta E_t\hat{\varpi}_{t+1} + \theta(1+\omega)\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \left(1 - \bar{\alpha}\bar{\vartheta}\beta\right)(\omega+\sigma)\hat{x}_t + (1-\sigma)\left(\hat{x}_t - \hat{x}_{t-1}\right) \\
\widehat{\mathfrak{u}}_t &= \bar{\alpha}\bar{\vartheta}\beta E_t\widehat{\mathfrak{u}}_{t+1} + \left(\bar{\vartheta} - 1\right)\beta E_t\hat{\xi}_{t+1} \\
\hat{\xi}_t &= \bar{\kappa}_{\varpi}\tfrac{(1+\omega)}{(\omega+\sigma)}\left[(1-\sigma)\left(\widehat{\mathcal{A}}_t - \widehat{\mathcal{A}}_{t-1}\right) + \left(\hat{\epsilon}_t - \hat{\epsilon}_{t-1}\right)\right]
\end{aligned} \tag{10}$$

where $\hat{\pi}_t^{ind} = \gamma_\pi\hat{\pi}_{t-1}$ is the indexation term, $\hat{\varpi}_t$ is an ancillary variable with no obvious interpretation,[4] $\hat{\xi}_t$ is an aggregate shock term that collects the effects of the technology

[3] I am aware that the degree of price rigidity α is likely to endogenously decrease as the trend inflation rises. I assume, however, that the parameter remains constant for all values of trend inflation as long as it is sufficiently small (less than 5% year, for instance).

[4] In the literature on trend inflation, there are two usual ways to describe trend inflation Phillips

shock $\widehat{\mathcal{A}}_t$ and the utility shock $\hat{\epsilon}_t$, and $\widehat{\mathfrak{u}}_t$ is the endogenous *trend inflation cost-push shock*, which ultimately depends only on the technology and preference shocks. As for the composite parameters, $\bar{\vartheta} \equiv \bar{\Pi}^{(1+\theta\omega)(1-\gamma_\pi)}$ is a positive transformation of the level $\bar{\pi}$ of trend inflation and $\bar{\alpha} \equiv \alpha\bar{\Pi}^{(\theta-1)(1-\gamma_\pi)}$ is the effective degree of price stickiness.[5] Since $\bar{\alpha}$ and $\bar{\vartheta}$ increase as trend inflation rises, the trend inflation cost-push shock $\widehat{\mathfrak{u}}_t$ amplifies, by means of $\left(\bar{\vartheta}-1\right)$ and the coefficient $\bar{\alpha}\bar{\vartheta}\beta$ on $E_t\widehat{\mathfrak{u}}_{t+1}$, the effect of the aggregate shock $\hat{\xi}_t$ and transmits it through the inflation dynamics. The remaining composite parameters are

$$\bar{\kappa} \equiv \frac{(1-\bar{\alpha})\left(1-\bar{\alpha}\beta\bar{\vartheta}\right)}{\bar{\alpha}}\frac{(\omega+\sigma)}{(1+\theta\omega)} \quad ; \; \bar{\kappa}_{\varpi} \equiv \frac{(1-\bar{\alpha})}{(1+\theta\omega)} \quad ; \; \omega \equiv \frac{(1+\nu)}{\varepsilon}-1 \tag{11}$$

As well documented in the literature on trend inflation, the GNKPC becomes flatter ($\bar{\kappa}$ decreases) and more forward looking ($\left(\bar{\vartheta}-1\right)\bar{\kappa}_{\varpi}\beta$ and $\bar{\alpha}\bar{\vartheta}\beta$ increases) with trend inflation.[6] The effect of $\hat{\varpi}_t$ on the inflation dynamics is to make it even more forward looking. This is due to the fact that the coefficients $\left(\bar{\vartheta}-1\right)$ on $E_t\hat{\varpi}_{t+1}$, in the first equation, and $\bar{\alpha}\bar{\vartheta}\beta$ on $E_t\hat{\varpi}_{t+1}$, in the second equation, increase as trend inflation rises.

As for the aggregates, we have:

$$\hat{\upsilon}_t = (1+\omega)\left(\hat{y}_t - \widehat{\mathcal{A}}_t - \theta\hat{\mathcal{P}}_t\right) \quad ; \; \hat{h}_t = (1+\tilde{\omega})\left(\hat{y}_t - \widehat{\mathcal{A}}_t - \theta\hat{\mathcal{P}}_{ht}\right)$$
$$\hat{\mathcal{P}}_t = \bar{\alpha}\bar{\vartheta}\hat{\mathcal{P}}_{t-1} - \frac{\left(\bar{\vartheta}-1\right)\bar{\alpha}}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) \quad ; \; \hat{\mathcal{P}}_{ht} = \bar{\alpha}\tilde{\vartheta}\hat{\mathcal{P}}_{ht-1} - \frac{\left(\tilde{\vartheta}-1\right)\bar{\alpha}}{(1-\bar{\alpha})}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$$

where $\tilde{\vartheta} \equiv \bar{\Pi}^{(1+\theta\tilde{\omega})(1-\gamma_\pi)}$ and $\tilde{\omega} \equiv \frac{1}{\varepsilon}-1$.

3 Probability of hitting the ZLB

Given the information set $\mathfrak{I}_t$ at period t, the probability $\mathfrak{p}_{\mathfrak{o},t} \equiv \mathbb{P}\left(I_t \leq 1|\mathfrak{I}_t\right)$ of hitting the ZLB at period t is an endogenous variable. In this regard, monetary policy has an important role, for it influences $\mathfrak{p}_{\mathfrak{o},t}$ by means of the expectations channel.

I assume that the preference (demand) shock ϵ_t follows an AR(1) process $\epsilon_t = \epsilon_{t-1}^{\rho_\mathfrak{u}}\epsilon_{\mathfrak{u},t}$, where $\epsilon_{\mathfrak{u},t}$ is a unit-meaned white noise disturbance term. Let us first consider the natural (flexible prices) equilibrium, in which the inflatin rate is kept fixed at the trend inflation level $\bar{\pi}$. In this case, natural output Y_t^n evolves according to $Y_t^{n(\omega+\sigma)} = \frac{\varepsilon}{\chi\mu}\epsilon_t\mathcal{A}_t^{(1+\omega)}$, where I assume that the technology shock follows an AR(1) process $\mathcal{A}_t = \mathcal{A}_{t-1}^{\rho_\mathfrak{a}}\epsilon_{\mathfrak{a},t}$, where $\epsilon_{\mathfrak{a},t}$ is a unit-meaned white-noise disturbance, independent of $\epsilon_{\mathfrak{u},t}$. Using the Euler equation (2) and the marginal utility definition, I compute $\mathfrak{p}_{\mathfrak{o},t}^n$ as follows (see Appendix B for more details):

$$\mathfrak{p}_{\mathfrak{o},t}^n = \mathbb{F}_{\mathfrak{ua}}\left(\frac{\beta}{\bar{\Pi}}\left(\epsilon_{t-1}\right)^{-\frac{\omega\rho_\mathfrak{u}(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\mathcal{A}_{t-1}\right)^{\frac{\sigma(1+\omega)\rho_\mathfrak{a}(1-\rho_\mathfrak{a})}{(\omega+\sigma)}}\right)$$

where $\mathbb{F}_{\mathfrak{ua}}(\varkappa) \equiv \mathbb{P}\left(\epsilon_{\mathfrak{ua},t} \leq \varkappa\right)$ and $\mathfrak{f}_{\mathfrak{ua}}(\varkappa)$ are the cdf and density function of the aggregate shock $\epsilon_{\mathfrak{ua},t} \equiv \left(\epsilon_{\mathfrak{u},t}\right)^{\frac{\omega(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\epsilon_{\mathfrak{a},t}\right)^{-\frac{\sigma(1+\omega)(1-\rho_\mathfrak{a})}{(\omega+\sigma)}}$. If $\epsilon_{\mathfrak{u},t} \overset{iid}{\sim} LN\left(0,\mathfrak{s}_\mathfrak{u}^2\right)$ is independent of $\epsilon_{\mathfrak{a},t} \overset{iid}{\sim}$

curves: (*i*) with ancillary variables (e.g. Ascari and Ropele (2007)); and (*ii*) with infite sums (e.g. Cogley and Sbordone (2008) and Coibion and Gorodnichenko (2011)).

[5] The composite parameters $\bar{\alpha}$ and ϑ are bounded by $\max\left(\bar{\alpha},\bar{\alpha}\vartheta\right) < 1$ to guarantee the existence of an equilibrium with trend inflation.

[6] As Ascari and Ropele (2007) show, the GNKPC reduces to the usual form when the level of trend inflation is zero. In this case, the ancillary variable $\hat{\varpi}_t$ become irrelevant and the trend inflation cost-push shock $\widehat{\mathfrak{u}}_t$ vanishes to zero.

$LN\left(0,\mathfrak{s}_{\mathfrak{a}}^2\right)$, where $\mathfrak{s}_{\mathfrak{u}}^2$ and $\mathfrak{s}_{\mathfrak{a}}^2$ are dispersion parameters, then $\epsilon_{\mathfrak{ua},t} \overset{iid}{\sim} LN\left(0,\mathfrak{s}_{\mathfrak{ua}}^2\right)$, where $\mathfrak{s}_{\mathfrak{ua}}^2 \equiv \left(\frac{\omega(1-\rho_{\mathfrak{u}})}{(\omega+\sigma)}\right)^2 \mathfrak{s}_{\mathfrak{u}}^2 + \left(\frac{\sigma(1+\omega)(1-\rho_{\mathfrak{a}})}{(\omega+\sigma)}\right)^2 \mathfrak{s}_{\mathfrak{a}}^2$.

At the steady state, with $\bar{\epsilon}=1$ and $\overline{\mathcal{A}}=1$, I obtain:[7]

$$\bar{\mathfrak{p}}_{\mathfrak{o}}^n = \mathbb{F}_{\mathfrak{ua}}\left(\tfrac{\beta}{\bar{\Pi}}\right) = \tfrac{1}{2}\left[1+\operatorname{erf}\left(\tfrac{-1}{\sqrt{2}}\tfrac{\mathring{i}}{\mathfrak{s}_{\mathfrak{ua}}}\right)\right] \quad ; \; \bar{\mathfrak{f}}_{\mathfrak{ua}} = \mathfrak{f}_{\mathfrak{ua}}\left(\tfrac{\beta}{\bar{\Pi}}\right) = \frac{\bar{I}}{\sqrt{2\pi\mathfrak{s}_{\mathfrak{ua}}^2}}\exp\left(-\tfrac{1}{2}\left(\tfrac{\mathring{i}}{\mathfrak{s}_{\mathfrak{ua}}}\right)^2\right)$$

where $\bar{I}=\bar{\Pi}/\beta$ and $\mathring{i}\equiv\log\left(\bar{I}\right)$ is $-\hat{\imath}_t$ evaluated at $i_t=0$.

A linear approximation of $\mathfrak{p}_{\mathfrak{o},t}^n$ about the trend inflation steady state is what I call the natural ZLB Probability curve:

$$\mathfrak{p}_{\mathfrak{o},t}^n \approx \bar{\mathfrak{p}}_{\mathfrak{o}}^n - \phi_\epsilon\left[\frac{\omega\rho_{\mathfrak{u}}\left(1-\rho_{\mathfrak{u}}\right)}{(\omega+\sigma)}\hat{\epsilon}_{t-1} - \frac{\sigma\left(1+\omega\right)\rho_{\mathfrak{a}}\left(1-\rho_{\mathfrak{a}}\right)}{(\omega+\sigma)}\widehat{\mathcal{A}}_{t-1}\right] \tag{12}$$

where ϕ_ϵ is the shock-elasticity of ZLB probability.

$$\phi_\epsilon = \frac{\beta}{\bar{\Pi}}\bar{\mathfrak{f}}_{\mathfrak{ua}} = \frac{1}{\sqrt{2\pi\mathfrak{s}_{\mathfrak{ua}}^2}}\exp\left(-\frac{1}{2}\left(\frac{\mathring{i}}{\mathfrak{s}_{\mathfrak{ua}}}\right)^2\right)$$

Note now that, conditional on the expected paths of output and inflation in any equilibrium with trend inflation, $\mathfrak{p}_{\mathfrak{o},t}$ satisfies (see Appendix B for more details):

$$\mathfrak{p}_{\mathfrak{o},t} = \mathbb{F}_{\mathfrak{u}\rho}\left(\left(\epsilon_{t-1}\right)^{-\rho_{\mathfrak{u}}(1-\rho_{\mathfrak{u}})} E_t\left(\frac{\beta}{\Pi_{t+1}}\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\epsilon_{\mathfrak{u},t+1}\right)\right)$$

where $\epsilon_{\mathfrak{u}\rho,t}\equiv\left(\epsilon_{\mathfrak{u},t}\right)^{(1-\rho_{\mathfrak{u}})}$, whose distribution is $\epsilon_{\mathfrak{u}\rho,t} \overset{iid}{\sim} LN\left(0,\mathfrak{s}_{\mathfrak{u}\rho}^2\right)$, where $\mathfrak{s}_{\mathfrak{u}\rho}^2 \equiv \left(1-\rho_{\mathfrak{u}}\right)^2\mathfrak{s}_{\mathfrak{u}}^2$.

There is no closed-form solution for $\mathfrak{p}_{\mathfrak{o},t}$, as it depends on the joint distribution of the expected path of the endogenous variables and the exogenous shocks. However, I it is easy to conclude that $\bar{\mathfrak{p}}_{\mathfrak{o}} = \bar{\mathfrak{p}}_{\mathfrak{o}}^n$ once we account that the distorsive contribution of non-zero levels of trend inflation is offset in the steady-state value of $\left(Y_{t+1}/Y_t\right)$.

In this context, the log-linearization of $\mathfrak{p}_{\mathfrak{o},t}$ is what I call the ZLB Probability curve:

$$\mathfrak{p}_{\mathfrak{o},t} \approx \bar{\mathfrak{p}}_{\mathfrak{o}} - \phi_\epsilon E_t\left[\sigma\left(\hat{Y}_{t+1}-\hat{Y}_t\right)+\hat{\pi}_{t+1}\right] - \phi_\epsilon\rho_{\mathfrak{u}}\left(1-\rho_{\mathfrak{u}}\right)\hat{\epsilon}_{t-1} \tag{13}$$

As expected, the conditional probability $\mathfrak{p}_{\mathfrak{o},t}$ of hitting the ZLB falls when we expect output and inflation to rise and have had positive demand shocks.

3.1 Monetary policy

In Alves (2014), I derive a trend-inflation welfare based TIWeB loss function, which implies the following second order log-approximation of the (negative) welfare function:

$$\mathcal{W}_t = -\frac{1}{2}\bar{\mathcal{V}}E_t\sum_{\tau=0}^{\infty}\beta^\tau\bar{\mathcal{L}}_{t+\tau} + \overline{tip}_t^{\mathcal{W}} \tag{14}$$

[7] Recall that the cdf and pdf of a log-normal distributed random variable $\varkappa \sim LN\left(\mu,\mathfrak{s}^2\right)$ are $F\left(x\right)=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\log(\varkappa)-\mu}{\sqrt{2\mathfrak{s}^2}}\right)\right]$ and $f\left(x\right)=\frac{1}{\varkappa\sqrt{2\pi\mathfrak{s}^2}}\exp\left(-\frac{1}{2}\frac{(\log(\varkappa)-\mu)^2}{\mathfrak{s}^2}\right)$.

where

$$\bar{\mathcal{L}}_t \equiv \left(\hat{\pi}_t - \hat{\pi}_t^{ind} + \bar{\phi}_\pi\right)^2 + \bar{\mathcal{X}}\left(\hat{x}_t - \bar{\phi}_x\right)^2 \tag{15}$$

is the trend inflation welfare-based (TIWeB) loss function, $\overline{tip}_t^{\mathcal{W}}$ stands for terms independent of policy at period t, $\bar{\phi}_\pi$ and $\bar{\phi}_x$ are constants that depend on the inefficiency parameters $\bar{\Phi}_\vartheta \equiv \left(\bar{\vartheta} - 1\right)$ and $\bar{\Phi}_y \equiv 1 - \bar{v}_Y / \bar{u}_Y$, and $\bar{\mathcal{V}}$ corrects for the aggregate reduction in the welfare when trend inflation increases. Those composite parameters are defined as follows:

$$\bar{\phi}_\pi \equiv \frac{(1-\bar{\alpha})}{\left(1-\bar{\alpha}\bar{\vartheta}\right)(1+\theta\omega)}\bar{\Phi}_\vartheta \quad ; \bar{\phi}_x \equiv \frac{1}{(\omega+\sigma)}\bar{\Phi}_y \quad ; \bar{\mathcal{V}} \equiv \frac{(\omega+\sigma)}{\bar{\mathcal{X}}}\bar{Y}^{1-\sigma} \quad ; \bar{\mathcal{X}} \equiv \frac{(1-\bar{\alpha})}{\left(1-\bar{\alpha}\bar{\vartheta}\right)}\frac{\bar{\kappa}}{\theta} \tag{16}$$

Assume that the central bank implements inflation targeting by keeping the unconditional mean of the inflation rate at the central target $\bar{\pi}$, or $E\pi_t = \bar{\pi}$. When log-linearizing around the inflation target, $E\hat{\pi}_t = 0$. As for the ZLB constraint $I_t > 1$, its log-linearized form is $\hat{\imath}_t \geq -\mathring{i}$, where again $\mathring{i} \equiv \log\left(\bar{I}\right)$ is $-\hat{\imath}_t$ evaluated at $i_t = 0$.

Here, I expand optimal policies results I obtained in Alves (2014) by internalizing the influence monetary policy has in gauging $\mathfrak{p}_{\mathfrak{o},t}$ when deriving trend inflation optimal policies rules under unconditionally commitment (e.g. Damjanovic et al. (2008)), which is unconditionally time-consistent. That is, there is no inconsistency arising from first-order conditions obtained at first periods of optimization steps. As a consequence, unconditionally, the monetary authority has no incentive to deviate from the optimal policy rule. I assume that the welfare-concerned central bank minimizes the unconditional expectation of the Lagrangian problem formed by the discounted sum of the TIWeB loss function, subject to the IS curve (9), GNKPC (10), ZLB Probability curve (13), $E\hat{\pi}_t = 0$ and the constraint $\hat{\imath}_t \geq -\mathring{i}$.

In order to make it easier to derive optimal policies rules under unconditionally commitment, I use the ancillary variable $\hat{\varrho}_t$ and split the IS curve into $\hat{x}_t = \hat{\varrho}_t - \frac{1}{\sigma}\hat{\imath}_t + \frac{1}{\sigma}\hat{r}_t^n$ and $\hat{\varrho}_t = E_t\left(\hat{x}_{t+1} + \frac{1}{\sigma}\hat{\pi}_{t+1}\right)$.

Since it is the unconditional expectation which is minimized, optimal policy rules derived this way are unconditionally time consistent.

The use of unconditional expectations allows us to decompose the problem in periods for which $\hat{\imath}_t \geq -\mathring{i}$ is biding, with probability $\mathfrak{p}_{\mathfrak{o},t}$, and those in which the restriction is loose, with probability $(1 - \mathfrak{p}_{\mathfrak{o},t})$. When the restriction binds, I simply impose $\hat{\imath}_t = -\mathring{i}$ into the IS curve, which is the only one affected by the restriction. The remaining equations are not affected. Analogously, the only loss function quadratic term affect by the restriction is $\bar{\mathcal{X}}\left(\hat{x}_t - \bar{\phi}_x\right)^2$. When building the Lagrangian form, the simplest approach is to directly impose the restricted IS curve $\hat{x}_t = \hat{\varrho}_t + \frac{1}{\sigma}\mathring{i} + \frac{1}{\sigma}\hat{r}_t^n$ into $\bar{\mathcal{X}}\left(\hat{x}_t - \bar{\phi}_x\right)^2$ when the restriction binds.

In addition, the whole Lagrangian problem must be of order $\mathcal{O}(2)$, for this is the order to which the welfare function is log-approximated. Since log-linearized equations are used as restrictions, Lagrangian multipliers must be of order $\mathcal{O}(1)$. This order issue is relevant when adding the ZLB Probability curve (13), i.e. first order approximation of $\mathfrak{p}_{\mathfrak{o},t}$, into the problem. The issue arises when multiplying this approximation by the second order components from the loss function. We must disregard all $\mathcal{O}(3)$ terms from the resulting multiplication. In Alves (2014), I show that the distortion parameters $\bar{\phi}_\pi$ and $\bar{\phi}_x$ must be of order $\mathcal{O}(1)$ in order for the trend inflation welfare-based loss function to be properly used with log-linearized equations when deriving optimal policy rules. With the same logic, I assume that $\mathring{i}$ is of order $\mathcal{O}(1)$. This assumption is reasonable once we consider that any hatted variable is assumed to be of order $\mathcal{O}(1)$ and $\mathring{i}$ is $-\hat{\imath}_t$ evaluated at $i_t = 0$.

After taking in consideration the fact that the Lagrangian problem may only have $\mathcal{O}(3)$ terms, I derive the trend inflation optimal policy rules under unconditionally commitment (based on e.g. Damjanovic et al. (2008)), for the case in which the monetary authority internalizes its influence over episodes of occasionally hitting the ZLB on nominal interest rates, as described by proposition 1.[8]

Proposition 1 *When a welfare-concerned central bank targets $\bar{\pi}$ as the inflation target, follows the recommendations of the TIWeB loss function, and recognizes its role in influencing occasionally binding episodes of hitting the zero-lower bound (ZLB) on nominal interest rates, the optimal precautionary policy under unconditionally commitment are described by the following targeting rule, when the ZLB constraint is not binding:*

$$0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \left(1 - \bar{\mathfrak{p}}_{\mathfrak{o}}\right) \tfrac{1}{\mathfrak{c}_1} \tfrac{\bar{\mathcal{X}}}{\bar{\kappa}} \left[\hat{x}_t - \beta \hat{x}_{t-1} - \left(\mathfrak{c}_2 - \mathfrak{c}_1\right) \hat{\varkappa}_{1,t-1}\right] + \bar{\mathfrak{p}}_{\mathfrak{o}} \bar{\mathcal{X}} \left(\tfrac{1}{\sigma} \hat{\partial}_{1,t} + \tfrac{1}{\bar{\kappa}} \hat{\partial}_{2,t}\right) \tag{17}$$

where $\hat{\varkappa}_t$, $\hat{\partial}_{1,t}$ and $\hat{\partial}_{2,t}$ are ancillary variables, whose dynamics are described by

$$\begin{aligned}
\hat{\varkappa}_t &= \tfrac{\mathfrak{c}_4}{\mathfrak{c}_1} \hat{\varkappa}_{t-1} + \tfrac{\mathfrak{c}_3}{\mathfrak{c}_1} \hat{x}_t - \tfrac{\beta}{\mathfrak{c}_1} \left(1 - \bar{\alpha}\beta\bar{\vartheta}\right) \hat{x}_{t-1} \\
\hat{\partial}_{1,t} &= \gamma_\pi E_t \hat{\partial}_{1,t+1} + \left(\hat{x}_t + \tfrac{1}{\sigma} \hat{\imath}_t\right) \\
\hat{\partial}_{2,t} &= \tfrac{\mathfrak{c}_4}{\mathfrak{c}_1} \hat{\partial}_{2,t-1} + \tfrac{1}{\mathfrak{c}_1} \left(\hat{x}_{t-1} + \tfrac{1}{\sigma} \hat{\imath}_{t-1}\right) \\
&\quad - \tfrac{1}{\mathfrak{c}_1} \left[\left(1 + \bar{\alpha}\bar{\vartheta}\right) \beta + \bar{\kappa}\theta \left(\mathfrak{c}_2 - \mathfrak{c}_1\right)\right] \left(\hat{x}_{t-2} + \tfrac{1}{\sigma} \hat{\imath}_{t-2}\right) + \tfrac{1}{\mathfrak{c}_1} \bar{\alpha}\bar{\vartheta}\beta^2 \left(\hat{x}_{t-3} + \tfrac{1}{\sigma} \hat{\imath}_{t-3}\right)
\end{aligned}$$

and the composite parameters are defined as follows:

$$\begin{aligned}
\mathfrak{c}_1 &\equiv 1 - \left(\bar{\vartheta} - 1\right) \beta \tfrac{\bar{\kappa}_{\varpi}}{\bar{\kappa}} \left(1 - \sigma\right) \quad ; \quad \mathfrak{c}_2 \equiv 1 + \left(\bar{\vartheta} - 1\right) \beta \tfrac{\bar{\kappa}_{\varpi}}{\bar{\kappa}} \left(\omega + \sigma\right) \\
\mathfrak{c}_3 &\equiv \theta \bar{\kappa} \mathfrak{c}_1 + \left(1 - \bar{\alpha}\beta\bar{\vartheta}\right) \quad ; \quad \mathfrak{c}_4 \equiv \mathfrak{c}_1 - \left(1 - \bar{\alpha}\beta\bar{\vartheta}\right) \mathfrak{c}_2
\end{aligned} \tag{18}$$

The proof is shown in Appendix C.

Of course, $\hat{\imath}_t \to -\mathring{i}$ when the ZLB constraint binds. Therefore, the full targeting rule must be understood as the one to be pursued in between occasionally binding episodes when the monetary authority is internalizes its role of influencing the probability of hitting the ZLB by means of the expectations channel. Note that under low steady level of the (gross) nominal interest rate $\bar{I} = \bar{\Pi}/\beta$, $\bar{\mathfrak{p}}_{\mathfrak{o}}$ and ϕ_ϵ fast increase and the full targeting rule becomes more and more history dependent and more directly dependent on the history of nominal interest rates. As I find, the central bank consciously and directly adopts precautionary behavior in normal times in order not to cut nominal interest rates so fast after negative demand shocks and taking longer to increase the rate after the shock has dissipated. And $\bar{\mathfrak{p}}_{\mathfrak{o}}$ gauges the optimal degree to which this behavior is to be used.

These terms do not arise when the monetary authority do not directly internalize its role in affecting the probability of hitting the ZLB, as in e.g. Eggertsson and Woodford (2003a,b) and Nakov (2008). In their approach, policy precautionary behavior arises indirectly in general equilibrium, and so the central bank does not benefit as much.

[8] In Alves (2014), I find that the trend inflation optimal policy under unconditionally commitment slightly dominates the one from timeless perspective, even though both optimal policy rules imply almost indistinguishable dynamics and unconditional moments. Due to this result, I choose the aproach of deriving optimal policy under unconditionally commitment to deal with ZLB occasionally binding constraints.

Recall that $\bar{\mathfrak{p}}_{\mathfrak{o}}$ and ϕ_{ϵ} are direct functions of the steady level of the (gross) nominal interest rate $\bar{I} = \bar{\Pi}/\beta$, which can change by either changing the steady state level of (gross) real interest rate $\bar{R} = 1/\beta$ or the level of (gross) trend inflation $\bar{\Pi}$. Therefore, the effects of rising $\bar{\Pi}$ or $\bar{R}$ are perfectly substitutes on what regards $\bar{\mathfrak{p}}_{\mathfrak{o}}$ and ϕ_{ϵ}. However, the effects of both margins are different on the targeting rule are very different and not substitutes. And so, rising the trend inflation (inflation target) level is not a perfect remedy to instances in which the real interest rate is falling. That is, it is not enough to rise the trend inflation target as it wold create more distortions.

The effects of rising trend inflation on trend inflation composite parameters, such as $\bar{\alpha}$, $\bar{\vartheta}$, $\mathfrak{c}_1$, $\mathfrak{c}_2$, $\mathfrak{c}_3$, $\mathfrak{c}_4$, and $\bar{\kappa}$, do not parallel those obtained by increasing $1/\beta$. As a matter of fact, rising levels of trend inflation might create more instability, as shown in the literature of trend inflation.

For larger steady state levels of nominal interest rate, as we used to have in the past, $\bar{\mathfrak{p}}_{\mathfrak{o}}$ shrinks down to zero. Therefore, the rule returns to the trend inflation form obtained in Alves (2014), here written using ancillary variable $\hat{\varkappa}_t$:

$$0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{1}{\mathfrak{c}_1}\frac{\bar{\mathcal{X}}}{\bar{\kappa}}\left[\hat{x}_t - \beta\hat{x}_{t-1} - \left(\mathfrak{c}_2 - \mathfrak{c}_1\right)\hat{\varkappa}_{1,t-1}\right] \tag{19}$$

in which the targeting rule becomes more history dependent as trend inflation rises. The results obtained by Damjanovic et al. (2008) refer to the particular case $\bar{\pi} = 0$, for which $\bar{\alpha} = \alpha$, and $\bar{\vartheta} = \mathfrak{c}_1 = \mathfrak{c}_2 = 1$. In that case, the targeting rule under unconditionally commitment is $0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \frac{\bar{\mathcal{X}}}{\bar{\kappa}}\left(\hat{x}_t - \beta\hat{x}_{t-1}\right)$, which that authors show to slightly dominate the Woodford (2003) Timeless perspective targeting rule. The latter has $\left(\hat{x}_t - \hat{x}_{t-1}\right)$ instead of $\left(\hat{x}_t - \beta\hat{x}_{t-1}\right)$ as its last term. As a consequence, Timeless perspective optimal policy is equivalent to price level targeting, while unconditionally commitment optimal policy is not.

It is easy to verify that (19) is the obtained optimal policy under unconditionally commitment when expanding Nakov (2008) approach to the trend inflation case. Again, $\hat{\imath}_t \rightarrow -\mathring{i}$ when the ZLB constraint binds. Note that this policy rule does not directly internalize policy precautionary behavior – it does under general equilibrium, however. Therefore, I call this targeting rule the Standard Optimal Policy under occasionally binding ZLB constraint, in order to distinguish it from the Precautionary Optimal Policy (17) under occasionally binding ZLB constraint.

If $\bar{\pi} = 0$ and the economy has low levels of nominal interest rates, $\bar{\mathfrak{p}}_{\mathfrak{o}}$ fast increase and so the precautionary targeting rule becomes even more history dependent and more directly dependents on the history of nominal interest rates:

$$0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \left(1 - \bar{\mathfrak{p}}_{\mathfrak{o}}\right)\tfrac{\bar{\mathcal{X}}}{\bar{\kappa}}\left(\hat{x}_t - \beta\hat{x}_{t-1}\right) + \bar{\mathfrak{p}}_{\mathfrak{o}}\bar{\mathcal{X}}\left(\tfrac{1}{\sigma}\hat{\partial}_{1,t} + \tfrac{1}{\bar{\kappa}}\hat{\partial}_{2,t}\right)$$

$$\hat{\partial}_{1,t} = \gamma_{\pi}E_t\hat{\partial}_{1,t+1} + \left(\hat{x}_t + \tfrac{1}{\sigma}\hat{\imath}_t\right)$$

$$\hat{\partial}_{2,t} = \alpha\beta\hat{\partial}_{2,t-1} + \left(\hat{x}_{t-1} + \tfrac{1}{\sigma}\hat{\imath}_{t-1}\right) - \left(1 + \alpha\right)\beta\left(\hat{x}_{t-2} + \tfrac{1}{\sigma}\hat{\imath}_{t-2}\right) + \alpha\beta^2\left(\hat{x}_{t-3} + \tfrac{1}{\sigma}\hat{\imath}_{t-3}\right)$$

Using the lag $L(\cdot)$ operator, note that we can rewrite the rule as follows:

$$0 = \left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \left(1 - \bar{\mathfrak{p}}_{\mathfrak{o}}\right) \frac{\bar{\mathcal{X}}}{\bar{\kappa}} \left(1 - \beta L\right) \hat{x}_t + \bar{\mathfrak{p}}_{\mathfrak{o}} \bar{\mathcal{X}} \left(\frac{1}{\sigma} \hat{\eth}_{1,t} + \frac{1}{\bar{\kappa}} \hat{\eth}_{2,t}\right)$$
$$\hat{\eth}_{1,t} = \gamma_\pi E_t \hat{\eth}_{1,t+1} + \left(\hat{x}_t + \frac{1}{\sigma} \hat{\imath}_t\right)$$
$$\left(1 - \alpha\beta L\right) \hat{\eth}_{2,t} = \left(1 - \alpha\beta L\right) \left(1 - \beta L\right) \left(\hat{x}_{t-1} + \frac{1}{\sigma} \hat{\imath}_{t-1}\right)$$

Consider the case in which the (gross) real interest rate $\bar{R} = 1/\beta$ has been reducing over time, i.e. β is approaching closer and closer to unity. In this case, $(1 - L)$ is a good approximation for $(1 - \beta L)$. Moreover, empirical microevidence strongly suggests that there is none or very small degree of price stickiness in the US, i.e. γ_π is very small. Since $\hat{\pi}_t = (1 - L) \hat{p}_t$ and $\hat{\pi}_t^{ind} = (1 - L) \hat{p}_t^{ind}$, for $\hat{p}_t^{ind} = \gamma_\pi \hat{p}_{t-1}$, the targeting rule is reasonably approximated by the following expression when $\bar{R}$ is small:

$$0 \approx (1 - L) \left(\hat{p}_t - \hat{p}_t^{ind}\right) + \left(1 - \bar{\mathfrak{p}}_{\mathfrak{o}}\right) \frac{\bar{\mathcal{X}}}{\bar{\kappa}} (1 - L) \hat{x}_t + \bar{\mathfrak{p}}_{\mathfrak{o}} \frac{\bar{\mathcal{X}}}{\bar{\kappa}} \left[\frac{\bar{\kappa}}{\sigma} + (1 - L) L\right] \left(\hat{x}_t + \frac{1}{\sigma} \hat{\imath}_t\right)$$

Since macroevidence suggests that $\sigma >> \bar{\kappa}$, i.e. $\frac{\bar{\kappa}}{\sigma}$ is very small, the second term dominates the expression inside brackets. We then follow to "divide" the expression by $(1 - L)$, and obtain the simplification:

$$0 \approx \left(\hat{p}_t - \hat{p}_t^{ind}\right) + \left(1 - \bar{\mathfrak{p}}_{\mathfrak{o}}\right) \frac{\bar{\mathcal{X}}}{\bar{\kappa}} \hat{x}_t + \bar{\mathfrak{p}}_{\mathfrak{o}} \frac{\bar{\mathcal{X}}}{\bar{\kappa}} \left(\hat{x}_{t-1} + \frac{1}{\sigma} \hat{\imath}_{t-1}\right) \tag{20}$$

That is, under low levels of steady state real interest rate, a central bank who internalizes its role in influencing the probability of hitting the ZLB would implement monetary policy according to a optimal (unconditionally) time-consistent policy rule that closely resembles price level targeting.

In order to understand the effect of monetary policy internalizing its influence over the probability $\mathfrak{p}_{\mathfrak{o},t}$ of hitting the zero lower bound, suppose that the economy was hit by a negative demand shock. Under negligible values of $\bar{\mathfrak{p}}_{\mathfrak{o}}$, the standard result is that monetary policy lowers the rate in order to compensate the fall in current output gap so that $\left(\hat{p}_t - \hat{p}_t^{ind}\right)$ remains equal to zero, i.e. prices remain stable and monetary policy pursues price level targeting.

Under non-negligible steady state probability $\bar{\mathfrak{p}}_{\mathfrak{o}}$, current probability $\hat{\mathfrak{p}}_{\mathfrak{o},t}$ of hitting the ZLB rises by means of the ZLB Probability curve (13). Monetary policy now also look further into the past history of output gap and nominal interest rates. As a consequence, monetary policy does not react as strong on impact, for it also looks into periods in which the output gap was not yet hit by the shock. This slow down in reducing the rate makes more room for monetary policy to avoid hitting the ZLB. On the other hand, after the negative demand shock has dissipated, monetary policy continues to look longer in the past and still take in consideration that the output gap has fallen in the past. Therefore, nominal rates take longer to return. I highlight that this duration depends on $\bar{\mathfrak{p}}_{\mathfrak{o}}$. Hence, forward guidance is always optimized under this policy rule.

Finally, the presence of $\hat{\imath}_{t-1}$ in the targeting rule is a novelty in the literature of optimal policy prescriptions. It naturally arises as $\bar{\mathfrak{p}}_{\mathfrak{o}}$ rises and it serves to smooth optimal changes on nominal interest rates. As a consequence, the rate does not fall (rise) as fast under negative (positive) demand shocks when compared to responses under standard optimal policy prescriptions such as in Damjanovic et al. (2008), Woodford (2003),and Nakov (2008).

Addressing the case under larger values of trend inflation $\bar{\pi}$, under low real interest rates, is more entangled. The targeting rules does not resembles price level targeting anymore. In this case, only numerical analyses are feasible.

4 Calibration

The calibration is described as follows. As in Cooley and Prescott (1995), I set the elasticity to hours at the production function at $\varepsilon = 0.64$. As in ,Coibion et al. (2012), I set the elasticity of substitution at $\phi = 7$, which implies a steady state price markup of $\mu = 1.17$.[9] Recall that the (log-deviation) technology shock evolves according to $\widehat{\mathcal{A}}_t = \rho_{\mathfrak{a}}\widehat{\mathcal{A}}_{t-1} + \hat{\epsilon}_{\mathfrak{a},t}$, where $\hat{\epsilon}_{\mathfrak{a},t} \overset{iid}{\sim} N\left(0, \mathfrak{s}_{\mathfrak{a}}^2\right)$. Using the central estimate obtained by Smets and Wouters (2007) for the larger sample, I set the autoregressive coefficient of the technology shock at $\rho_{\mathfrak{a}} = 0.95$ and the shock's standard deviation at $\mathfrak{s}_{\mathfrak{a}} = 0.0045$. The remaining parameters were based on central estimates obtained by Smets and Wouters (2007), for the Great Moderation. I set the reciprocal of the intertemporal elasticity of substitution at $\sigma = 1.47$. As for the elasticity ν of the disutility from hours $h_t(z)$, i.e. the reciprocal of the Frisch elasticity, I use $\nu = 2.30$. Note that this value is consistent with micro evidence, as reported by Chetty et al. (2011).[10] I set the degree of price stickiness at $\alpha = 0.73$, while the price indexation parameter is fixed at $\gamma_\pi = 0.21$. In addition, I set the disutility nuisance parameter at $\chi = 1$.

Recall that the (log-deviation) demand shock evolves according to $\hat{\epsilon}_t = \rho_{\mathfrak{u}}\hat{\epsilon}_{t-1} + \hat{\epsilon}_{\mathfrak{u},t}$, where $\hat{\epsilon}_{\mathfrak{u},t} \overset{iid}{\sim} N\left(0, \mathfrak{s}_{\mathfrak{u}}^2\right)$, and that I do not assume consumtion habit persistence in my model. Therefore, $\rho_{\mathfrak{u}}$ will play a similar role as the degree of habit persistence in this model. Therefore, based on the authors' estimated habit persistence parameter, I set the persistence of the demand shock $\rho = 0.68$. In order to adjust the implied dynamics implied by this assumption, I estimate $\mathfrak{s}_{\mathfrak{u}}$ using quarterly US data from the *Great Moderation* period 1985:Q1-2005:Q4. For that, I fix $\beta = 0.995$ (consistent with annual real interest rate $\bar{r} = 2\%$) and (annual) $\bar{\pi} = 3.05\%$ (consistent with the sample average of the CPI inflation rate).

For estimation, I considered the following observed variables: (*i*) inflation rate $\hat{\pi}_t$ is the (log) BLS CPI inflation rate (US city average, all urban consumers), demeaned from its sample average; (*ii*) output $\hat{y}_t$ is the (log) BLS GDP, detrended by its linear trend; and (*iii*) nominal interest rate $\hat{\imath}_t$ is the (log) quarterly average of the Federal Funds Rate, demeaned from its sample average.

Since I observe the nominal interest rate in the estimation, I assume that monetary policy followed a simplifyed Trend Inflation Taylor rule, based on Coibion and Gorodnichenko (2011):

$$\hat{\imath}_t = \phi_{\mathfrak{i}}\hat{\imath}_{t-1} + (1 - \phi_{\mathfrak{i}})\left[\phi_\pi E_t\hat{\pi}_{t+1} + \phi_x\hat{x}_t + \phi_{gy}\left(\hat{y}_t - \hat{y}_{t-1}\right)\right] + \hat{\epsilon}_{\mathfrak{i},t} \tag{21}$$

where $\hat{\epsilon}_{\mathfrak{i},t} \overset{iid}{\sim} N\left(0, \mathfrak{s}_{\mathfrak{i}}^2\right)$ is the monetary policy shock, $\phi_{\mathfrak{i}}$ is the policy smoothing parameter, and ϕ_π, ϕ_x and ϕ_{gy} are response parameters consistent with stability and determinacy in equilibria with rational expectations in a equilibrium with positive trend inflation. The authors find that reacting to the observed output growth has two major advantages: (*i*) it has more stabilizing properties when the trend inflation is not zero; and (*ii*) it is empir-

[9] For instance, Ravenna and Walsh (2008, 2011) set the steady state price markup to 1.2.

[10] The authors conduct meta analyses of existing micro evidence. Their point estimate of the Frisch elasticity of intensive margin is $(1/\nu) = 0.54$.

ically more relevant. Based on Coibion and Gorodnichenko (2011) central estimations, I keep $\phi_{\mathfrak{i}} = 0.92$ and estimate the response parameters so that the estimated model adjusts to a possible misscalibration and absence of additional shocks. Since I am focused in inferring $\mathfrak{s}_{\mathfrak{u}}$ and $\mathfrak{s}_{\mathfrak{a}}$, this strategy is fairy reasonable.

Using Bayesian MCMC estimation, with flat priors and 200000 draws, table 1 reports posterior means and 95% credible intervals for ϕ_{π}, ϕ_x, ϕ_{gy}, $\mathfrak{s}_{\mathfrak{i}}$, and $\mathfrak{s}_{\mathfrak{u}}$. For simulations shown in Section 5, I set $\mathfrak{s}_{\mathfrak{u}}$ and $\mathfrak{s}_{\mathfrak{a}}$ at the posterior means. Since all exercises are focused on optimal policy rules, I did not consider outcomes under monetary policy shocks.

Table 1: Posterior Distributions

	PostMean	95% *cred.int.*
ϕ_{π}	1.2283	1.2158 − 1.2400
ϕ_x	0.0000	0.0000 − 0.0000
ϕ_{gy}	0.9918	0.9896 − 0.9941
$\mathfrak{s}_{\mathfrak{i}}$	0.0017	0.0014 − 0.0021
$\mathfrak{s}_{\mathfrak{u}}$	0.0269	0.0233 − 0.0304

Using the calibration set, note that the steady-state levels of the ZLB probability $\bar{\mathfrak{p}}_{\mathfrak{o}}$ and probability-elasticity of shocks ϕ_{ϵ} increase very fast as the steady state level of the annual nominal interest rate $\bar{\imath}$ falls towards the ZLB, as depicted in Figure 1. I highlight the fact that $\bar{\mathfrak{p}}_{\mathfrak{o}}$ is reference level, for the expected frequency $E\mathfrak{p}_{\mathfrak{o},t}$ according to which the ZLB binds is highly policy-dependent. In Section 5, I show simulations under different monetary policy frameworks.

For instance, the observed frequency at which the effective annualized Federal Funds rate is below 0.15%, which I call the Effective Lower Bound (ELB), from 2006Q1 to 2016Q4 (after the Great Moderation period) was 41%. During this period, the average Fed Funds rate was 1.19%. In this context, even though $\bar{\mathfrak{p}}_{\mathfrak{o}} = 32\%$, the Standard Optimal policy delivers $E\mathfrak{p}_{\mathfrak{o},t}$ between 43% and 44% depending on the combination of β and $\bar{\pi}$ such that $\bar{\imath} = 1.19$. In a similar exercise, for the larger sample 1985Q1 to 2016Q4, the average Fed Funds rate was 3.73% and the rate was below the ELB at 14% of the time. In this context, $\bar{\mathfrak{p}}_{\mathfrak{o}} = 7.5\%$ and $E\mathfrak{p}_{\mathfrak{o},t}$ ranges between 14% and 16%, depending on the combination of β and $\bar{\pi}$ such that $\bar{\imath} = 3.73$.

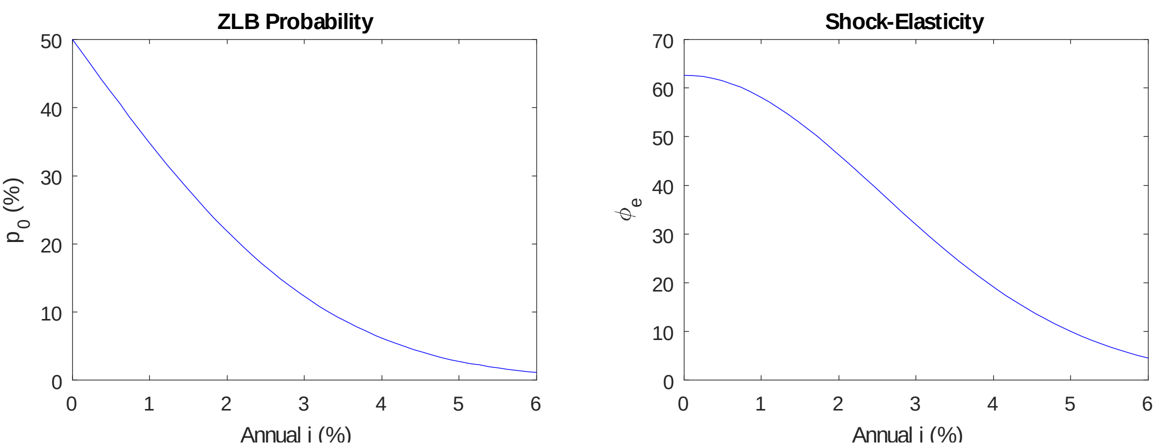

Figure 1: ZLB Probability and Probability-Elasticity

5 Simulations

This section studies the welfare gains and dynamics implied by trend inflation optimal policies under unconditionally commitment. I perform simulations using Occbin, by Guerrieri and Iacoviello (2015), to account for the occasionally biding ZLB constraint on the nominal interest rate. Due to ZLB restrictions, there is no closed form solution to compute the model's unconditional moments. Therefore, in order to infer them, I simulate artificial equilibria with 10,000 periods simultaneously using fixed sequences of exogenous demand and technology shocks, based on the distribution detailed in the last section.

In the first exercise, I compute welfare gains from using the TIWeB precautionary optimal policy under unconditionally commitment (PrOP) over the TIWeB standard optimal policy under unconditionally commitment (StOP), obtained by extending the Nakov (2008) analysis to a trend inflation economy, and estimated TTrend Inflation Taylor Rule (TayR). I do not compare with the Trend Inflation optimal policy under discretion, which I derive in Alves (2014), for it is only compatible with stability and determinacy at very small levels of trend inflation (see Alves (2014) for more details). In the second exercise, I compare impulse responses to negative demand shocks obtained under different policy frameworks.

5.1 Policy evaluation

As for studying the welfare gains, I follow Schmitt-Grohe and Uribe (2007) and Alves (2014) by computing welfare cost rates, in terms of consumption equivalence results, of each optimal monetary policy framework. The analysis is done in terms of assessing the gains from commitment as trend inflation rises from 0 percent to 2 percent.[11], paralleling the exercises done by Ascari and Ropele (2007).

I assess the gains from using the PrOP optimal policy under unconditionally commitment against the alternative StOP optimal policy under unconditionally commitment, considering the welfare cost rate λ of adopting each specific policy framework. In order to simplify the evaluation, I consider the TIWeB loss function to compute the unconditional expected value of the second order log-approximation of the welfare function, under occasionally biding ZLB restrictions:

$$E\mathcal{W}_t \approx \bar{\mathcal{W}} - \frac{1}{2}\frac{\bar{\mathcal{V}}}{(1-\beta)}E\bar{\mathcal{L}}_t$$

where $E\bar{\mathcal{L}}_t = Var\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \bar{\mathcal{X}}Var\left(\hat{x}_t\right) + \left[E\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)^2 + \bar{\mathcal{X}}E\hat{x}_t^2\right]$. Note that the term inside brackets might be relevant as occasionally biding ZLB restrictions induces non-zero values for $E\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right)$ and $E\hat{x}_t$.

The welfare cost rate λ is interpreted as a tax rate that must be applied to the steady state output level $\bar{Y}^0$ under the equilibrium with flexible prices ($\bar{\pi} = 0$) in order to the representative household to be indifferent between this equilibrium and a stochastic one with non-zero trend inflation and occasionally binding ZLB constraints over the nominal

[11] If nominal interest rates were allowed to be negative, optimal monetary policy under unconditionally commitment would fully stabilize the economy under zero trend inflation, and the model would not be disturbed by exogenous shocks. The reason is that the endogenous trend inflation cost push shock is zero at this level of trend inflation. If the ZLB is occasionally binding, on the other hand, optimal policy fails to always stabilize the economy. In this case, even at zero trend inflation, the uconditional variances of inflation and output gap are not simultaneously zero.

interest rate:

$$\frac{1}{(1-\beta)}\left[u\left(\left(1-\lambda\right)\bar{Y}^{0}\right)-\bar{v}^{0}\right]=E\mathcal{W}_{t}$$

Tables 2 and 3 report welfare cost rates λ, for different optimal policy frameworks and different levels of (annual) real interest rates, $\bar{r}=2\%$ and $\bar{r}=1\%$, as trend inflation rises from $\bar{\pi}=0\%$ to $\bar{\pi}=2\%$. The compared policy structures are TIWeB precautionary unconditional commitment (PrOP), TIWeB standard unconditional commitment (StOP) and TI Taylor Rule (TayR). For benchmark purposes, the tables also show the outcome in the ficticious economy where the ZLB constraint is not at play.

The tables also compare steady state levels $\bar{\mathfrak{p}}_{\mathfrak{o}}$ of the probability of hitting the ZLB with average probabilities $E\mathfrak{p}_{\mathfrak{o},t}$ obtained under different policy rules. Two lessons are learned from the tables: (*i*) if the ZLB constraint occasionally binds, relative gains from precautionary (PrOP) optimal commitment over standard (StOP) optimal commitment increase as trend inflation rises and the real interest rate falls; (*ii*) in the ficticious economy where the ZLB constraint is not at play, even though I obtain the expected result that the StOP optimal policy always dominates, the losses from adoting the PrOP optimal policy are negligible; (*iii*) the PrOP optimal policy delivers larger probabilities of hitting the ZLB, as it finds it optimal to remaining longer at the ZLB even after large negative shocks have dissipated (see Section 5.2); (*iv*) even though the taylor Rule delivers much smaller probabilities of hitting the ZLB, its implyied losses are much larger than those of both optimal policies.

Table 2 - Gains from Precautionary Optimal Policy at $\bar{r}=2\%$ $(\beta=0.995)$

A) Under ZLB constraints

Steady States $\bar{r}$=2%			PrOP Rates (%)		StOP Rates (%)		TayR Rates (%)	
$\bar{\pi}$	$\bar{\imath}$	$\bar{\mathfrak{p}}_{\mathfrak{o}}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$
0	2	21.8	0.05	32.1	0.06	31.4	0.40	0.8
1	3	12.1	0.54	19.5	0.55	18.3	0.84	0.0
2	4	6.0	2.49	12.2	2.54	11.7	2.76	0.0

B) No ZLB constraints

Steady States $\bar{r}$=2%			PrOP Rates (%)		StOP Rates (%)		TayR Rates (%)	
$\bar{\pi}$	$\bar{\imath}$	$\bar{\mathfrak{p}}_{\mathfrak{o}}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$
0	2	21.8	0.00	26.9	0.00	28.0	0.35	0.9
1	3	12.1	0.48	18.9	0.48	19.7	0.84	0.0
2	4	6.0	2.40	12.6	2.40	13.1	2.76	0.0

Note: TIWeB precautionary unconditional commitment (PrOP), TIWeB standard unconditional commitment (StOP), TI Taylor Rule (TayR), welfare loss (λ), trend inflation ($\bar{\pi}$), steady state annual real interest rate ($\bar{r}$), steady state annual nominal interest rate ($\bar{\imath}$), steady state probability of hitting the policy rate ZLB constraint ($\bar{\mathfrak{p}}_{\mathfrak{o}}$), expected policy-based probability of hitting the policy rate ZLB constraint ($E\mathfrak{p}_{\mathfrak{o},t}$).

Table 3 - Gains from Precautionary Optimal Policy at $\bar{r} = 1\%$ $(\beta = 0.9975)$

A) Under ZLB constraints

Steady States $\bar{r}$=1%			PrOP *Rates* (%)		StOP *Rates* (%)		TayR *Rates* (%)	
$\bar{\pi}$	$\bar{\imath}$	$\bar{\mathfrak{p}}_{\mathfrak{o}}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$
0	1	34.8	0.12	52.3	0.13	49.9	0.96	8.9
1	2	21.7	0.59	31.2	0.59	25.0	0.91	0.8
2	3	12.1	2.56	18.9	2.65	19.0	2.78	0.0

B) No ZLB constraints

Steady States $\bar{r}$=1%			PrOP *Rates* (%)		StOP *Rates* (%)		TayR *Rates* (%)	
$\bar{\pi}$	$\bar{\imath}$	$\bar{\mathfrak{p}}_{\mathfrak{o}}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$	λ	$E\mathfrak{p}_{\mathfrak{o},t}$
0	1	34.8	0.00	37.2	0.00	38.1	0.35	11.3
1	2	21.7	0.48	26.8	0.48	28.0	0.84	0.9
2	3	12.1	2.42	18.9	2.42	19.7	2.78	0.0

Note: TIWeB precautionary unconditional commitment (PrOP), TIWeB standard unconditional commitment (StOP), TI Taylor Rule (TayR), welfare loss (λ), trend inflation ($\bar{\pi}$), steady state annual real interest rate ($\bar{r}$), steady state annual nominal interest rate ($\bar{\imath}$), steady state probability of hitting the policy rate ZLB constraint ($\bar{\mathfrak{p}}_{\mathfrak{o}}$), expected policy-based probability of hitting the policy rate ZLB constraint ($E\mathfrak{p}_{\mathfrak{o},t}$).

5.2 Impulse Responses

In order to clearly illustrate the role of a precautionary optimal policy under unconditionally commitment, Figures 2 to 5 depict responses after a one-period $(t = 2)$ negative demand innovation shocks, with amplitudes varying from $\epsilon_{\mathrm{u},t} = -(0.5)\,\mathfrak{s}_{\mathrm{u}}$ to $\epsilon_{\mathrm{u},t} = -(3.0)\,\mathfrak{s}_{\mathrm{u}}$, where again $\mathfrak{s}_{\mathrm{u}}$ is the estimated standard deviation of the demand shock. In all simulations, I consider $\bar{r} = 1\%$ and trend inflation fixed at $\bar{\pi} = 2\%$. At those levels, there are distinct responses differences under the precautionary and standard optimal policy rules. In each exercise, I compare the responses obtained under the Precautionary Optimal Policy (17), Standard Optimal Policy (19), estimated Trend Inflation Taylor Rule (21) and under the Equilibrium with Flexible Prices without ZLB constraints. In this equilibrium, I assume that the nominal interest rate adjusts in order to keep the nominal interest rate constant at $\bar{\pi} = 2\%$, given the path of the real interest rate, i.e. $I_t^n = R_t^n \bar{\Pi}$.

The figures depict the responses of output $\hat{Y}_t$, annualized inflation rate π_t, annualized nominal interest rate i_t and the expected probability of hitting the ZLB in the next period $E_t\mathfrak{p}_{\mathfrak{o},t+1}$. Six lessons are learned from the responses: (i) output losses and inflation falls are smaller under precautionary (PrOP) optimal commitment over standard (StOP) optimal commitment and Taylor Rule; (ii) the PrOP policy delays the reduction in the nominal interest rate as the shock hits, making more room for policy efficacy, and delays even further the nominal rate return to normal levels after the shock dissipates; (iii) in line with the conclusions obtained in the analyses from the last section, the PrOP optimal policy deliver larger probabilities of hitting the ZLB, as it finds it optimal to remaining longer at the ZLB even after the negative shocks have dissipated; (iv) when not binded, the interest rate response under the PrOP policy tends to mimic that of the nominal interest rate under the equilibrium with flexible prices; (v) even though the Taylor Rule generates very low probability of hitting the ZLB, it generates costs in terms of larger declines in output and inflation when compared to the PrOP optimal policy; (vi) under large enough negative demand shocks, the Taylor Rule starts to dominate the StOP optimal policy.

The second and third lessons characterizes the precautionary nature of PrOP policies. In this exercise, the resulting optimal forward guidance structure depends on the size of the negative shock. For small shocks, which prevents the ZLB to actually bind, both optimal policy frameworks deliver very similar results in terms of output and inflation. Indeed, they virtually succeed in bringing price stability. For larger negative shocks, the differences become very clear, as the precautionary nature of PrOP dominates.

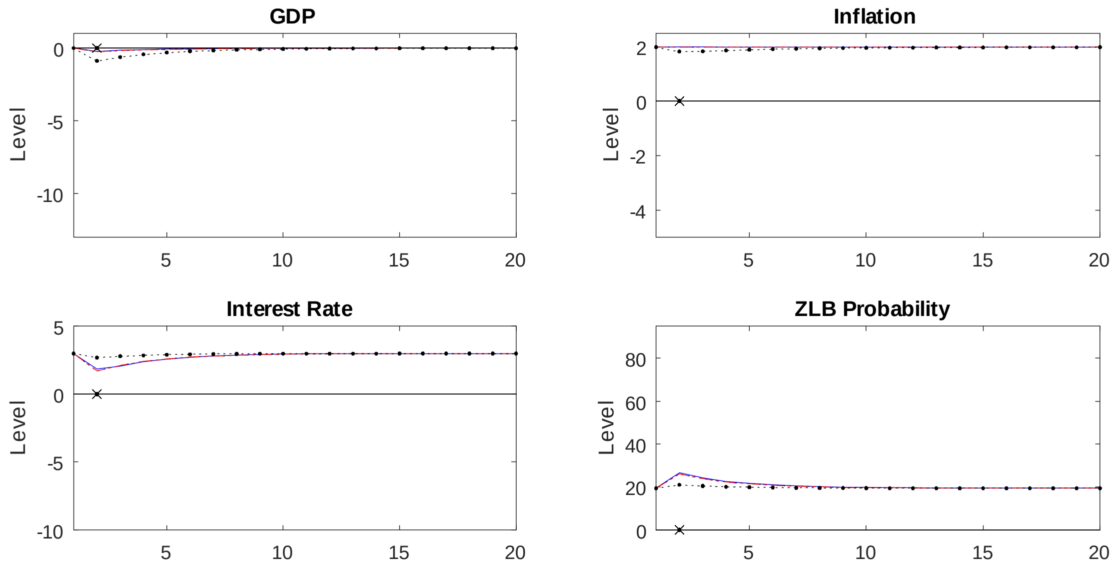

Figure 2: Responses to a one-period negative demand shock of 0.5 Std. Dev.

Note: $\bar{r} = 2$, $\epsilon_{u,t} = -(0.5)\mathfrak{s}_u$, Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and $\bar{\pi} = 2$ (black dotted)

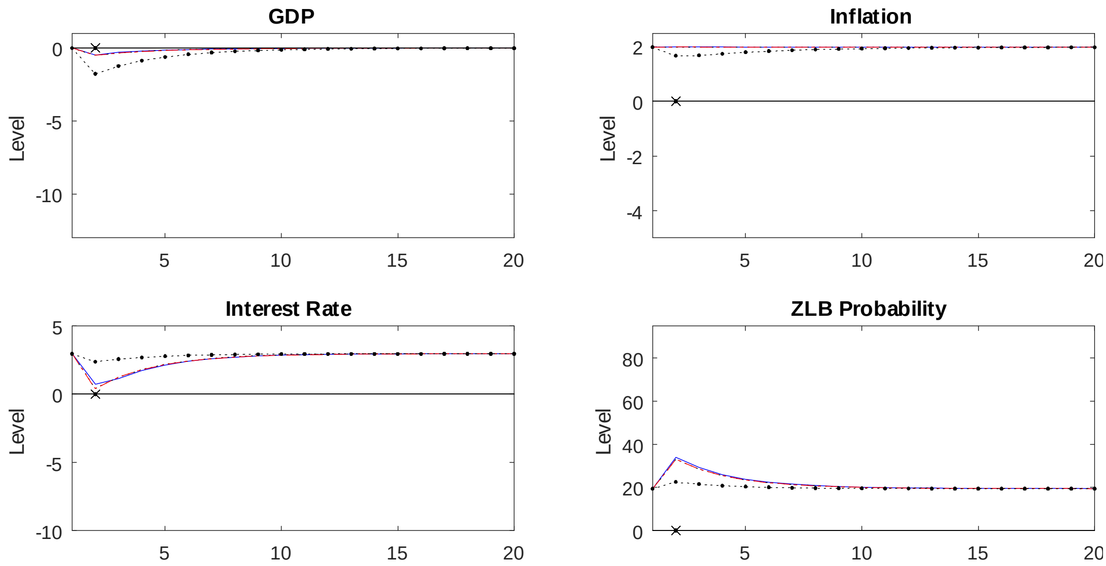

Figure 3: Responses to a one-period negative demand shock of 1.0 Std. Dev.

Note: $\bar{r} = 2$, $\epsilon_{u,t} = -(1.0)\mathfrak{s}_u$, Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and $\bar{\pi} = 2$ (black dotted)

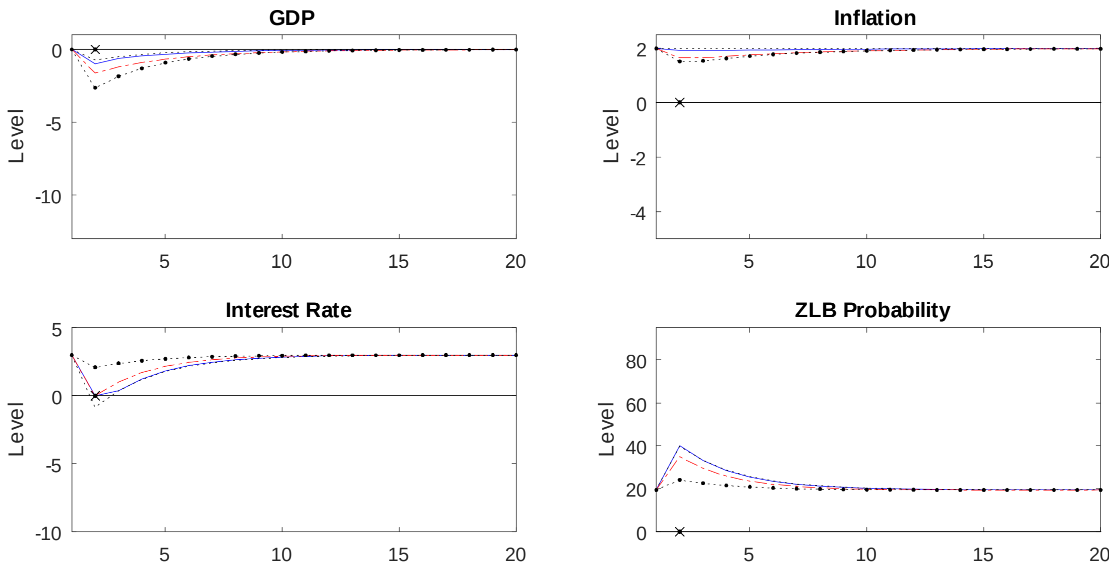

Figure 4: Responses to a one-period negative demand shock of 1.5 Std. Dev.

Note: $\bar{r} = 2$, $\epsilon_{u,t} = -(1.5)\mathfrak{s}_u$, Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and $\bar{\pi} = 2$ (black dotted)

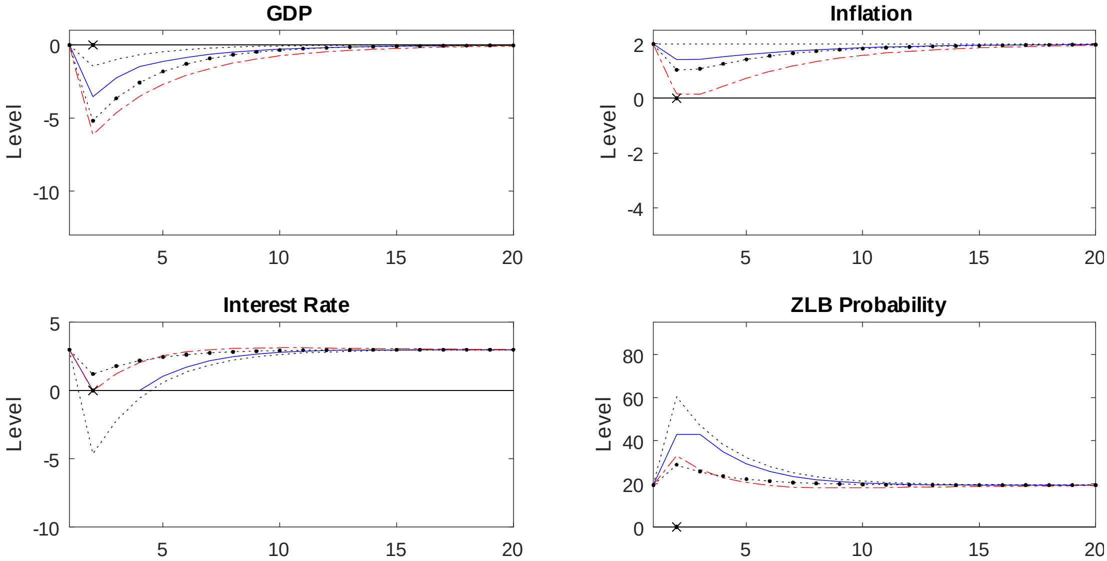

Figure 5: Responses to a one-period negative demand shock of 3.0 Std. Dev.

Note: $\bar{r} = 2$, $\epsilon_{\mathrm{u},t} = -(3.0)\mathfrak{s}_{\mathrm{u}}$, Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and $\bar{\pi} = 2$ (black dotted)

6 Conclusions

I derive a precautionary optimal policy under unconditionally commitment and occasionally binding ZLB constraint on the nominal interest rate, for a standard New Keynesian model with continously-distributed demand and technology shocks. I depart from the literature by directly considering the unconditional probabilities of hitting the ZLB constraint and avoiding the issue of modelling transition probabilities of entering and leaving binding states.

The optimal policy directly internalizes a precautionary behavior arising at occasionally binding ZLB constraint. My approach allows for keeping the direct precautionary behavior even under the first-order (loglinearization) approximation of the model equilibrium.

Finally, I show that the precautionary optimal policy dominates, in welfare terms, the standard optimal policy occasionally binding ZLB constraints (see e.g. Nakov (2008)). In addition, I find that optimal precautionary responses to negative demand shocks induces a slower reduction in the nominal rate as the shock hits, making more room implementing monetary policy in the future. As the negative shock dissipates, optimal precautionary policy takes longer to return the rate to normal values than what the standard optimal policy prescribes. As a consequence, optimal precautionary policy induces the probability of hitting the ZLB to be higher than what the standard optimal policy implies.

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A Steady state levels

Tables 2 and 3 define the structural and composite parameters. Table 4 describes the steady state levels under trend inflation.

Table 2: Structural parameters

$\sigma \equiv$ reciprocal of intertemp elast substit	$\gamma_\pi \equiv$ coeff lag inf on index rule
$\nu \equiv$ reciprocal of the Frisch elasticity	$\varepsilon \equiv$ labor elasticity prod function
$\chi \equiv$ scale parameter on labor disutility	$\alpha \equiv$ Calvo degree of price rigidity
$\theta \equiv$ elasticity of substit between goods	$\bar{\pi} \equiv$ level of trend inflation

Table 3: Composite parameters

$\mu \equiv \frac{\theta}{\theta-1}$	$\bar{\kappa} \equiv \frac{(1-\bar{\alpha})(1-\bar{\alpha}\beta\vartheta)}{\bar{\alpha}}\frac{(\omega+\sigma)}{(1+\theta\omega)}$	$\bar{\mathcal{X}} \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)}\frac{\bar{\kappa}}{\theta}$
$\omega \equiv \frac{1+\nu}{\varepsilon} - 1$	$\bar{\kappa}_{\varpi} \equiv \frac{(1-\bar{\alpha})}{(1+\theta\omega)}$	$\bar{\mathcal{V}} \equiv \frac{(\omega+\sigma)}{\bar{\mathcal{X}}}\bar{Y}^{1-\sigma}$
$\delta \equiv \frac{1}{1-\gamma_\pi}$	$\bar{\Phi}_y \equiv 1 - \frac{(1-\bar{\alpha}\beta\vartheta)(1-\bar{\alpha})}{\mu(1-\bar{\alpha}\beta)(1-\bar{\alpha}\vartheta)}$	$\mathfrak{c}_1 \equiv 1 - (\vartheta - 1)\,\beta\frac{\bar{\kappa}_{\varpi}}{\bar{\kappa}}\,(1-\sigma)$
$\bar{\alpha} \equiv \alpha\left(\bar{\Pi}\right)^{(\theta-1)(1-\gamma_\pi)}$	$\bar{\Phi}_\vartheta \equiv (\vartheta - 1)$	$\mathfrak{c}_2 \equiv 1 + (\vartheta - 1)\,\beta\frac{\bar{\kappa}_{\varpi}}{\bar{\kappa}}\,(\omega+\sigma)$
$\vartheta \equiv \left(\bar{\Pi}\right)^{(1+\theta\omega)(1-\gamma_\pi)}$	$\bar{\phi}_x \equiv \frac{1}{(\omega+\sigma)}\bar{\Phi}_y$	$\mathfrak{c}_3 \equiv \theta\bar{\kappa}\mathfrak{c}_1 + (1-\bar{\alpha}\beta\vartheta)$
$\max(\bar{\alpha}, \bar{\alpha}\vartheta) < 1$	$\bar{\phi}_\pi \equiv \frac{(1-\bar{\alpha})}{(1-\bar{\alpha}\vartheta)(1+\theta\omega)}\bar{\Phi}_\vartheta$	$\mathfrak{c}_4 \equiv \mathfrak{c}_1 - (1-\bar{\alpha}\beta\vartheta)\,\mathfrak{c}_2$

Table 4: Steady state levels

$\bar{I} = \beta^{-1}\left(\bar{\Pi}\right) = \frac{1}{\bar{q}}$	; $\bar{\Pi}^{ind} = \bar{\Pi}^{(\gamma_\pi+\gamma)}$	; $\bar{Y}^n = \left(\frac{\varepsilon}{\chi\mu}\right)^{\frac{1}{(\omega+\sigma)}}$	; $\bar{Y}^{ef} = \left(\frac{\varepsilon}{\chi}\right)^{\frac{1}{(\omega+\sigma)}}$
$\bar{\wp}^* = \left(\frac{1-\alpha}{1-\bar{\alpha}}\right)^{\frac{1}{\theta-1}}$	; $\frac{\bar{N}}{\bar{D}} = \left(\bar{\wp}^*\right)^{1+\theta\omega}$	; $\bar{X}^{\omega+\sigma} = \frac{1-\bar{\alpha}\beta\vartheta}{1-\bar{\alpha}\beta}\frac{\bar{N}}{\bar{D}}$	; $\bar{Y} = \bar{X}\bar{Y}^n$
$\bar{\mathcal{W}} = \frac{\bar{u}-\bar{v}}{1-\beta}$	; $\bar{u} = \frac{\bar{Y}^{1-\sigma}}{1-\sigma}$	; $\bar{v} = \frac{\chi\bar{Y}^{(1+\omega)}\bar{\mathcal{P}}^{-\theta(1+\omega)}}{1+\nu}$	; $\bar{\mathcal{P}} = \left(\frac{1-\bar{\alpha}\vartheta}{1-\alpha}\right)^{\frac{1}{\theta(1+\omega)}}\bar{\wp}^*$

B Deriving the probability of hitting the ZLB

Using the Euler equation (2) and the marginal utility definition, I rewrite the natural probability $\mathfrak{p}^n_{\mathfrak{o},t}$ as

$$\begin{aligned}
\mathfrak{p}^n_{\mathfrak{o},t} &= \mathbb{P}\left(u_t^{n\prime} \leq \beta E_t\left(\frac{u_{t+1}^{n\prime}}{\bar{\Pi}}\right)\right) = \mathbb{P}\left(\epsilon_t\left(Y_t^n\right)^{-\sigma} \leq \beta E_t\left(\frac{\epsilon_{t+1}\left(Y_{t+1}^n\right)^{-\sigma}}{\bar{\Pi}}\right)\right) \\
&= \mathbb{P}\left(\epsilon_t\left(\frac{\varepsilon}{\chi\mu}\epsilon_t\mathcal{A}_t^{(1+\omega)}\right)^{-\frac{\sigma}{(\omega+\sigma)}} \leq \beta E_t\left(\frac{\epsilon_{t+1}\left(\frac{\varepsilon}{\chi\mu}\epsilon_{t+1}\mathcal{A}_{t+1}^{(1+\omega)}\right)^{-\frac{\sigma}{(\omega+\sigma)}}}{\bar{\Pi}}\right)\right) \\
&= \mathbb{P}\left(\left(\epsilon_t\right)^{\frac{\omega}{(\omega+\sigma)}}\left(\mathcal{A}_t\right)^{-\frac{\sigma(1+\omega)}{(\omega+\sigma)}} \leq \frac{\beta}{\bar{\Pi}}E_t\left(\left(\epsilon_{t+1}\right)^{\frac{\omega}{(\omega+\sigma)}}\left(\mathcal{A}_{t+1}\right)^{-\frac{\sigma(1+\omega)}{(\omega+\sigma)}}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p}^n_{\mathfrak{o},t} &= \mathbb{P}\left(\left(\epsilon_t\right)^{\frac{\omega}{(\omega+\sigma)}}\left(\mathcal{A}_t\right)^{-\frac{\sigma(1+\omega)}{(\omega+\sigma)}} \leq \frac{\beta}{\bar{\Pi}}E_t\left(\left(\epsilon_t^{\rho_\mathfrak{u}}\epsilon_{\mathfrak{u},t+1}\right)^{\frac{\omega}{(\omega+\sigma)}}\left(\mathcal{A}_t^{\rho_\mathfrak{a}}\epsilon_{\mathfrak{a},t+1}\right)^{-\frac{\sigma(1+\omega)}{(\omega+\sigma)}}\right)\right) \\
&= \mathbb{P}\left(\left(\epsilon_t\right)^{\frac{\omega(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\mathcal{A}_t\right)^{-\frac{\sigma(1+\omega)(1-\rho_\mathfrak{a})}{(\omega+\sigma)}} \leq \frac{\beta}{\bar{\Pi}}E_t\left(\left(\epsilon_{\mathfrak{u},t+1}\right)^{\frac{\omega}{(\omega+\sigma)}}\left(\epsilon_{\mathfrak{a},t+1}\right)^{-\frac{\sigma(1+\omega)}{(\omega+\sigma)}}\right)\right) \\
&= \mathbb{P}\left(\left(\epsilon_{t-1}^{\rho_\mathfrak{u}}\epsilon_{\mathfrak{u},t}\right)^{\frac{\omega(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\mathcal{A}_{t-1}^{\rho_\mathfrak{a}}\epsilon_{\mathfrak{a},t}\right)^{-\frac{\sigma(1+\omega)(1-\rho_\mathfrak{a})}{(\omega+\sigma)}} \leq \frac{\beta}{\bar{\Pi}}\right) \\
&= \mathbb{P}\left(\left(\epsilon_{t-1}\right)^{\frac{\omega\rho_\mathfrak{u}(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\mathcal{A}_{t-1}\right)^{-\frac{\sigma(1+\omega)\rho_\mathfrak{a}(1-\rho_\mathfrak{a})}{(\omega+\sigma)}}\left(\epsilon_{\mathfrak{u},t}\right)^{\frac{\omega(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\epsilon_{\mathfrak{a},t}\right)^{-\frac{\sigma(1+\omega)(1-\rho_\mathfrak{a})}{(\omega+\sigma)}} \leq \frac{\beta}{\bar{\Pi}}\right) \\
&= \mathbb{P}\left(\left(\epsilon_{\mathfrak{u},t}\right)^{\frac{\omega(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\epsilon_{\mathfrak{a},t}\right)^{-\frac{\sigma(1+\omega)(1-\rho_\mathfrak{a})}{(\omega+\sigma)}} \leq \frac{\beta}{\bar{\Pi}}\left(\epsilon_{t-1}\right)^{-\frac{\omega\rho_\mathfrak{u}(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\mathcal{A}_{t-1}\right)^{\frac{\sigma(1+\omega)\rho_\mathfrak{a}(1-\rho_\mathfrak{a})}{(\omega+\sigma)}}\right)
\end{aligned}$$

Let $\epsilon_{\mathfrak{ua},t} \equiv \left(\epsilon_{\mathfrak{u},t}\right)^{\frac{\omega(1-\rho_\mathfrak{u})}{(\omega+\sigma)}}\left(\epsilon_{\mathfrak{a},t}\right)^{-\frac{\sigma(1+\omega)(1-\rho_\mathfrak{a})}{(\omega+\sigma)}}$ denote the aggregate shock. If $\epsilon_{\mathfrak{u},t} \overset{iid}{\sim} LN\left(0, \mathfrak{s}_\mathfrak{u}^2\right)$

is independent of $\epsilon_{\mathfrak{a},t} \overset{iid}{\sim} LN\left(0, \mathfrak{s}_{\mathfrak{a}}^2\right)$, where $\mathfrak{s}_{\mathfrak{u}}^2$ and $\mathfrak{s}_{\mathfrak{a}}^2$ are dispersion parameters, then $\epsilon_{\mathfrak{ua},t} \overset{iid}{\sim} LN\left(0, \mathfrak{s}_{\mathfrak{ua}}^2\right)$, where $\mathfrak{s}_{\mathfrak{ua}}^2 \equiv \left(\frac{\omega(1-\rho_{\mathfrak{u}})}{(\omega+\sigma)}\right)^2 \mathfrak{s}_{\mathfrak{u}}^2 + \left(\frac{\sigma(1+\omega)(1-\rho_{\mathfrak{a}})}{(\omega+\sigma)}\right)^2 \mathfrak{s}_{\mathfrak{a}}^2$. Therefore, I compute $\mathfrak{p}_{\mathfrak{o},t}^n$ as follows:

$$\mathfrak{p}_{\mathfrak{o},t}^n = \mathbb{F}_{\mathfrak{ua}}\left(\frac{\beta}{\bar{\Pi}}\left(\epsilon_{t-1}\right)^{-\frac{\omega\rho_{\mathfrak{u}}(1-\rho_{\mathfrak{u}})}{(\omega+\sigma)}}\left(\mathcal{A}_{t-1}\right)^{\frac{\sigma(1+\omega)\rho_{\mathfrak{a}}(1-\rho_{\mathfrak{a}})}{(\omega+\sigma)}}\right)$$

Conditional on the expected paths of output and inflation in any equilibrium with trend inflation, $\mathfrak{p}_{\mathfrak{o},t}$ satisfies:

$$\begin{aligned}
\mathfrak{p}_{\mathfrak{o},t} &= \mathbb{P}\left(u_t' \leq \beta E_t\left(\frac{u_{t+1}'}{\Pi_{t+1}}\right)\right) = \mathbb{P}\left(\epsilon_t\left(Y_t\right)^{-\sigma} \leq \beta E_t\left(\frac{\epsilon_{t+1}\left(Y_{t+1}\right)^{-\sigma}}{\Pi_{t+1}}\right)\right) \\
&= \mathbb{P}\left(\left(\epsilon_{\mathfrak{u},t}\right)^{(1-\rho_{\mathfrak{u}})} \leq \left(\epsilon_{t-1}\right)^{-\rho_{\mathfrak{u}}(1-\rho_{\mathfrak{u}})} E_t\left(\frac{\beta}{\Pi_{t+1}}\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\epsilon_{\mathfrak{u},t+1}\right)\right) \\
&= \mathbb{F}_{\mathfrak{u}\rho}\left(\left(\epsilon_{t-1}\right)^{-\rho_{\mathfrak{u}}(1-\rho_{\mathfrak{u}})} E_t\left(\frac{\beta}{\Pi_{t+1}}\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\epsilon_{\mathfrak{u},t+1}\right)\right)
\end{aligned}$$

C Deriving the optimal precautionary optimal policy

The monetary authority solves the following problem, written in the Lagrangian form:

$$\begin{aligned}
\min_{\substack{\{\hat{\imath}_t,\hat{x}_t,\hat{\varpi}_t,\hat{\varrho}_t\} \\ \left\{\hat{\pi}_t,\hat{\pi}_t^{ind},\mathfrak{p}_{\mathfrak{o},t}\right\}}} \quad & \tfrac{1}{(1-\beta)}E\tfrac{1}{2}\left[\left(\hat{\pi}_t-\hat{\pi}_t^{ind}+\phi_\pi\right)^2+\left(1-\mathfrak{p}_{\mathfrak{o},t}\right)\bar{\mathcal{X}}\left(\hat{x}_t-\phi_x\right)^2+\mathfrak{p}_{\mathfrak{o},t}\bar{\mathcal{X}}\left(\hat{\varrho}_t+\tfrac{1}{\sigma}\mathring{i}+\tfrac{1}{\sigma}\hat{r}_t^n-\phi_x\right)^2\right. \\
+ \quad & \tfrac{1}{(1-\beta)}E\Lambda\left(\hat{\pi}_t-0\right) \\
+ \quad & \tfrac{1}{(1-\beta)}E\lambda_t^{\varrho}\left[\hat{\varrho}_t-\hat{x}_{t+1}-\tfrac{1}{\sigma}\hat{\pi}_{t+1}\right] \\
+ \quad & \tfrac{1}{(1-\beta)}E\lambda_t^{ind}\left[\hat{\pi}_t^{ind}-\gamma_\pi\hat{\pi}_{t-1}\right] \\
+ \quad & \tfrac{1}{(1-\beta)}E\lambda_t^{\pi}\left[\hat{\pi}_t-\hat{\pi}_t^{ind}-\beta\left(\hat{\pi}_{t+1}-\hat{\pi}_{t+1}^{ind}\right)-\bar{\kappa}\hat{x}_t-\left(\bar{\vartheta}-1\right)\bar{\kappa}_{\varpi}\beta\hat{\varpi}_{t+1}\right] \\
+ \quad & \tfrac{1}{(1-\beta)}E\lambda_t^{\varpi}\left[\hat{\varpi}_t-\bar{\alpha}\bar{\vartheta}\beta\hat{\varpi}_{t+1}-\theta\left(1+\omega\right)\left(\hat{\pi}_t-\hat{\pi}_t^{ind}\right)\right. \\
& \qquad\qquad \left.-\left(1-\bar{\alpha}\bar{\vartheta}\beta\right)\left(\omega+\sigma\right)\hat{x}_t-\left(1-\sigma\right)\left(\hat{x}_t-\hat{x}_{t-1}\right)\right] \\
+ \quad & \tfrac{1}{(1-\beta)}E\left(1-\mathfrak{p}_{\mathfrak{o},t}\right)\lambda_t^{x}\left[\hat{x}_t-\hat{\varrho}_t+\tfrac{1}{\sigma}\hat{\imath}_t\right] \\
+ \quad & \tfrac{1}{(1-\beta)}E\lambda_t^{\mathfrak{p}}\left[\left(\mathfrak{p}_{\mathfrak{o},t}-\bar{\mathfrak{p}}_{\mathfrak{o}}\right)+\phi_\epsilon\left[\sigma\left(\hat{Y}_{t+1}-\hat{Y}_t\right)+\hat{\pi}_{t+1}\right]\right] \\
+ \quad & \textit{constant and exogenous terms}
\end{aligned}$$

Consider a generic variable $\hat{\varkappa}_t$ and a generic Lagrangian multiplier $\lambda_t^{\varkappa}$. Using properties of unconditional expectations on stationary time series, we can substitute $E\lambda_{t+1}^{\varkappa}\hat{\varkappa}_t$ for $E\lambda_t^{\varkappa}\hat{\varkappa}_{t-1}$. Likewise, we substitute $E\lambda_{t-1}^{\varkappa}\hat{\varkappa}_t$ for $E\lambda_t^{\varkappa}\hat{\varkappa}_{t+1}$. With this strategy, we simplify the Lagrangian problem to set all relevant variables to current time, leaving Lagrangian multipliers at appropriate lags or leads:

$$\begin{aligned}
\min_{\substack{\{\hat{\imath}_t,\hat{x}_t,\hat{\varpi}_t,\hat{\varrho}_t\}\\ \left\{\hat{\pi}_t,\hat{\pi}_t^{ind},\mathfrak{p}_{\mathfrak{o},t}\right\}}} & \tfrac{1}{(1-\beta)}E\tfrac{1}{2}\left[\left(\hat{\pi}_t-\hat{\pi}_t^{ind}+\phi_\pi\right)^2+\left(1-\mathfrak{p}_{\mathfrak{o},t}\right)\bar{\mathcal{X}}\left(\hat{x}_t-\phi_x\right)^2+\mathfrak{p}_{\mathfrak{o},t}\bar{\mathcal{X}}\left(\hat{\varrho}_t+\tfrac{1}{\sigma}\mathring{i}+\tfrac{1}{\sigma}\hat{r}_t^n-\phi_x\right)^2\right.\\
+ & \tfrac{1}{(1-\beta)}E\Lambda\hat{\pi}_t\\
+ & \tfrac{1}{(1-\beta)}E\left[\lambda_t^\varrho\hat{\varrho}_t-\lambda_{t-1}^\varrho\hat{x}_t-\tfrac{1}{\sigma}\lambda_{t-1}^\varrho\hat{\pi}_t\right]\\
+ & \tfrac{1}{(1-\beta)}E\left[\lambda_t^{ind}\hat{\pi}_t^{ind}-\gamma_\pi\lambda_{t+1}^{ind}\hat{\pi}_t\right]\\
+ & \tfrac{1}{(1-\beta)}E\left[\lambda_t^\pi\hat{\pi}_t-\lambda_t^\pi\hat{\pi}_t^{ind}-\beta\lambda_{t-1}^\pi\left(\hat{\pi}_t-\hat{\pi}_t^{ind}\right)-\bar{\kappa}\lambda_t^\pi\hat{x}_t-\left(\bar{\vartheta}-1\right)\bar{\kappa}_\varpi\beta\lambda_{t-1}^\pi\hat{\varpi}_t\right]\\
+ & \tfrac{1}{(1-\beta)}E\left[\lambda_t^\varpi\hat{\varpi}_t-\bar{\alpha}\bar{\vartheta}\beta\lambda_{t-1}^\varpi\hat{\varpi}_t-\theta\left(1+\omega\right)\lambda_t^\varpi\left(\hat{\pi}_t-\hat{\pi}_t^{ind}\right)\right.\\
& \qquad\left.-\left(1-\bar{\alpha}\bar{\vartheta}\beta\right)\left(\omega+\sigma\right)\lambda_t^\varpi\hat{x}_t-\left(1-\sigma\right)\left(\lambda_t^\varpi\hat{x}_t-\lambda_{t+1}^\varpi\hat{x}_t\right)\right]\\
+ & \tfrac{1}{(1-\beta)}E\left(1-\mathfrak{p}_{\mathfrak{o},t}\right)\left[\lambda_t^x\hat{x}_t-\lambda_t^x\hat{\varrho}_t+\tfrac{1}{\sigma}\lambda_t^x\hat{\imath}_t\right]\\
+ & \tfrac{1}{(1-\beta)}E\left[\lambda_t^{\mathfrak{p}}\left(\mathfrak{p}_{\mathfrak{o},t}-\bar{\mathfrak{p}}_{\mathfrak{o}}\right)+\phi_\epsilon\left[\sigma\left(\lambda_{t-1}^{\mathfrak{p}}\hat{Y}_t-\lambda_t^{\mathfrak{p}}\hat{Y}_t\right)+\lambda_{t-1}^{\mathfrak{p}}\hat{\pi}_t\right]\right]\\
+ & \textit{constant and exogenous terms}
\end{aligned}$$

In addition, the whole Lagrangian problem must be of order $\mathcal{O}(2)$, for this is the order to which the welfare function is log-approximated. Since log-linearized equations are used as restrictions, Lagrangian multipliers must be of order $\mathcal{O}(1)$. This order issue is relevant when adding the ZLB Probability curve (13), i.e. first order approximation of $\mathfrak{p}_{\mathfrak{o},t}$, into the problem. The issue arises when multiplying this approximation by the second order components from the loss function. We must disregard all $\mathcal{O}(3)$ terms from the resulting multiplication. In Alves (2014), I show that the distortion parameters $\bar{\phi}_\pi$ and $\bar{\phi}_x$ must be of order $\mathcal{O}(1)$ in order for the trend inflation welfare-based loss function to be properly used with log-linearized equations when deriving optimal policy rules. With the same logic, I assume that $\mathring{i}$ is of order $\mathcal{O}(1)$. This assumption is reasonable once we consider that any hatted variable is assumed to be of order $\mathcal{O}(1)$ and $\mathring{i}$ is $-\hat{\imath}_t$ evaluated at $i_t = 0$.

Since I assume that $\mathring{i}$ is of order $\mathcal{O}(1)$, the whole Lagrangian problem is equivalent to the one in which $\bar{\mathfrak{p}}_{\mathfrak{o}}$ substitutes $\mathfrak{p}_{\mathfrak{o},t}$ everywhere. As a consequence, the last constaint is easily shown not to bind, i.e. $\lambda_t^{\mathfrak{p}} = 0$. The remaining first-order conditions are then obtained in the traditional way, and so are not shown here.