# Commodity Price Beliefs, Financial Frictions and Business Cycles *

Jesús Bejarano Franz Hamann Enrique G. Mendoza Diego Rodríguez

This version: August 29 2016
(Preliminary and incomplete)

## Abstract

After a long period of high international commodity prices, their sudden collapse has had important macroeconomic implications for commodity-exporting economies. This paper conducts a quantitative assessment of the macroeconomic impact of persistent changes in commodity prices for small but resource rich economies, in environments in which agents face financial and informational frictions. Agents in the model face an informational friction by which they have to learn about the true persistence of international commodity price fluctuations, while at the same time they face foreign borrowing constraints in international financial markets. We find that the macroeconomic effects of commodity price shifts vary depending on the interaction of these frictions with the incentives to exploit the commodity because the short and long-run net foreign asset position of the economy is jointly determined by its natural resource wealth. Under incomplete international financial markets, commodity price fluctuations affect not only the aggregate patterns of consumption and saving but also the private sector incentives to exploit the natural resource inter-temporally. Moreover, the impact on the economy of shifts from high-price times to the low ones depends on the agents beliefs of the duration of the good times. More optimistic beliefs of agents regarding the duration of commodity price booms turn out to imply wider boom-boost external borrowing cycles. We take the case of oil and calibrate a small-scale model to the Colombian data and find that the informational and financial friction model reproduces the boom-bust foreign borrowing pattern typically associated to the business cycle in emerging economies.

**Keywords: commodity prices, precautionary savings, financial frictions, learning, information asymmetries, current account, oil, Colombia**

**JEL Classifications: C61, E31, E37, E52, F41**

---

[*] The views expressed in this document are those of the authors and not necessarily those of the Bank of the Republic, Colombia or its Board of Governors (*Junta Directiva*). We would like to thank Martin Uribe for his comments to an earlier version of this work. Also, to Rafael Hernández, Joao Hernández and Carlos-Leonardo Barreto, who assisted us at different stages of this project. Of course, any mistake in this paper is our full responsibility.

# 1 Introduction

Large international commodity price swings have been a source of instability in the global economy. In particular, commodity exporters have shown to be vulnerable to commodity price cycles. These economies enjoy the gains from the run-up and remain apparently strong during the good times but suffer and appear weak with the fall. How do commodity exporters react to changes in commodity prices? The conventional answer is that it depends on whether commodity price changes are known to be transitory or permanent. There is some degree of consensus that long lasting changes in global conditions pose a different challenge for small, open and commodity dependent economies. Persistent changes in oil prices tend to affect permanent income more than short-lived ones, with potential effects on aggregate consumption and savings decisions and have implications for resource allocations between tradable and nontradable sectors which show up in the real exchange rate, real wages and the country's net foreign asset position in the long term. These long lasting changes may have different macroeconomic consequences than temporary shocks.

However, accurately identifying whether commodity price changes will be temporary or permanent is difficult in practice and agents have to learn over time. For instance, when oil prices collapsed in the second half of 2014, after five years of staying at historically high levels, some expected this change to last a few quarters (as it did in the previous collapse in 2008) while others expected a decade-long of low oil prices. These discrepancies highlight the uncertainty surrounding the persistence of commodity prices. In commodity exporting economies, an uncertain duration of commodity price shifts implies an uncertain duration of the good or the bad times. During the high price phase, optimism about the duration of the good times stimulates spending, widens current account deficits and increases net external indebtedness. Higher indebtedness becomes a key risk factor in the presence of borrowing constraints to foreign financing. If net external debt reaches levels that are sufficiently close to some borrowing limit, a sudden change in commodity prices may trigger a sharp macroeconomic adjustment. This adjustment is more abrupt than the one triggered under normal times, because the income reduction cannot be financed by additional borrowing. Therefore aggregate demand collapses. Overconfidence about the duration of the good times leads to over-borrowing and increases the risk of a sharp macroeconomic adjustment.

The literature has documented that commodity exporters experienced credit booms and busts which have been commonly associated with changes in the international environment, especially in commodity prices. For instance, many of the countries which enjoyed higher commodity prices, increased national income, creating a surge in demand for tradable and nontradable goods, inducing a real exchange rate appreciation and a shift of economic resources from the tradable sector to the nontradable sector. Overall economic activity and demand booms, move in tandem with asset prices. However, sharp commodity price reversals truncate this process and a reallocation of resources happen together with a collapse in asset prices and the currency. (e.g. Chile, Colombia, Ecuador, Peru, Russia, Venezuela).[1] Mendoza and Terrones (2012) document more systematically this boom-bust cycle. They found that 35% of the credit booms observed in the 1960–2010 period across developed and emerging economies occurred after surges in capital flows, which are commonly driven by high commodity prices.

We conduct a quantitative assessment of the impact of commodity price fluctuations in a small, open and commodity-exporting economy. We proceed in two stages. In the first stage we present the model under rational expectations and perfect information. The use of these family of quantitative models in the international economics literature has its roots in Mendoza [1991] and it is related to the emerging market business cycles literature including Mendoza (1995), Neumeyer and Perri (2005), Uribe and Yue (2006), Aguiar and Gopinath (2007), Garcia-Cicco et al. (2010), and among others. The dynamics of the real exchange rate adjustment has been quantified in Mendoza and Uribe [2001]. More recently, the macroeconomic interaction with financial frictions has been investigated in Mendoza [2006] and Mendoza [2010]. The main insights and lessons of this strand of the literature have been reviewed in Korinek and Mendoza [2014].

The rational expectations perfect information setup allows us to understand the basic mechanisms at work. Since commodity price changes bring about changes in the terms of trade, GDP and the associated equilibrium adjustment of the real exchange rate, this economy shares some features of those in which agents can borrow and lend to smooth random fluctuations in income. Differences between interest and discount rates and precautionary saving motives drive the determination of net foreign assets in the long run. However, we complement this framework by introducing a resource extracting sector into the model. Unlike the majority of models in the literature, resource extraction responds to economic incentives and the determination of natural wealth also plays a fundamental role

---

[1] **?** have documented some empirical regularities around *transitory* oil price shocks in Colombia. The study performs an oil price shock identification analysis, which analyzes how a key set of macroeconomic variables behave around such events. In that work the focus is to study large and temporary increases in international oil prices. The paper describes how country risk, output, private consumption, domestic credit, trade balance and the real exchange rate evolve during oil price surges as well as during the corrections. Their sample covered episodes from 1988 to 2012 and the event analysis was carried out at quarterly frequency. Following Hamilton [2003] the study finds the quarters during which there were oil price shocks, defined as large increases in oil prices. The paper documents that before the peak of a large and steady oil price hikes, country risk falls, output rises, private consumption increases, domestic credit booms, trade balance improves and the real exchange rate appreciates. In general, after the sudden oil price reversal all these patterns shift back in the opposite direction.

in consumption and saving decisions. We model the resource sector as a dynamic optimal extracting problem as in Sickles and Hartley [2001] and Pesaran [1990]. The economy owns a natural resource, extracts the optimal portion of it to sell it in international competitive commodity markets. Thus optimal extraction rules depend on the stock of reserves of the natural resource, commodity prices, interest rates and current and future marginal costs and revenues of extraction. We calibrate this model to the Colombian data and taking oil as the natural resource for extraction.

In the second stage, we introduce imperfect information and learning. Following Boz and Mendoza [2014] we propose a model in which the true persistence of commodity prices can only be discovered with time, and this learning process interacts with an international borrowing constraint that limits the economy's external debt not to exceed a fraction of GDP. The commodity price process is modeled as two-state Markov chain: high price and low price. Agents in this small open economy know that one of these two regimes can materialize in any given period: one in which high prices continues, and one in which the price is lower. We assume that agents do not know the true probability of this new low-price regime, because they lack data with which to estimate accurately the switching probabilities across the two regimes (i.e. the true commodity price persistence). We also assume that agents learn over time as they observe price realizations (Bayesian learning), and in the long-run their beliefs eventually converge to the true probabilities of the Markov chain. As in Boz and Mendoza [2014], in the long-run the model converges to the rational expectations perfect information (REPI) solution. However, in the short-run optimal allocations deviate from the REPI equilibrium, because agents beliefs that differ from those of the REPI solution lead to misapprehension of the macroeconomic consequences of commodity price fluctuations. Here, since international financial markets are incomplete, this miss-appreciation leads to less precautionary saving with respect to the one that would be observed in the RE allocation, in the short-run. As a result, the small open economy over-borrows in the short-run (with respect to the REPI allocation).

Our quantitative analysis points to two main findings about the long-run adjustment of a small open economy in response to changes of international commodity prices. First, the natural response to high prices is to increase extraction and reduce the reserves of the commodity. The opposite happens during low prices. Thus, natural wealth matters for the determination of NFA in the short run and in the long run. The adjustment of a naturally rich economy to terms of trade shocks is different to an endowment economy, where the stock of the natural resource is irrelevant by assumption. Precautionary savings coupled with incomplete financial markets imply that uncertainty in the resource sector translates into the private agents income uncertainty affecting their motives to spend, save and borrow but also determine wealth the long run. Moreover, the aggregate consequences of commodity price swings translate into domestic prices, like the price of non-tradable goods (the real exchange rate). In the presence of financial frictions, like constraints to foreign borrowing, these swings relax or tighten the agents ability to borrow abroad through different channels. Therefore, the share of the resource sector in the economy is important.

Second, the introduction of asymmetries of information and uncertainty about the true persistence of commodity prices fluctuations shows that the process of learning in the presence of foreign borrowing constraints leads to a period of booming activity, followed by a sharp, sudden collapse of consumption and the exchange rate. We conduct an experiment in this economy subject to informational and financial frictions, calibrated to Colombian data and taking the oil sector as the resource sector, in which we date the start of the high oil price regime in the second quarter of 2009 and the beginning of the oil price reversal in the third quarter of 2014. Hence, during five years we assume that the economy experienced the high-oil price regime, followed by a switch to the low-oil price regime in the second quarter of 2014. We evaluate the quantitative predictions of the calibrated model. In this paper, however, we focus exclusively on the role of commodity prices affecting the economy's ability to borrow in an environment with imperfect information and incomplete and imperfect financial markets, because we aim to show how these frictions alone cause a sharp boom–bust business cycle in an oil exporting economy.

Also, at the core of the adjustment mechanism lies the external borrowing constraint that the economy faces in international financial markets. The presence of the borrowing constraint introduces a Fisherian amplification mechanism that interacts with the learning mechanism and the precautionary savings motive. For instance, a change in the commodity price regime from high to low has a direct effect of tightening the borrowing constraint because it reduces real GDP (in units of tradables) via the relative price of the commodity in units of the tradable goods. There are also at least two indirect effects: one that tightens further the constraint because the commodity extraction falls, reducing further GDP (in units of tradables); second, because the relative price of nontradables in units of the tradable good also falls, reducing even further GDP (in units of tradables). This financial amplification mechanism gets amplified even further because of the presence of learning. Periods of optimism about the persistence of the high price regime induce under-extraction of the commodity, over-consumption, over-borrowing and an overvalued real exchange rate (relative to the rational expectations perfect information case), while the opposite happens during periods of pessimism. Yet the precautionary saving forces at work in these economies make Sudden Stops unlikely events because households in the commodity-exporting economy self insure against them by lower levels of indebtedness in the long run.

We model learning following the approach proposed by Cogley and Sargent [2008] and applied to the 2008 U.S. financial crisis by Boz and Mendoza [2014]. They offer an explanation of the U.S. financial crisis based on financial innovation and overconfidence about the risk of new financial products and show that a model with a collateral constraint, in which learning about the risk of a new financial environment interacts with Fisherian amplification, produces a boom-bust cycle in debt, asset prices and consumption. Early realizations of a financially lax regime turn agents optimistic about the duration of the good times. Conversely, the first realization of a financially tight regime turns agents very pessimistic. Their model predicts large increases in household debt, land prices and excess returns during 1998-2006 followed by a collapse. Similarly, in our setup, the true probabilities of switching across international commodity price regimes are unknown, and agents in the commodity-exporting economy learn about them over time.

This paper primarily contributes to one strand of the literature in international macroeconomics that aims to explain the documented boom-bust pattern of business cycles in emerging economies. Our work is also closely related to Boz [2009] and Boz et al. [2011]. These models explore the role of optimism in explaining key business cycle differences between emerging and developed economies. They highlight that uncertainty regarding the duration of structural breaks explain these differences. Boz [2009] models these changes in structural breaks by introducing a learning problem about persistent productivity shocks while Boz et al. [2011] does so by learning to decompose total TFP into trend and cycle.

We provide an alternative explanation that also accounts for the observed relationship between commodity price fluctuations and the business cycle in commodity exporting economies. In this paper, our focus is not exogenous shifts in TFP but changes in commodity prices. Under-appreciating the true process of commodity prices is natural because of the lack of data on the duration of the high/low price regimes, as well as the lack of knowledge about the true factors behind commodity price fundamentals. Natural resources are known to be affected by political instability, changes in their market structure, structural changes in technology to exploit them, shifts in global demand, among other factors. As a result, discovering the true process of commodity prices is an ongoing process as the academic debate about the true nature of oil price swings illustrates (see Hamilton [2003], Rebucci and Spatafora [2006], Kilian [2009] and Kilian et al. [2009], among others.)

It is thus a natural extension to think that with time agents learn about the true process of commodity prices. Fornero et al. [2015] examine quantitatively the widening of the current account through the lens of a large-scale small open economy monetary policy model with a commodity sector and the assumption that agents have imperfect information and learn about the persistence of commodity price shocks. They show that during a persistent copper price increase, agents believe at first that this increase is temporary but eventually revise their expectations upward as they are surprised by higher-than-expected price levels. Investment and production in the commodity sector (and in the overall economy) expands while domestic savings deteriorating the current account.

Our main contribution is to show that commodity prices and the changes that they induce on incentives to exploit a natural resource interact with informational and financial frictions in the short and the long-run. Unlike Fornero et al. [2015] our model does not fix long-run stocks of NFA and natural wealth at exogenously predetermined levels. Here both short and long-run stocks of natural and financial assets are an equilibrium outcome. As our model features precautionary savings by households, the long-run level of the economy's assets are determined by the true properties of commodity prices. As time passes, the discovery of this process also determines the short-term dynamics of real and financial assets and the adjustment is very different with respect to the REPI case, because the extraction incentives are substantially different. In the REPI case a high realization of the price is expected to be followed by a lower price in the future, because the true process is mean reverting. Therefore in a REPI economy extraction tends to be higher in the current period and lower in the future. The extraction pattern is crucially different in optimist Bayesian learning environment. If there is optimism, a high realization of the price is expected to be followed by a high price in the future, because the agents are in the process of learning and remain optimists. Therefore in a Bayesian learning economy extraction tends to be lower in the current period and higher in the future. The economy borrows more in the Bayesian learning allocation (relative to the REPI) because households are optimist and the positive price effect (higher international prices) of the commodity revenues dominate the negative quantity effect (less extraction). When a negative realization of the commodity price hits the economy, agents become very pessimistic in the Bayesian learning economy and cut extraction dramatically, more than in the REPI economy, tightening even further the borrowing constraint. This process of over-borrowing (compared to the REPI allocation) puts the economy closer to the international borrowing constraint. So when commodity prices collapse, a sharp macroeconomic adjustment follows, because the constraint becomes suddenly binding not only because of commodity prices, but also because commodity extraction tanks and the real exchange rate depreciates. A sharp consumption adjustment follows, reproducing the over-borrowing and the macroeconomic adjustment that follows of the business cycles in emerging economies.

The rest of the paper proceeds as follows. In Section 2 we present the model under both rational expectations perfect information and Bayesian learning setups. In Section 3 we present our quantitative analysis which includes the baseline calibration of the model and analyze the implications of a sudden fall in commodity prices after a long period of high commodity prices. We conclude with Section 4.

# 2 A Commodity Exporting Economy

In this section we present a small open and naturally-rich economy model to explore the implications of uncertainty in the persistence of international commodity prices. In the model the true persistence of oil prices can only be discovered with time, and this learning process interacts with an international borrowing constraint that limits the economy's external debt not to exceed a fraction of GDP. We proceed in two steps. First, we present the rational expectations perfect information setup and then we present the learning imperfect information one.

## 2.1 Rational expectations and perfect information

Consider a small open economy with three sectors: tradable, non-tradable and a resource sector (for example, coal, oil or copper). The economy is populated by a representative household, who owns a stock of the commodity and a representative firm, which exploits it. Time is discrete, $t = 0, 1, 2, \dots$ . Every period $y^T$ units of the tradable and $y^N$ units of the non-tradable goods are available in fixed supply to the household. The stock of reserves is $\bar{s} > 0$ units of the commodity and $d \geq 0$ units are discovered every period. At the beginning of period $t$ the stock of reserves available for extraction during that period is $s_t \in [0, \bar{s}]$ and $x_t \in X = [0, s_t]$ units can be extracted. Thus, reserves evolve over time according to $s_{t+1} = s_t - x_t + d$.

The commodity has a random relative price $p_t$ (in units of tradables), which is determined in this competitive market and it is the only source of uncertainty in the model. The price follows a Markov process characterized by a time-invariant two-point regime switching process. There is a high-price regime $p_h$ and a low-regime price, $p_l$, with $p_h > p_l$. The transition probabilities given by $Q = q\left(p_{t+1} \mid p_t\right)$. The continuation probabilities are given by $q_{hh}\left(p_{t+1} = p_h \mid p_t = p_h\right)$ and $q_{ll}\left(p_{t+1} = p_l \mid p_t = p_l\right)$ , and switching probabilities $q_{hl} = 1 - q_{hh}$ and $q_{lh} = 1 - q_{ll}$. The long-run probabilities are $\Pi^h = \left(\frac{q_{lh}}{q_{lh}+q_{hl}}\right)$ and $\Pi^l = \left(\frac{q_{hl}}{q_{lh}+q_{hl}}\right)$. Agents do not know the true price process and have to learn about it. Later we explain the nature and consequences of commodity-price uncertainty under different assumptions about the information available to agents.

### 2.1.1 The commodity extracting firm

There is a commodity extracting firm in the small open economy with frictionless access to international financial and commodity markets. The firm extract $x_t$ units of the commodity for exporting to international commodity markets. Total revenues of the firm are $p_t x_t$. Commodity extraction is a costly activity. The total cost of extracting $x_t$ units of the commodity in any period, given that the stock is $s_t$ units at the beginning of period $t$, is $e(s_t, x_t)$ units of the tradable good. The cost function $e$ is decreasing in $s_t$ (total extraction cost falls the larger the reserves) and increasing in $x_t$ (total cost grows the higher the extraction rate). The total cost function has the following properties: $e_s < 0$, $e_x > 0$ and $e_s\left(s, 0\right) = 0$. The marginal cost of an additional units of reserves, conditioned on not extracting oil, is zero: $e_s\left(s, 0\right) = 0$. These assumptions are conventional in the natural resource literature. See for instance Salant [1976], Pindyck [1980] and Pindyck [1981].

Under these assumptions, the value of the firm, given that the economy has $s_t$ units of reserves at the beginning of the period, satisfies the Bellman equation:

$$v(s_t, p_t) = \max_{x \in X} \left\{ p_t x_t - e(s_t, x_t) + \delta \mathbb{E}_t^B \left[ v\left(s_{t+1}, p_{t+1}\right)\right]\right\} \tag{1}$$

where $v$ is the value function of the firm, $\delta \in (0,1)$ is the discount factor of the firm. We set the discount factor equal to $\delta = 1/R$, where $R$ is a constant international risk-free (gross) interest rate. $\mathbb{E}_t^B$ represents the expectations conditional on the representative agent's beliefs formulated with the information available up to and including date $t$. These beliefs are different from the rational expectations formulation with perfect information, denoted $\mathbb{E}_t$.

To gain some intuition on the inter-temporal tradeoff that the commodity extracting firm faces it is convenient to derive the Euler equation of the firm's problem:

$$p_t - e_x\left(s_t, x_t\right) = \frac{\mathbb{E}_t^B \left[p_{t+1} - e_x\left(s_{t+1}, x_{t+1}\right) - e_s\left(s_{t+1}, x_{t+1}\right)\right]}{R}, \tag{2}$$

which states that the current marginal profit of extracting a unit of the commodity today (the LHS of 2) must be equal to the *perceived* expectation of future marginal profits (the RHS of 2). Future marginal profits have three components. The first term is the belief about future prices. The second term is the marginal cost of extracting a unit of the commodity in the future. The third term is the marginal cost associated to the fact that extracting a unit of the commodity today implies less reserves tomorrow and therefore it will be more costly to extract it in the future. Optimist beliefs of future prices imply more extraction tomorrow and less extraction today, which means more reserves today relative to tomorrow.

Thus, as we will see later in the quantitative experiments, beliefs about commodity prices are fundamental for the determination of commodity extraction and the stock of reserves.

In a rational expectations perfect-information setup, the firm knows the commodity price process. Associated with the solution to the firm program (1) there is a time-invariant optimal commodity extraction policy, $\tilde{x}\left(s, p\right)$. The controlled-state process of the firm with optimal policy function $\tilde{x}$, is a stationary Markov chain with transition probability matrix $\mathbf{H}$ whose typical element in the position $(i, j)$ is the probability of jumping from state $i$ in the current year to state $j$ next year, conditioned on following the optimal policy $\tilde{x}\left(i\right)$: $\mathbf{H}_{ij} = \Pr\left(s_{t+1} = j | s_t = i, x_t = \tilde{x}(i)\right)$. However, under Bayesian learning, as we will describe later, the optimal extraction policy becomes a function of the beliefs about commodity prices.

### 2.1.2 The household

The household consumes $c$ units of a composite non-storable good and the preferences are given by

$$\mathbb{E}_0^B\left[\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma}\right] \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor and $\sigma$ is the parameter that determines the degree of risk aversion. The consumption bundle is

$$c_t = \left[a\left(c_t^T\right)^{-\mu} + (1-a)\left(c_t^N\right)^{-\mu}\right]^{-\frac{1}{\mu}}, a > 0, \mu \geq -1 \tag{4}$$

where $c_t^T$ and $c_t^N$ represent the consumption of tradable and non-tradable goods, $a$ is the standard CES weighing factor and the parameter $\mu$ determines the elasticity of substitution between consumption of tradable goods and consumption of non-tradable goods, which is given by $\frac{1}{1+\mu}$.

We assume that the firm pays $\tau_S p_t \tilde{x}_t$ to the household, with $\tau_S \in (0, 1)$, for the right to exploit the commodity. Also, the firm pays $\tau_E e\left(s_t, \tilde{x}_t\right)$ to the household, with $\tau_E \in (0, 1)$, for the extraction costs. Thus the household budget constraint is:

$$c_t^T + p_t^N c^N = y^T + \tau_\pi \pi_t + \tau_E e(s_t, \tilde{x}_t) + p_t^N y^N - b_{t+1} + R b_t$$

where $\pi_t \equiv p_t \tilde{x}_t - e\left(s_t, \tilde{x}_t\right)$ and $\tau_\pi \equiv \frac{\tau_S p_t \tilde{x}_t - \tau_E e(s_t, \tilde{x}_t)}{\pi_t}$ represents the fraction of the profits that the household receives, $p_t^N$ denotes the price of non-tradables relative to tradables, $b_t$ represents the net foreign asset position (denominated in units of tradable goods) in terms of the only internationally traded asset: one-period bonds that pay the world-determined gross real interest rate, $R$.

Unlike the firm, the household has imperfect access to international financial markets. This imperfection takes the form of a borrowing constraint that limits the value of future net foreign asset position, $b_{t+1}$, to a fraction of GDP:

$$b_{t+1} \quad \geq \quad -\phi\left(y^T + \tau_\pi \pi_t + \tau_E e(s_t, \tilde{x}_t) + p_t^N y^N\right),$$

where $\phi \geq 0$ represents the degree of tightening of the borrowing constraint: $\phi = 0$ means that the borrowing constraint is not binding. The above credit constraint can arise from a variety of imperfections, which are not explicitly modeled here, but allows us to focus on their implications for the equilibrium allocations and prices.

The solution to the household problem can be derived by maximizing (3) subject to the budget and the borrowing constraints. The first order conditions of the problem are:

$$\begin{aligned} p_t^N &= \left(\frac{1-a}{a}\right)\left[\frac{c_t^T}{c_t^N}\right]^{1+\mu} && (5)\\ c_t^{-\sigma} &= \beta R \mathbb{E}_t^B\left[c_{t+1}^{-\sigma}\right] + \lambda_t && (6)\\ c_t^T + p_t^N c^N &= y^T + \tau_\pi \pi_t + \tau_E e(s_t, \tilde{x}_t) + p_t^N y^N - b_{t+1} + R b_t + A^T + p_t^N A^N && (7)\\ b_{t+1} &\geq -\phi\left(y^T + \tau_\pi \pi_t + \tau_E e(s_t, \tilde{x}_t) + p_t^N y^N\right) && (8) \end{aligned}$$

where $\lambda_t$ denotes the Lagrange multiplier on the credit constraint. $A^T$ and $A^N$ are auxiliary variables which captures the part of tradable and non-tradable absorption which are not included in private consumption $c$ and that it is not modeled, but it is present in the National Accounts data. Equation (5) determines the household's marginal rate of substitution between tradable and non-tradable goods. Equation (6) is the standard Euler equation in the presence of the borrowing constraint. Equation (7) is the budget constraint, which is influenced by the firm's decisions. The household receives a fraction of the profits and the payment for the resources sold to the firm to perform the commodity extraction. Because extraction incentives respond to commodity prices, a resource extracting economy responds differently from a stochastic endowment economy. Changes in commodity prices affect the household decisions because both the firm's profits and the payments received by extraction respond to commodity prices. If the firm expects higher future commodity prices, current extraction will be lower relative to the future, reducing current revenues (at a given price), profits are and payments for extraction. Moreover, since the household has only a single financial asset to deal with uncertainty and international financial markets are incomplete, the small open economy's wealth varies with the nature of the randomness of international commodity prices, as in the stochastic endowment case, but its level is very different.

It is also important to highlight the role of the borrowing constraint. The household value of income in the right-hand-side of (8) is equal *at equilibrium* to the economy's total GDP valued at tradables goods prices, so (8) can be viewed as a constraint on the economy's NFA to GDP ratio. The constraint is endogenous, however, because NFA, commodity extraction and the price of nontradables are endogenous. Thus, in addition to the feedback between borrowing decisions and the value of nontradables GDP there is also feedback from the commodity extraction. As in the stochastic endowment economy, note that the household takes as given the price of nontradables. Since the firm is not influenced by the price of non-tradable goods, the firm decisions are not altered by real exchange movements. Hence, the household does not internalize the effect of their consumption and bond decisions on the equilibrium price, and therefore the effects of changes in the equilibrium price and production of nontradables on the ability to borrow are also not internalized. As a result, the model features a credit-market externality by which individual choices affect the economy's ability to borrow (see Uribe [2006].)

### 2.1.3 Equilibrium under REPI

In equilibrium, because we assume that the firm pays $\tau_S p_t \tilde{x}_t$ to the household for the right to exploit the commodity and $\tau_E e\left(s_t, \tilde{x}_t\right)$ for the extraction costs and because the market of non-tradable goods clear, the resource constraint of the economy becomes $c_t^T = y^T + \tau_S p_t \tilde{x}_t - b_{t+1} + R b_t + A^T$. To see this, replace $\pi_t \equiv p_t \tilde{x}_t - e\left(s_t, \tilde{x}_t\right)$ and $\tau_\pi \equiv \frac{\tau_S p_t \tilde{x}_t - \tau_E e(s_t, \tilde{x}_t)}{\pi_t}$ in (7) and set $c_t^N = y^N + A^N$.

Under the assumptions stated above, we characterize the REPI competitive equilibrium as follows. Given a history of realizations $p^t$, a recursive competitive equilibrium is a set of time-invariant policy functions $\tilde{x}\left(s,p\right)$, $b'\left(b,s,p\right)$, $c\left(b,s,p\right)$, $c^T\left(b,s,p\right)$, $c^N\left(b,s,p\right)$, $\lambda\left(b,s,p\right)$ and non-tradable prices $p^N\left(b,s,p\right)$ such that: $(i)$ the firm solves problem (1); $(ii)$ the household maximizes (3) subject to (5)-(8) conditional on $\mathbb{E}_t\left[p'|p\right]$; $(iii)$ markets of tradable and non-tradable goods clear.

The model's equilibrium conditions are:

$$\begin{aligned}
p_t^N &= \left(\frac{1-a}{a}\right)\left[\frac{c_t^T}{c_t^N}\right]^{1+\mu} \\
c_t^{-\sigma} &= \beta R \mathbb{E}_t^B\left[c_{t+1}^{-\sigma}\right]+\lambda_t \\
b_{t+1} &= y^T+\tau_S p_t \tilde{x}_t-c_t^T+R b_t+A^T \\
b_{t+1} &\geq -\phi\left(y^T+\tau_S p_t \tilde{x}_t+p_t^N y^N\right) \\
c_t^N &= y^N+p_t^N A^N .
\end{aligned}$$

Note that the representative agent takes as given the optimal policy of the extracting firm, $\tilde{x}\left(s_t, p_t\right)$. However, since $p_t$ varies randomly $\tilde{x}\left(s_t, p_t\right)$ does too. So the value of commodity production varies because of changes in prices and optimal adjustments to extraction. Changes in terms of trade stemming from international commodity price movements imply changes in GDP not only through valuation effects (the change in $p_t$ and the equilibrium response in $p_t^N$) but also because of the optimal response of extraction. A negative commodity price shock tightens the borrowing constraint through these channels.

Another point to note is that we can do some algebra to derive the balance of payments of the economy. Using (7), $\pi_t \equiv p_t \tilde{x}_t-e\left(s_t, \tilde{x}_t\right)$ and $\tau_\pi \equiv \frac{\tau_S p_t \tilde{x}_t-\tau_E e\left(s_t, \tilde{x}_t\right)}{\pi_t}$, we get the resource constraint of tradable goods:

$$c^T=y^T-b_{t+1}+b_t R+p_t x_t-(1-\tau_\pi) \pi_t-(1-\tau_E) e(s_t, x_t)$$

and using the definition of the trade balance

$$N X_t=y^T+p_t x_t-c^T$$

and $R=1+r$, the net interest rate, $r$, we get:

$$b_{t+1}-b_t-b_t r+(1-\tau_\pi) \pi_t+(1-\tau_E) e(s_t, x_t)=N X_t,$$

which is the balance of payments identity: the current account is $C A_t=b_{t+1}-b_t$, the net factor payments are $N F P_t=-b_t r+(1-\tau_\pi) \pi_t+(1-\tau_E) e(s_t, x_t)$, which correspond to interest payments (if $b_t<0$) and payments for profits obtained from extraction activities as well as the payments to foreign costs of extraction.

## 2.2 Bayesian Learning and Recursive Anticipated Utility Competitive Equilibrium

As in Boz and Mendoza [2014] we model learning following the approach proposed by Cogley and Sargent [2008] but applied to commodity prices. In our case when a sequence of high price realizations is observed, agents become optimistic about the duration of this regime one would expect agents to over-borrow relative to the case in which agents know the true probability of the Markov switching process. This Bayesian learning setup has the property that, if agents observe a long enough sample with sufficient regime switches, their beliefs converge to the true transition probabilities ($Q = q\left(p_{t+1} \mid p_t\right)$).

The solution strategy has two stages. First, we use Bayesian learning to generate the sequence of posterior density functions $\left\{f\left(Q^B \mid p^t\right)\right\}_{t=1}^{T}$ over $T$ periods in which a history of realizations of $p$ is observed. Each of these density functions is a probability distribution over possible transition matrices $Q^B$. The density function changes with the history of realizations observed up to date $t$, that is $p^t = (p_t, p_{t-1}, p_{t-2}, \ldots, p_1)$ with the date $t = 0$ priors explained below. Second, we solve the agent's optimal plans and the model's recursive equilibrium by Anticipated Utility (AU) approach to model dynamic optimization with Bayesian learning.

### 2.2.1 Bayesian Learning

At every date $t$, agents form posteriors using a Bayesian beta-binomial probability model and the information set up to $t$, $(p^t)$. The posteriors obtained from this procedure are imperfect estimates of the true probabilities $q_{hh}$ and $q_{ll}$ . The posteriors have distributions $q^B_{hh} \propto Beta\left(n^{hh}_t, n^{hl}_t\right)$ and $q^B_{ll} \propto Beta\left(n^{ll}_t, n^{lh}_t\right)$ where $n^{ij}_t$ $i, j = h, l$ are counters of the number of regime switches observed up to $t$.

To define the initial prior distribution we follow Cogley and Sargent [2008] who construct the prior distribution for $q^B_{hh}$ and $q^B_{ll}$ in period $t = 0$ as a function on the number of regime switches observed before period $t = 1$, $\left(n^{ij}_0\right)$.

From period $t + 1$ onwards the regime counters are updated according to the following rules

$$n^{hh}_{t+1} = \begin{cases} n^{hh}_t + 1 & \text{if } p_{t+1} = p_h \text{ and } p_t = p_h \\ n^{hh}_t & \text{otherwise} \end{cases}$$

$$n_{t+1}^{hl} = \begin{cases} n_t^{hl} + 1 & \text{if}\, p_{t+1} = p_h \,\text{and}\, p_t = p_l \\ n_t^{hl} & \text{otherwise} \end{cases}$$

$$n_{t+1}^{lh} = \begin{cases} n_t^{lh} + 1 & \text{if}\, p_{t+1} = p_l \,\text{and}\, p_t = p_h \\ n_t^{lh} & \text{otherwise} \end{cases}$$

$$n_{t+1}^{ll} = \begin{cases} n_t^{ll} + 1 & \text{if}\, p_{t+1} = p_l \,\text{and}\, p_t = p_l \\ n_t^{ll} & \text{otherwise} \end{cases}$$

Finally, from the counters the posterior mean are calculated as

$$\mathbb{E}_t\left[q_{hh}^B\right] = \frac{n_t^{hh}}{n_t^{hh} + n_t^{hl}} \tag{9}$$

and

$$\mathbb{E}_t\left[q_{ll}^B\right] = \frac{n_t^{ll}}{n_t^{ll} + n_t^{lh}}. \tag{10}$$

### 2.2.2 Recursive Anticipated Utility Competitive Equilibrium

We formulate the AU competitive equilibrium in recursive form. The state variables are $(s, b, p)$ and because of the law of iterated expectations still holds, the problem can be divided into a sequence of AU optimization problems (AUOP) for $t = 1, 2, \ldots, T$, each conditional on $\mathbb{E}_t\left[q_{hh}^B\right]$ and $\mathbb{E}_t\left[q_{ll}^B\right]$, where the time indexes identify the date of the beliefs that match the corresponding AUOP. Thus, solving this sequence of AUOPs means finding a sequence of equilibrium policy functions for commodity extraction and next-period's NFA, one for each set of beliefs at each date $t = 1, 2, \ldots, T$.

Consider the AUOP at date $t$. Agents observe $p_t$ and update their beliefs. Using (9) and (10) the transition matrix at date $t$ is:

$$\mathbb{E}_t^B\left[p'|p\right] = \begin{bmatrix} \mathbb{E}_t\left[q_{hh}^B\right] & 1 - \mathbb{E}_t\left[q_{hh}^B\right] \\ 1 - \mathbb{E}_t\left[q_{ll}^B\right] & \mathbb{E}_t\left[q_{ll}^B\right] \end{bmatrix}.$$

Now the optimal extraction policy of the commodity firm becomes a function of the beliefs about commodity prices.[2] Denote these policies as $\tilde{x}_t(s,p)$, one for each set of beliefs at each date $t = 1, 2, \ldots, T$, because they vary with time (as agents learn about the true process of commodity prices) so does the transition probability matrix $\mathbf{H}_t$, whose typical element in the position $(i,j)$ is the probability of jumping from state $i$ in the current year to state $j$ next period, conditioned on following the optimal policy $\tilde{x}_t(i)$ : $\mathbf{H}_{t,ij} = \Pr(s_{t+1} = j | s_t = i, x_t = \tilde{x}_t(i))$.

The solution to the AUOP at date $t$ is given by the optimal extraction policy $\tilde{x}_t(s,p)$, which solves

$$v_t(s,p) = \max_{x \in X} \left\{ [px - e(s,x)] + R^{-1} \mathbb{E}_t^B \left[ v_t(s - x + d, p') \right] \right\},$$

the policy functions $b'_t(b,s,p)$, $c_t(b,s,p)$, $c_t^T(b,s,p)$, $c_t^N(b,s,p)$, $\lambda_t(b,s,p)$ and a pricing function $p_t^N(b,s,p)$ that satisfy conditions (5)-(8) as well as the market clearing conditions for the tradable and non-tradable sectors rewritten in recursive form:

$$\begin{aligned} p_t^N(b,s,p) &= \left( \frac{1-a}{a} \right) \left[ \frac{c_t^T(b,s,p)}{c_t^N(b,s,p)} \right]^{1+\mu} & (11) \\ c_t(b,s,p)^{-\sigma} &= \beta R \mathbb{E}_t^B \left[ c_{t+1}(b,s,p)^{-\sigma} \right] + \lambda_t(b,s,p) & (12) \\ b'_t(b,s,p) &= y^T + \tau_S p_t \tilde{x}_t(s,p) - c_t^T(b,s,p) + Rb + A^T & (13) \\ b'_t(b,s,p) &\geq -\kappa \left( y^T + \tau_S p_t \tilde{x}_t(s,p) + p_t^N(b,s,p) y^N \right) & (14) \\ c_t^N(b,s,p) &= y^N + p_t^N(b,s,p) A^N. & (15) \end{aligned}$$

Note that the AUOP solution for period $t$ is conditional on the beliefs at date $t$ and takes a full set of optimal plans over the entire state space $(b,s,p)$. So, for period $t$, agents conjecture they would make optimal plans over the infinite future acting under the beliefs at period $t$.[3] As beliefs change as time passes and each subsequent $p_t$ is observed, therefore the policy and pricing functions that solve each AUOP also change, implying that history matters for the full solution of the model because different histories of commodity prices $p^t$ yield different sequences of beliefs, and hence different AUOP solutions. If at any two dates $t$ and $t+j$ we give the agents the same values for $(b,s,p)$, they in general will not choose the same bond holdings for the following period because $\mathbb{E}_t^B[p'|p]$ and $\mathbb{E}_{t+j}^B[p'|p]$ will differ.

---

[2] Unlike in the rational expectations perfect-information setup, where associated with the solution to program (1) there is a time-invariant optimal extraction policy, $\tilde{x}(s,p)$, in the Bayesian learning setup the controlled-state process of the firm's program with optimal extraction policy function $\tilde{x}$, is not a stationary Markov chain with a time invariant transition probability matrix $\mathbf{H}$.

[3] For example, the equilibrium decision rules for commodity reserves that the model predicts for $t = 1, 2, \ldots, T$ is obtained by chaining the relevant decision rules as follows: $s_2 = s'_1(s,p) =, s_3 = s'_2(s,p), \ldots, s_T = s'_{T-1}(s,p)$, where $s'$ denotes the next period's stock of reserves.

We characterize the recursive AU competitive equilibrium as follows. Given a history of realizations $p^t$, a recursive AU competitive equilibrium is a sequence of policy functions $\left\{\tilde{x}_t(s,p), b_t'(b,s,p), c_t(b,s,p), c_t^T(b,s,p), c_t^N(b,s,p), \mu_t(b,s,p), \lambda_t(b,s,p)\right\}_{t=1}^{T}$ and a sequence of non-tradable prices $\left\{p_t^N(b,s,p)\right\}_{t=1}^{T}$ such that: $(i)$ $\tilde{x}_t(s,p)$, $b_t'(b,s,p)$, $c_t(b,s,p)$, $c_t^T(b,s,p)$ $c_t^N(b,s,p)$, $\mu_t(b,s,p)$, $,\lambda_t(b,s,p)$ and $p_t^N(b,s,p)$ solve the firm and the representative agent AUOPs for date $t$, conditional on $\mathbb{E}_t^B[p'|p]$; $(ii)$ $\mathbb{E}_t^B[p'|p]$ is the transition probability matrix of $p$ produced by the posterior density of $Q^B$ at date $t$.

# 3 Quantitative Analysis

Our quantitative analysis is applied to the Colombian economy, which main natural resource is oil. Oil production in Colombia is significant and it is an important source of revenues for the government and the economy. In the last decade, oil production increased from 5% of GDP to 11% in 2014; the share of oil exports in GDP jumped from 3% in 2002 to 8% in 2014. In turn, fiscal revenues from oil (as a share from total public revenues) increased from under 10% in 2002 to close to 20% in 2011. Foreign direct investment in oil sector represented 32% (as a share from the total FDI in Colombia).

After four years of stable high nominal prices at slightly above \$100/bbl, oil prices fell sharply between the last quarter of 2014 and the first quarter of 2015. Compared to the previous episodes of oil price drops during the past three decades, the decline in oil prices was significant. Moreover, compared to the early 2011 commodity price peaks, the decline in oil prices was much larger than that in non-oil commodity price indices. This leads us to think that the price collapse of second half of 2014 is a nice candidate to test the quantitative properties of the model.

## 3.1 Baseline calibration

We start by characterizing the "true" oil price process. Our oil price model is a hidden Markov model $p_t = p(I_t) + \epsilon_t$ where $I_t$ is an indicator variable that records whether oil prices are high or low and $\epsilon_t$ is an identically and independently distributed normal random variable with mean 0 and variance $\sigma_\epsilon^2$. We apply Hamilton [1989] Markov switching estimator on quarterly real oil prices covering the period 1970:1 to 2014:2. As a proxy for real oil prices we take the BRENT crude oil price in nominal US dollars deflated by the United States Consumer Price Index. The base year for the US CPI is 1983. Table 1 show the estimates:

The low-price state is quite persistent, and the economy spends most of its time there. Price swings are large when comparing the high and the low states. Oil prices are on average

Table 1: Maximum likelihood estimates of real oil prices

| | $q_{hh}$ | $q_{ll}$ | $p_h$ | $p_l$ | $\sigma_\epsilon$ |
|---|---|---|---|---|---|
| Estimate | 0.9839 | 0.9830 | 56.9 | 22.5 | 5.5 |
| Standard error | - | - | 2.9 | 0.7 | 1.1 |

60% lower relative to the high price regime. Furthermore, because both the high-price state is very persistent, a run of expansions can occur with non-negligible probability, producing prolonged booms and busts of the oil sector. Furthermore, households will be reacting as if oil price changes were nearly permanent. Thus, even with 100 years of data, substantial model uncertainty endures. Agents in the model cope with uncertainty of this magnitude. The amount of uncertainty will be moderated by the size of the oil sector: the larger the sector the greater the impact of this type of uncertainty.

We simplify the oil price process by eliminating the innovation $\epsilon_t$ and assuming instead that oil prices follow a two-state process[4]:

$$p_t = \begin{cases} p_h & \text{if } I_t = 1 \\ p_l & \text{if } I_t = 0 \end{cases}$$

with the "true transitions" determined by $q_{hh}$ and $q_{ll}$. In the Bayesian learning setup, agents know the two values for oil price, $p_h$ and $p_l$, but do not know the transition probabilities $q_{hh}$ and $q_{ll}$. Instead, agents learn about them by applying Bayes' theorem. Agents adopt a beta-binomial probability model for learning about the oil price process. As Cogley and Sargent [2008] binomial likelihood is a natural representation for a two-state process such as this, and a beta density is the conjugate prior for a binomial likelihood.

There are some parameters that are standard in the literature while others deserve more explanation. We set the parameters that determines the degree of risk aversion $\sigma = 2$ (usually between 2 and 5) and the international interest rate at $R = 1.035$. We do not have an estimation for the elasticity of substitution between tradables and nontradables for Colombia so we take the value in Durdu et al. [2009] for Mexico, $\mu = 0.316$. We set $\beta = 0.99$ and $\phi$ to match as closely as possible both the level of NFA to GDP observed in the data (24.37% of GDP) and the fraction of the years that Colombia has been excluded from financial markets. Setting $\beta = 0.99$ and the borrowing limit at 50% of GDP ($\phi = 2$) we obtain a debt to GDP ratio of 22.63% and a frequency of international financial markets exclusion of 4.82% (vs. 16% in the data).

---

[4] We make this assumption to make the learning problem tractable. The noise term matters more in models in which agents cannot observe the state because agents must solve a signal extraction problem.

We work with units of tradables in the model and we follow the procedure described Durdu et al. [2009] and normalize the steady-state relative price of nontradables, the real price of oil and gross production in units of tradables, $p^N = 1$, $p = 1$ and $y^T + px + p^N y^N = 1$. The model is calibrated to match some ratios of the three-sector economy, using aggregate and sectoral data from Colombian national accounts. All the information is available from DANE (National Administrative Department of Statistics of Colombia).

The ratio of nontradable GDP to tradable GDP is $p^N y^N / y^T$, which yields an average of 1.74 for the 2000Q1-2014Q2 period. This ratio can be calculated using the GDP from the supply side, that decomposes gross production in ten main economic activities. The tradable sector excludes the "oil sector", the last encompassing the industry of oil, natural gas, and uranium and thorium minerals. Thereby, the tradable sector includes the following: manufacturing industries, mining sector (except the oil sector), agriculture, animal agriculture, forestry, and hunting. Some services can also be classified as tradable (as they have a large share of either exports or imports in gross production), such as air transportation, complementary services to transportation, and services to businesses different from financial and real estate services. The sectors that are classified as nontradable are personal, social, and community services, construction, electricity, water and gas, financial services, commercial services, terrestrial transportation, mailing, and telecommunications. We further assume that total taxes are proportionally distributed between the two sectors.

From $p^N y^N / y^T = 1.74$ and $y^T + px + p^N y^N = 1$, $y^T = 1/\left(1 + 1.74 + px/y^T\right)$. The ratio $px/y^T$ can also be retrieved from the data, yielding an average of 0.16 for the same sample period. Therefore, $y^T$ is 0.34, which implies $p^N y^N = \left(\left[1/y^T\right] - 1 - 0.16\right) y^T = 0.6$. The observed ratio of oil revenues to GDP is 5.6%, for a normalized value of total GDP and real oil price, we can solve for the implied long run value for oil extration. That is, $x = 0.056\frac{y}{p}$.

Table 2: Calibration for the three-sector model

| Notation | Variable | Value |
|---|---|---|
| $y^T + p^x x + p^{NT} y^{NT}$ | Output in units of non-oil tradables | 1.0000 |
| $p^{NT}$ | Relative price of nontradables | 1.0000 |
| $p$ | Relative price of oil | 1.0000 |
| $p^{NT} y^{NT} / y^T$ | Nontradable to tradable output ratio | 1.7440 |
| $p^x x / y^T$ | Oil to Tradable output ratio | 0.1624 |
| $x$ | Oil extraction | 0.0560 |
| $c^T / y^T$ | Consumption to output ratio in the tradable sector | 0.9185 |
| $p^{NT} c^{NT} / p^{NT} y^{NT}$ | Consumption to output ratio in the nontradable sector | 0.5416 |

The other two ratios that are calibrated are the shares of sectoral consumption in each sector's GDP. The 2000Q1-2014Q2 average of $c^T/y^T$ is 0.92, while the $p^N c^N/p^N y^N$ average for the same period is 0.54. To construct these numbers, we can use the annual matrices of utilization at current prices from DANE. These matrices divide consumption, gross capital formation, exports and government expenditures between 61 sectors, that can be classified between tradable, nontradable, and oil sector. We assume that exports belong completely to the tradable sector (except for oil), and thus we have $y^T = c^T + g^T + i^T + x - m^T$ and $y^N = c^N + g^N + i^N - m^N$. Departing from these macroeconomic identities, we can construct the ratios mentioned above. It is worth mentioning that there is no consumption of oil, so $px$ does not enter in $c^T$. From these numbers, and using the normalization, we get $c^T = 0.3160$ and $c^N = 0.3249$. Finally, we introduce constant levels of absorption $A^T$ and $A^N$ that capture investment and government expenditures in both sectors, and are compatible with the budget constraint of households. Thus, we have $A^T = y^T + p^x y^x + b(R-1) - c^T$ and $A^N = y^N - c^N$. The full calibration is summarized in Table 2.

We also need to specify the functional form of the extraction cost function $e$ and its parameters. The function we use to perform the quantitative experiments is $e(s,x) = \frac{\kappa}{2}\frac{x^2}{s}$, where $\kappa$ determines the total cost elasticity to changes in the rate of extraction. We calibrate the parameter that determines the cost sensitivity of oil firms to the extraction rate, $\kappa$. Given that we assume that the representative oil firm's discount factor is $\delta = R^{-1}$, we fix $\kappa = 3.7702$ to match the years of reserves to exhaustion $(s/x)$ of Colombia (6.3 years or 25 quarters) at a price of oil barrel of US\$100. We set the price at high levels because our experiment simulates a change in regime to low oil prices. At the high prices observed during 2010-2014 oil reserves hovered around 2 billion barrels and production reached one million barrels per day.

We solve the model starting with the firm's problem to find the optimal policy rules. We solve it by discrete dynamic programming using a discrete grid of 340 equidistant nodes for $s$ on the interval $[0,1]$. We find the optimal policy rule $\tilde{x}_t(s,p)$ as well as the optimal Markov transition matrix $\mathbf{H}_t$ associated with the problem at time $t$. Taking matrix $\mathbf{H}_t$ as given, we solve the problem of the rest of the economy by time iteration. We find the optimal policy rules $b'_t(b,s,p)$ and as well as the rest of the policy rules as described in the previous section.

Using the optimal transition matrix $\mathbf{H}_t$ for $t = 1,2,\ldots,J$ (the high-price regime), we simulate the economy's path overtime and obtain the statistics shown on Tables 3. The model matches quite closely the oil sector statistics. It matches not only the targeted statistic: years of reserves, but also matches the stock of reserves at 2.4 billion barrels and annual production at one million barrels per day.

Table 3: Ratios of the Small Open Economy Model vs the Data

| | **Data** | **Model** |
|---|---|---|
| | Colombia | Steady State |
| Tradable Output | 0.34 | 0.34 |
| Tradable Consumption (% of $y^T$) | 0.92 | 0.94 |
| Oil revenues (% of $y$) | 0.056 | 0.056 |
| Non Tradable Output (% of $y^T$) | 1.744 | 2.02 |
| Net Foreign Assets | -0.24 | -0.23 |
| | **Oil Sector** | |
| | **Data** | **Model** |
| Years of Reserves | 6.3 | 6.3 |
| Extraction (TBPD) | 1028 | 1054 |
| Oil Stock (billion bl.) | 2.38 | 2.44 |

Note: Oil extraction is expressed in thousand barrels per day, Oil Stock in billions of barrels.

## 3.2 Quantitative findings

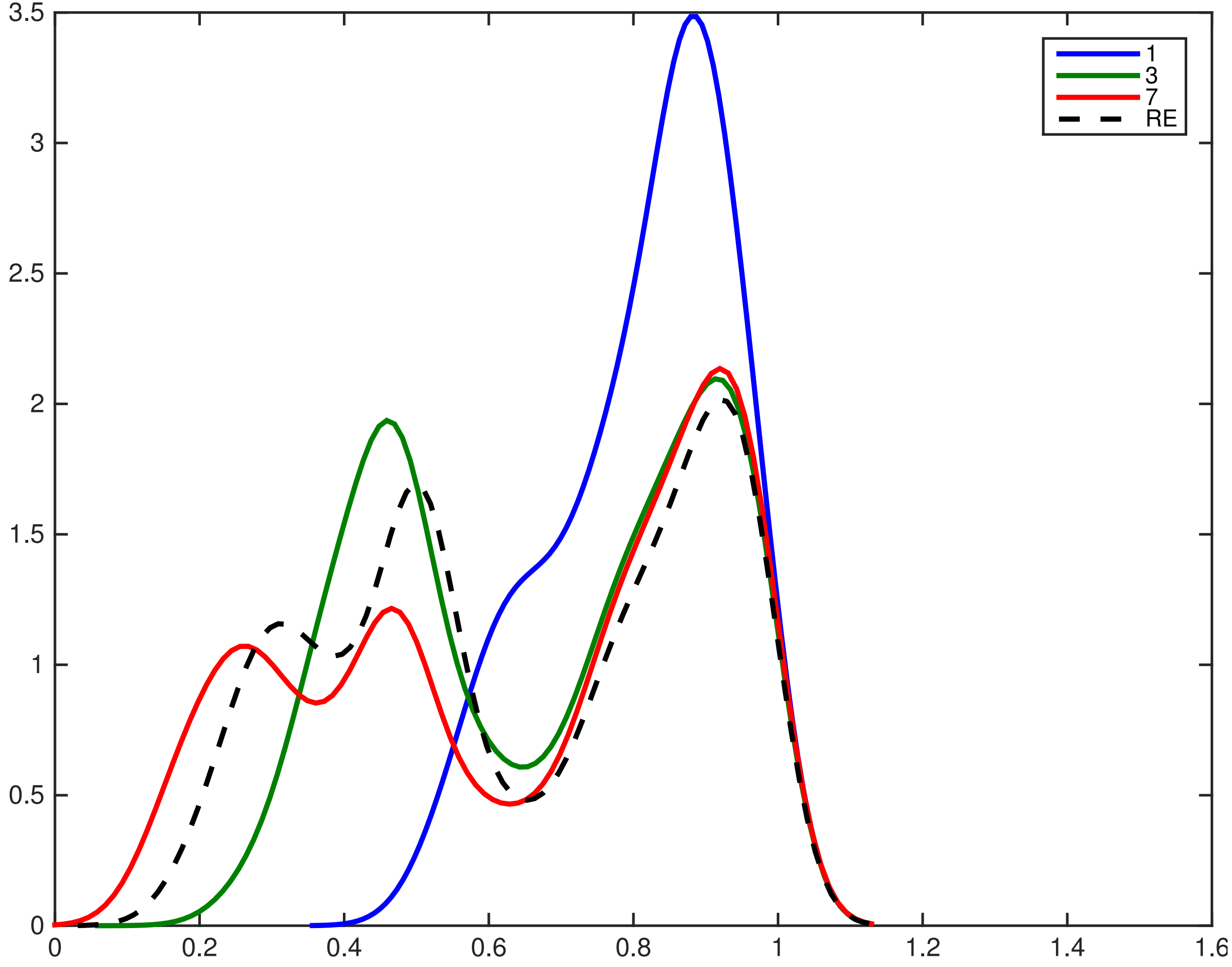
In this section we discuss three sets of numerical results: first, long-run distributions of net foreign asset positions (we report the level of external debt of the economy); second, the forecast functions of macroeconomic aggregates and the average changes in these aggregates at the moment prior to the collapse under alternative initial conditions of the economy and the information available to agents; and finally, we report a simulation of the economy during a long sequence of high oil prices and a posterior oil price collapse. In this last exercise, we compare the commodity extraction model with an endowment economy to illustrate the impact of the optimal extraction problem on a resource rich economy.

### 3.2.1 Ergodic distributions under a price collapse

To illustrate the dynamics of the model we simulate the following oil price sequence. In $t = 1$ the price is high and then for periods $t = 2, \ldots, 7$ a low price sequence is observed.

Figure 1 plots conjectured ergodic distributions of external debt as of $t = 1, 3$ and 7 in the BL model (that is, the agents' projections of what the long-run equilibrium would look like using their beliefs as of each date) and the "true" ergodic distribution of the REPI model (the black dashed line). Since in the BL agents learn $Q$ in the long-run, the actual ergodic distribution of the BL model should approach that of the RE model. These plots help to assess the impact of the optimism and pessimism driving the model's dynamics of the households willingness to borrow. Consider the conjectured distribution for $t = 1$ (the blue

Figure 1: Ergodic distributions of External Debt



line). Recall that the mean of external debt at the beginning of the high-oil price regime is close to 1. By period 1 agents are uninformed about the true distribution and have only seen a realization of high prices. They conjecture that the support of the long-run distribution of external debt will shift to the right (i.e., support higher debt levels). Comparing the initial distribution with the distributions for the high-price-regime, pre-oil price collapse with those of the low price regime ($t = 2$ to $t = 7$, only $t = 3$ and $t = 7$ are reported), note that the household cuts borrowing. In this case, in period $t = 1$ households are not assessing the risk of the oil price fully correctly, and in particular they are not aware that a long spell of low oil price regimes is possible. As low prices begin to realize, the subsequent distributions shift to the left. Compare now the RE ergodic distribution with the conjectured ergodic distribution for period 7 in the BL model. They are closer. This is because agents are learning as time passes with new realizations of oil prices.

### 3.2.2 Forecasting functions

We analyze the quantitative properties of the model by conducting impulse-response and transitional dynamics analysis. In both exercises, to calibrate the "true" oil price process, we take a long quarterly time series of oil price data from 1970:1 to 2014:2 and estimate a two-state Markov switching model. We assume this estimation is our "true" process and simulate the impact on the calibrated economy of a change in regime from high prices to low prices. We compare the response of the baseline calibrated model to a change in oil price regime from high to low prices under alternative assumptions regarding the learning mechanism.

In the impulse-response analysis we assume that there is a change in the oil price regime from high to low, starting from a level of debt and oil reserves, which correspond to their ergodic equilibrium expected value. In the transitional dynamics experiment, we assume once again that there is a change in the oil price regime from high to low, but we start from a level of debt and oil reserves, which correspond to the observed values for the Colombian economy in 2014:Q2. Our motivation for the second exercise is to assess the behavior of the calibrated economy during the high commodity price period and the subsequent unexpected change of regime in 2014.

Each quantitative experiment is performed under two extreme assumptions regarding the learning mechanisms: one in which agents have little previous knowledge of the true oil price process and another in which they have a lot of knowledge. This allows us to quantify the contribution of the informational friction to the model properties. More formally, the two assumptions are:

1. Agents do not have much information regarding the oil price process at the time of the oil price collapse. More formally, the prior distribution for $q_{hh}^B$ and $q_{ll}^B$ in period $t = 0$ as a function on the number of regime switches observed before period $t = 1$, $\left(n_0^{ij}\right)$ is set close to zero, $n_0^{ij} \approx 0$ for $i, j$.

2. Agents do have a lot of information about the true process and use the estimated Markov switching oil price regimes to learn about the oil price process. In this case, the prior distribution for $q_{hh}^B$ and $q_{ll}^B$ in period $t = 0$ as a function on the number of regime switches observed before period $t = 1$, $\left(n_0^{ij}\right)$ is set close very close to the true one. In other words, $q_{hh}^B = \hat{q}_{hh}$ and $q_{ll}^B = \hat{q}_{ll}$, where $\hat{q}_{hh}$ and $\hat{q}_{ll}$ are the estimated probabilities of the Markov switching process. See Table 1.

[TABLES AND GRAPHS TO BE REPORTED]

### 3.2.3 Price collapse after long period of high prices in an endowment economy

We conduct an experiment calibrated to Colombian data, in which we date the start of the high commodity price regime in the fourth quarter of 2009 and its end on the third quarter of 2014. The beginning of the oil price reversal is dated in the fourth quarter of 2014 and its end on the first quarter of 2016, the most recent quarter for which we have information. Hence, we assume that the economy experienced the high-oil price regime during five years, followed by a year and a half of low prices.

This exercise is motivated by the observation that after a realizations of high oil prices, agents tend to expect prices to remain high. Similarly, after low price realizations, agents tend to expect low prices. Figure 2 shows how future prices, a proxy of agents expectations at a given point in time for different horizons, are very similar to the spot price.

To understand the importance of taking into account the natural resource extraction incentives, we compare two cases. In the first one, we take the sequence of commodity revenues as exogenous fixing the extraction at a level that is consistent with the rational expectations perfect information (REPI) ergodic equilibrium of the model but allow commodity prices to fluctuate randomly. More specifically, the sequence of revenues is $\{p_t x_t\}_{t=1}^T$ with $\{x_t = d\}_{t=1}^T$. This case is analogous to a small open economy with a stochastic endowment of tradable goods. In the second case, commodity extraction is endogenous and responds to the random movements of commodity prices and we take the sequences $\{p_t\}_{t=1}^T$ and $\{\tilde{x}_t\left(p, s\right)\}_{t=1}^T$. In both cases we compare the REPI allocation with the Bayesian learning (BL) allocation.

Consider the first case, the one that is analogous to the small open economy stochastic endownment case. Figure 3 shows the results of the experiment by comparing the REPI path (red dashed line) with the Bayesian learning path (blue solid line). The results show

Figure 2: Spot price and future prices of oil at different horizons

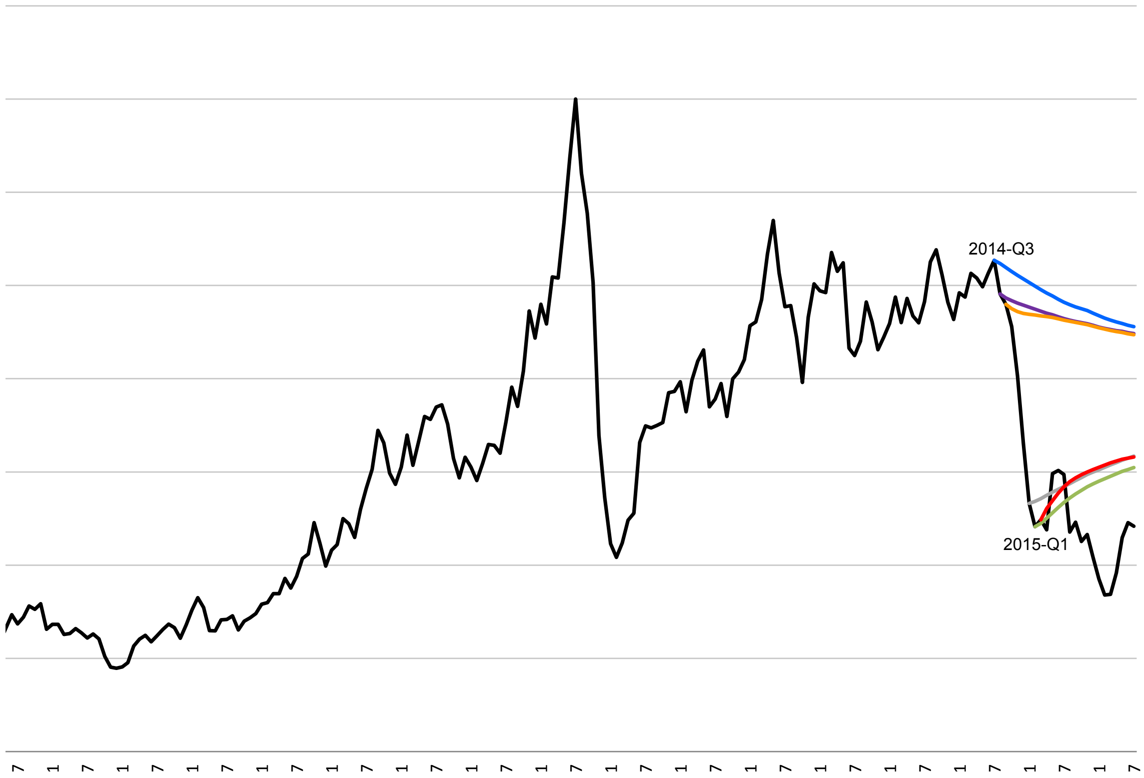
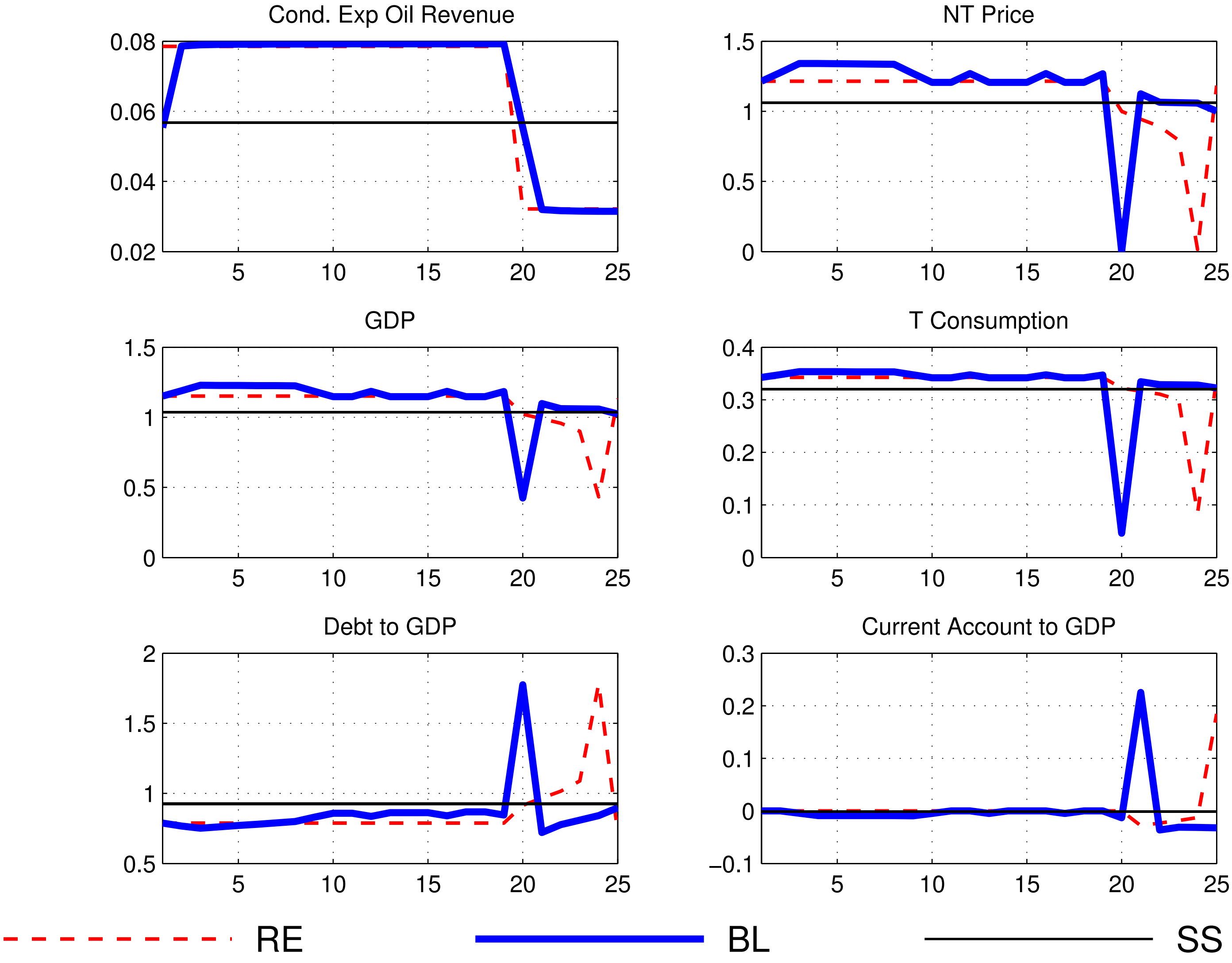


Figure 3: Conditional Forecast Functions: Exogenous Revenues Model



that during the high-revenue regime, the optimistic beliefs about revenues induces over-borrowing, over-consumption, a wider current account deficit and an RER overvaluation, with respect to to the REPI allocations.

By the end of the high-price regime, the level of indebtedness of the economy is so high that hits the borrowing limit. When the first realization of the low-price regime materializes, agents turn overly pessimistic and they would like to borrow to smooth consumption in light of the lower than expected revenues. Since the borrowing constraint is binding, they have to cut sharply consumption. As a result, the RER depreciates rapidly, and the current account deficit turns positive.

It is interesting to compare the timing and the magnitude of the adjustment in the REPI and the BL experiments. Note that the sudden stop, triggered by the commodity price reversion, occurs earlier in the BL case than in the REPI case. In the former, it happens as soon as the first realization of low prices materializes. In the REPI, it happens later. The reason is that when commodity prices suddenly collapse, the REPI economy can actually borrow to mitigate the fall in commodity revenues. Consumption falls, but by less than in the BL economy, which cannot borrow and therefore the fall in consumption is larger.

### 3.2.4 Price collapse after long period of high prices in an extraction economy

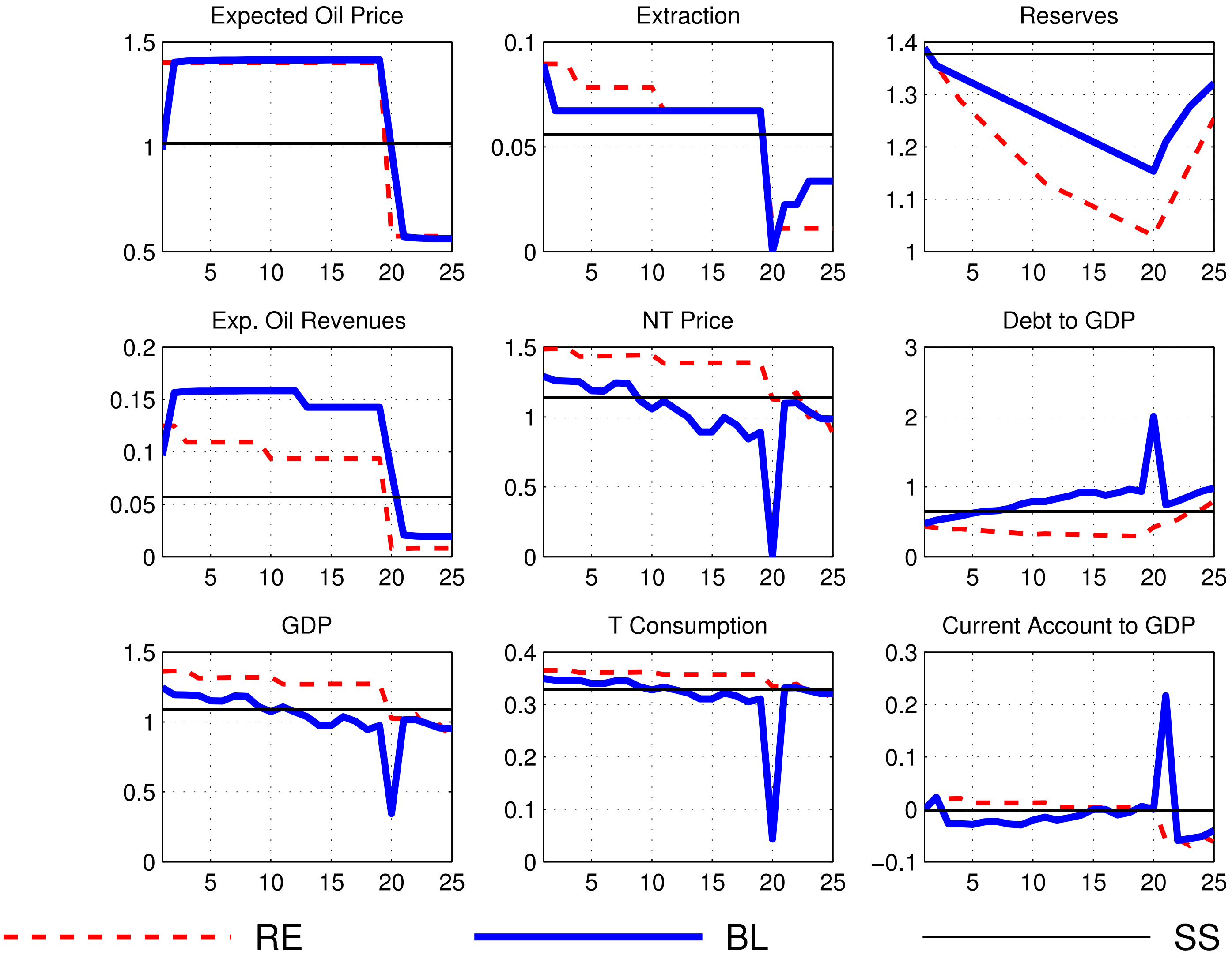
Now consider the second case. Figure 4 show the REPI and the BL allocations. In this case commodity extraction is endogenous and responds to price incentives, given the priors at time $t$. Recall that the optimal extraction problem of the firm has an inter-temporal dimension given by equation 2: given a commodity price level today, higher expected future prices imply lower extraction today relative to the future. This inter-temporal extraction allocation is present in both the REPI and the BL economies but the differences between the expectations of future prices imply different extraction rates.

Why is it that during the high price regime the extraction rate is lower in the BL economy than in the REPI economy? Note that both economies confront the same sequence of price realizations, $\{p_t\}_{t=1}^{T}$. In the REPI economy the expected price is known to all agents to be at $p = 1$, our calibrated level. Agents also know the true process and they know that it is a mean reverting process. Therefore, during the high price regime agents expect prices to fall and revert to the mean. With those expectations, the optimality condition for extraction implies a higher extraction rate in the present than in the future. As the firm extracts the stock of the commodity falls, because the discovery rate is fixed (by the model's assumption).

In the BL economy, however, the true expected value of the commodity price has to be discovered with time. We assume that agents in the economy assign a 50-50% chance to be high or low. Thus, in the first period the price expectation is also 1. With the first realization of the price, the firm extracts the same quantity as in the REPI economy. However, the high realization of the price leads to a reassessment of the expectations. That realization has made agents optimistic and the firm expects a higher price in the second period and thereafter. Note that after a few realizations of the price, agents have a quite firm believe that prices will be expected to remain at high levels. This high expectation of future prices in the BL leads the firm to optimally extract less than the REPI economy. Therefore, a more optimistic perception of future prices will imply a lower relative extraction in the present.

In $t = J + 1$, when the firm observes the first realization of the low-price regime, they

Figure 4: Conditional Forecast Functions: Endogenous Revenues Model



turn overly pessimistic about future prices. Therefore, the firm frontloads extraction. As more negative price realization materialize, which tend to confirm the now lower expected price, extraction is increased in the BL much more than in the REPI economy, which expect this low prices to revert to the mean.

The larger the importance of the commodity sector in the economy, the stronger will be the impact on households income, which receive the commodity revenues, $p_t x_t$. Therefore, the macroeconomic effect of optimism about future prices depends on the direct effect of prices (given a level of extraction) and the indirect effect that this optimism has on extraction incentives, which is negative. This key mechanism indeed changes the response of the economy. In an endowment economy, the impact of optimism on extraction works only through prices. In an extraction economy which exploits a resource optimally, optimism of a brighter future leads the economy to postpone extraction.

And it does in our calibrated experiment. In the calibrated economy the response of the representative agent during the high-price regime is to borrow to mitigate the lower commodity revenues. Inspite of the higher expected commodity price path, the expected lower commodity extraction induces the consumer to consume less in the BL economy than in the REPI allocation. As a result, during the high-price regime the RER appreciates less and the current account is less negative. Net foreign debt is increasing because the consumer borrows to mitigate the downward path of expected commodity revenues.

Note also that when agents observe the first realization of the low-oil price regime, they respond with a sharp correction in their beliefs and become very pessimistic, causing sharp downward adjustments in the current account, non-tradable prices and consumption. The transition to the low-oil price regime is exogenous, and thus part of the Sudden Stop in the model is because of exogenous forces. However, the equilibrium adjustment in the current account and prices in the model also reflect the endogenous amplification operating through the interaction of the external borrowing constraint and the agents' beliefs. This amplification mechanism is significant and accounts for most of the drop in the current account and prices predicted by the model.

It is also interesting to compare the timing and the magnitude of the adjustment in the REPI and the BL experiments when the extraction is endogenous. Note that the sudden stop, triggered by the commodity price reversion also occurs earlier in the BL case than in the REPI case. The reason is also that when commodity prices collapse, the REPI economy can borrow to mitigate the fall in commodity revenues. Consumption falls, but by less than in the BL economy, which cannot borrow because it has already hit the borrowing constraint and therefore the fall in consumption is larger.

To date, consumption level in Colombia has not fallen yet, partly because net external indebtedness has increased, but there has been a substantial consumption growth slowdown and a real depreciation of the peso against the U.S. dollar. It is important to note that in our baseline calibration, the economy hits the borrowing constraint. There is no evidence yet that this is the case in Colombia, however the model highlights that this event is a potential risk. We also acknowledge that several factors beyond the scope of this paper have played a role in the Colombian macroeconomic dynamics of recent years (e.g. global financial liquidity and low international real interest rates, weak external demand, domestic factors as climate shocks to inflation, etc.)

### 3.3 Sensitivity analysis

To be written

## 4 Conclusions

To be written

## References

Emine Boz. Can miracles lead to crises? the role of optimism in emerging markets crises. *Journal of Money, Credit and Banking*, 41(6):1189–1215, 09 2009. URL http://ideas.repec.org/a/mcb/jmoncb/v41y2009i6p1189-1215.html.

Emine Boz and Enrique G. Mendoza. Financial innovation, the discovery of risk, and the U.S. credit crisis. *Journal of Monetary Economics*, 62(C):1–22, 2014. URL https://ideas.repec.org/a/eee/moneco/v62y2014icp1-22.html.

Emine Boz, Christian Daude, and C. Bora Durdu. Emerging market business cycles: Learning about the trend. *Journal of Monetary Economics*, 58(6):616–631, 2011. URL https://ideas.repec.org/a/eee/moneco/v58y2011i6p616-631.html.

Timothy Cogley and Thomas J. Sargent. Anticipated Utility And Rational Expectations As Approximations Of Bayesian Decision Making. *International Economic Review*, 49(1):185–221, 02 2008. URL https://ideas.repec.org/a/ier/iecrev/v49y2008i1p185-221.html.

C. Durdu, E. Mendoza, and M. Terrones. Precautionary Demand for Foreign Assets in Sudden Stop Economies: An Assessment of the New Mercantilism. *Journal of Development Economics*, 89(2):194–209, 2009.

Jorge Fornero, Markus Kirchner, and Andres Yany. Terms of Trade Shocks and Investment in Commodity-Exporting Economies. In Rodrigo Caputo and Roberto Chang, editors, *Commodity Prices and Macroeconomic Policy*, volume 22 of *Central Banking, Analysis, and Economic Policies Book Series*, chapter 5, pages 135–193. Central Bank of Chile, 06 2015. URL `https://ideas.repec.org/h/chb/bcchsb/v22c05pp135-193.html`.

J. Hamilton. “What is an Oil Shock". *Journal of Econometrics*, 113(2):263–398, 2003.

James D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. 57:357–384, 1989.

Lutz Kilian. Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market. *American Economic Review*, 99(3):1053–69, June 2009.

Lutz Kilian, Alessandro Rebucci, and Nikola Spatafora. Oil shocks and external balances. *Journal of International Economics*, 77(2):181–194, 2009.

A. Korinek and E. Mendoza. From sudden stops to fisherian deflation: Quantitative theory and policy implications. *Annual Review of Economics*, 6, August 2014.

Enrique G. Mendoza. Real business cycles in a small open economy. 81:797–818, September 1991.

Enrique G. Mendoza. Lessons from the debt-deflation theory of sudden stops. *American Economic Review*, 96(2):411–416, May 2006. URL `http://ideas.repec.org/a/aea/aecrev/v96y2006i2p411-416.html`.

Enrique G. Mendoza. Sudden stops, financial crises, and leverage. *American Economic Review*, 100(5):1941–66, December 2010. URL `http://ideas.repec.org/a/aea/aecrev/v100y2010i5p1941-66.html`.

Enrique G. Mendoza and Martin Uribe. *Devaluation Risk and the Business Cycle Implications of Exchange Rate -Management*, chapter 53. Carnegie-Rochester Conference Series on Public Policy, 2001.

M. Hashem Pesaran. An econometric analysis of exploration and extraction of oil in the u.k. continental shelf. *The Economic Journal*, 100(401):367–390, 1990.

Robert S Pindyck. Uncertainty and Exhaustible Resource Markets. *Journal of Political Economy*, 88(6):1203–25, December 1980. URL https://ideas.repec.org/a/ucp/jpolec/v88y1980i6p1203-25.html.

Robert S Pindyck. The Optimal Production of an Exhaustible Resource When Price is Exogenous and Stochastic. *Scandinavian Journal of Economics*, 83(2):277–88, 1981. URL https://ideas.repec.org/a/bla/scandj/v83y1981i2p277-88.html.

Alessandro Rebucci and Nikola Spatafora. Oil prices and global imbalances. *IMF World Economic Outlook*, 4:71–96, 2006.

Stephen W Salant. Exhaustible Resources and Industrial Structure: A Nash-Cournot Approach to the World Oil Market. *Journal of Political Economy*, 84(5):1079–93, October 1976. URL https://ideas.repec.org/a/ucp/jpolec/v84y1976i5p1079-93.html.

Robin Sickles and Peter Hartley. A model of optimal dynamic oil extraction: Evidence from a large middle eastern field. *Journal of Productivity Analysis*, 15:59–71, 2001.

Martin Uribe. On overborrowing. *American Economic Review*, 96(2):417–421, May 2006. URL http://ideas.repec.org/a/aea/aecrev/v96y2006i2p417-421.html.

# Appendix 1: Data set

Oil Production: We took the monthly average of the daily crude oil production (in barrels) and averaged it for each quarter. This data is available from 1993Q1 to 2013Q2.

Oil Price: Quarterly prices are calculated from daily data by taking an unweighted average of the daily closing spot prices for BRENT. We took the seasonally adjusted series and deflate it by the United States CPI. This data is available from 1970Q1 to 2014Q2. The base year for the US CPI is 1983.

Consumption: We took disaggregated quarterly data of total private consumption from 2000Q1 to 2013Q2. In particular, this disaggregation divides consumption in non durable, durable and semi durable goods, and services. We then approximate tradable consumption as the sum between consumption in durable and semi-durable goods, and non tradable consumption as the sum between consumption non durable goods and services.

Gross fixed capital formation: We took disaggregated quarterly data of total gross fixed capital formation from 2000Q1 to 2013Q2. In particular, this disaggregation divides fixed capital formation by sector: agricultural, machinery, transportation, construction, civil project building and services. We then approximate tradable fixed capital formation as the sum of this among the following sectors: agricultural, machinery and transportation. We approximate non tradable fixed capital formation as the sum of this among the following sectors: construction, civil project building and services.

GDP: We build a measure of tradable and non tradable GDP using sectoral data. Specifically, tradable GDP is approximated using the sum between agriculture, silviculture, hunting and fishing, mining, manufacture, air transportation, supplementary transportation services, mail and communication services, financial services to firms (excluding real estate) and total taxes. Non tradable GDP is then computed as the difference between total and tradable GDP. We also compute a measure of tradable GDP excluding the mining sector. This data is available from 2000Q1 to 2013Q2.

Inflation: We build a measure of tradable and non tradable inflation based on the CPI of the same sectoral data as that of the GDP. These CPI measures (tradable and non tradable) are then seasonally adjusted using Census x12 and then turned to quarterly frequency by taking the value for the last month in the quarter. This CPI data is then used to compute quarterly inflation. These inflation measures are available from 1999Q2 to 2013Q2.