Risk and Liquidity in a System Context

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Pricing claims in a system context

Some assets (e.g. loans) are claims against other parties

Value of my claim against $A$ depends on value of $A$’s claims against $B, C,$ etc.

But $B$ or $C$ may have claim against me.

Balance sheet strength, spreads, asset prices fluctuate together

Equity value of financial system as a whole is value of “fundamental assets”
“While many believe that irresponsible borrowing is creating a bubble in housing, this is not necessarily true. At the end of 2004, U.S. households owned $17.2 trillion in housing assets, an increase of 18.1% (or $2.6 trillion) from the third quarter of 2003. Over the same five quarters, mortgage debt (including home equity lines) rose $1.1 trillion to $7.5 trillion. The result: a $1.5 trillion increase in net housing equity over the past 15 months.”

Value of fundamental assets is tide that lifts all boats

Housing $\Rightarrow$ mortgages $\Rightarrow$ CDOs $\Rightarrow$ claims against CDO holders . . .
Balance Sheet Approach

Financial system is a network of interlinked balance sheets

Everything is marked to market

Risk-neutrality in pricing

- no role for risk aversion, but spreads fluctuate due to fluctuations in fundamental asset price

- fluctuations in risk appetite arising from solvency constraints

Spreads can fall when debt rises (“reaching for yield”).

Spreads can rise when debt falls (financial crises).
Related literature

- Balance sheet propagation


Framework

- $n$ entities in financial system
- risky endowments realized at date $T$ with means $\{w_i\}$
- single fundamental asset, price $v$
- zero coupon debt of $i$ with face value $\bar{x}_i$ payable at $T$
- risk-free interest rate is zero
Balance Sheets

$x_i$ is market value of $i$’s debt

$a_i$ is market value of $i$’s assets

$e_i$ is market value of $i$’s equity

\[ a_i = e_i + x_i \]

If $i$ holds proportion $\pi_{ji}$ of $j$’s debt,

\[ a_i = w_i + vy_i + \sum_j \pi_{ji}x_j \]
Merton (1974)

\[ x_i = f_i(a_i, \theta) \]

Figure 1: Market value of total debt $x_i$ of investor $i$
Lemma 1. \textit{There exist functions} $\{f_i\}$ \textit{such that}

$$x_i = f_i (a_i, \theta)$$

(1)

\textit{where each} $f_i$ \textit{is non-decreasing in} $a_i$, \textit{and is bounded above by} $\bar{x}_i$ \textit{and}

$$\theta = (v, w, \bar{x})$$

Lemma 2. \textit{The market value of equity is non-decreasing in} $a_i$. \textit{That is, the function} $e_i$ \textit{defined as}

$$e_i \equiv a_i - \sum_j f_i (a_i, \theta)$$

(2)

\textit{is non-decreasing in} $a_i$. 
System

\[ x_1 = f_1 (a_1 (x), \theta) \]
\[ x_2 = f_2 (a_2 (x), \theta) \]
\[ \vdots \]
\[ x_n = f_n (a_n (x), \theta) \]

where \( x = (x_1, x_2, \cdots, x_n) \).

Solve for fixed point \( x \) in:

\[ x = F (x, \theta) \]
Iterative approach

\[
x^1 = F(0, \theta) \\
x^{t+1} = F(x^t, \theta)
\]

“Pessimistic” case

\[0 \preceq x^1 \preceq x^2 \preceq x^3 \preceq \cdots\]

“Optimistic” case

\[x^1 = F(\bar{x}, \theta)\]

\[\bar{x} \succeq x^1 \succeq x^2 \succeq x^3 \succeq \cdots\]

Are the limits the same?
Unique solution

Theorem 3. There is a unique profile of debt prices $x(\theta)$ that solves $x = F(x, \theta)$.

Theorem 4. $x(\theta)$ is increasing in $\theta$.

Result follows from

(i) Tarski’s fixed point theorem

(ii) fact that $\{f_i\}$ are contraction mappings
A complete lattice is partially ordered set \((X, \leq)\) such that each subset \(S \subseteq X\) has both a greatest lower bound \(\inf(S)\) and a least upper bound \(\sup(S)\) in the set \(X\).

In our context, complete lattice with the set \(X\) given by

\[ X \equiv [0, \bar{x}_1] \times [0, \bar{x}_2] \times \cdots \times [0, \bar{x}_n] \]

and ordering \(\leq\) given by the usual component-wise order.

**Lemma 5.** (Tarski’s Fixed Point Theorem) Let \((X, \leq)\) be a complete lattice and \(F\) be a non-decreasing function on \(X\). Then there are \(x^*\) and \(x_*\) such that \(F(x^*) = x^*\), \(F(x_*) = x_*\), and for any fixed point \(x\), we have \(x_* \leq x \leq x^*\).
Proof. Define the set $S$ as

$$S = \{x | x \leq F(x)\}$$  \hspace{1cm} (3)$$

and define $x^*$ as $x^* \equiv \sup S$. For any $x \in S$, $x \leq x^*$. Since $F$ is non-decreasing, $x \leq F(x) \leq F(x^*)$. Thus, $F(x^*)$ is also an upper bound for $S$. But $x^*$ is defined as the least upper bound of $S$. Thus

$$x^* \leq F(x^*)$$  \hspace{1cm} (4)$$

Applying $F$ to both sides of (4), we have $F(x^*) \leq F(F(x^*))$. But this implies that $F(x^*) \in S$, so that $F(x^*)$ is bounded by $x^*$. That is, $F(x^*) \leq x^*$. Taken together with (4), this means that $F(x^*) = x^*$. Any other fixed point of $F$ must belong to $S$, and so $x^*$ is the largest fixed point. The smallest fixed point $x_*$ is defined as $\inf \{x | x \geq F(x)\}$, and the argument is exactly analogous.
Argument for Uniqueness

Suppose there are distinct solutions $x$, $x'$. 

By Tarski, $x \leq x'$ and $x_i < x'_i$ for some $i$

Equity value of the system under $x$ is strictly lower than under $x'$. 

Equity value of the system is value of fundamental assets 

Contradiction.
**Solvency Constraints**

Value of all assets and liabilities determined by

\[ \theta = (v, w, \bar{x}) \]

Constraints on equity/debt ratio

\[ \frac{a_i - x_i}{x_i} \geq r^* \]

Spreads

\[ 1 - \frac{x_i}{\bar{x}_i} \]
\[ \bar{x} \]

\[ \theta', \theta \]

solvency region

\[ V, W \]
Restoring Solvency
Two Scenarios for Spreads
Sale $s_i$ to restore solvency

\[
\frac{w_i + v(y_i - s_i) + b_i - (x_i^0 - vs_i)}{x_i^0 - vs_i} \geq r^* 
\]  \hspace{1cm} (5)

\[
s_i = \min \left\{ y_i, \max \left\{ 0, \frac{(1 + r^*)x_i^0 - w_i - vy_i - b_i}{r^*v} \right\} \right\} \]  \hspace{1cm} (6)
Minimum Leverage Constraint

Constraints on equity/debt ratio

\[
\frac{a_i - x_i}{x_i} \leq r^{**}
\]
Scenarios for Spreads and Reaching for Yield

\[ \bar{x} \]

\[ \theta' \]

\[ \theta \quad \rightarrow \quad \theta'' \]

\[ V, W \]
Example of Housing

Property Price

\[ v' \]

\[ v \]

Supply of property from old

property stock held by young
Applications and Extensions

- Value at risk
- Financial stability
- Correlations in downturns
- “Risk appetite”
- Seniority structure
- Pricing of credit derivatives