Risk and Liquidity in a System Context

Hyun Song Shin

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Pricing claims in a system context

Some assets (e.g. loans) are claims against other parties

Value of my claim against A depends on value of A's claims against $B, C,$ etc.

But B or C may have claim against me.

Balance sheet strength, spreads, asset prices fluctuate together

Equity value of financial system as a whole is value of "fundamental assets"

Marking to market

> "While many believe that irresponsible borrowing is creating a bubble in housing, this is not necessarily true. At the end of 2004, U.S. households owned $17.2 trillion in housing assets, an increase of 18.1% (or $2.6 trillion) from the third quarter of 2003. Over the same five quarters, mortgage debt (including home equity lines) rose $1.1 trillion to $7.5 trillion. The result: a $1.5 trillion increase in net housing equity over the past 15 months."

Value of fundamental assets is tide that lifts all boats

Housing $\Rightarrow$ mortgages $\Rightarrow$ CDOs $\Rightarrow$ claims against CDO holders . . .

Balance Sheet Approach

Financial system is a network of interlinked balance sheets

Everything is marked to market

Risk-neutrality in pricing

- no role for risk aversion, but spreads fluctuate due to fluctuations in fundamental asset price
- fluctuations in *risk appetite* arising from solvency constraints

Spreads can fall when debt rises ("reaching for yield").

Spreads can rise when debt falls (financial crises).

Related literature

- Balance sheet propagation
  - Borrower balance sheet: Bernanke and Gertler (1989), Kiyotaki and Moore (1998, 2001),
  - Lender balance sheet: Bernanke and Blinder (1988), Van den Heuvel (2002)
- Liquidity and asset prices: Gennotte and Leland (1990), Geanakoplos (2003), Morris and Shin (2004), Brunnermeier and Pedersen (2005a, 2005b), Acharya and Pedersen (2005)
- Lattice theory applications: Topkis (1978), Milgrom and Roberts (1990, 1994), Eisenberg and Noe (2001)

Framework

- n entities in financial system
- risky endowments realized at date T with means $\{w_i\}$
- single fundamental asset, price v
- zero coupon debt of i with face value $\bar{x}_i$ payable at T
- risk-free interest rate is zero

Balance Sheets

x_i is market value of i's debt

a_i is market value of i's assets

e_i is market value of i's equity

$$a_i = e_i + x_i$$

If i holds proportion π_{ji} of j's debt,

$$a_i = w_i + vy_i + \sum_j \pi_{ji} x_j$$

Merton (1974)

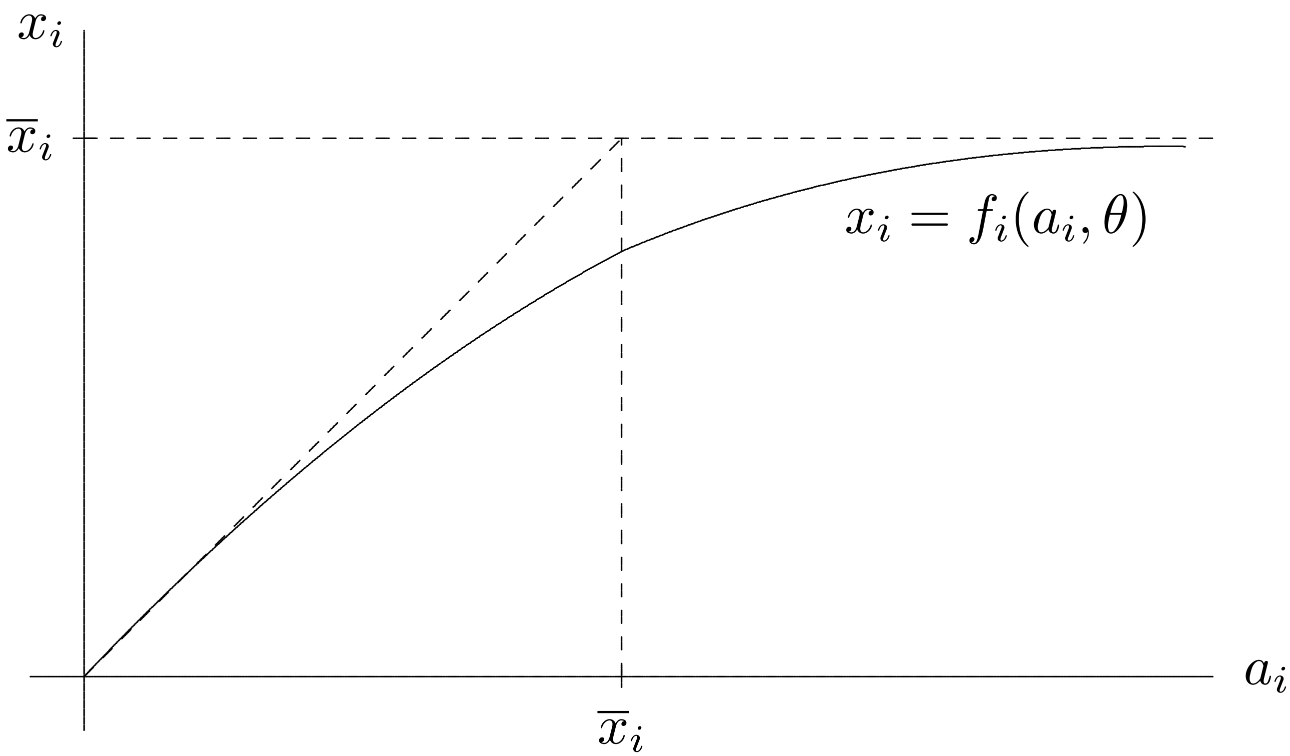


Figure 1: Market value of total debt x_i of investor i

Lemma 1. *There exist functions* $\{f_i\}$ *such that*

$$x_i = f_i(a_i, \theta) \tag{1}$$

where each f_i *is non-decreasing in* a_i*, and is bounded above by* $\bar{x}_i$ *and*

$$\theta = (v, w, \bar{x})$$

Lemma 2. *The market value of equity is non-decreasing in* a_i*. That is, the function* e_i *defined as*

$$e_i \equiv a_i - \sum_j f_i(a_i, \theta) \tag{2}$$

is non-decreasing in a_i*.*

System

$$
\begin{aligned}
x_1 &= f_1\left(a_1(x), \theta\right) \\
x_2 &= f_2\left(a_2(x), \theta\right) \\
&\vdots \\
x_n &= f_n\left(a_n(x), \theta\right)
\end{aligned}
$$

where $x = (x_1, x_2, \cdots, x_n)$.

Solve for fixed point x in:

$$x = F(x, \theta)$$

Iterative approach

$$
\begin{aligned}
x^1 &= F(0, \theta) \\
x^{t+1} &= F\left(x^t, \theta\right)
\end{aligned}
$$

"Pessimistic" case

$$0 \lneqq x^1 \lneqq x^2 \lneqq x^3 \lneqq \cdots$$

"Optimistic" case

$$x^1 = F(\bar{x}, \theta)$$

$$\bar{x} \gneqq x^1 \gneqq x^2 \gneqq x^3 \gneqq \cdots$$

Are the limits the same?

Unique solution

Theorem 3. *There is a unique profile of debt prices* $x(\theta)$ *that solves* $x = F(x, \theta)$.

Theorem 4. $x(\theta)$ *is increasing in* θ.

Result follows from

(i) Tarski's fixed point theorem

(ii) fact that $\{f_i\}$ are contraction mappings

A *complete lattice* is partially ordered set $(X, \leq)$ such that each subset $S \subseteq X$ has both a greatest lower bound $\inf(S)$ and a least upper bound $\sup(S)$ in the set X.

In our context, complete lattice with the set X given by

$$X \equiv [0, \bar{x}_1] \times [0, \bar{x}_2] \times \cdots \times [0, \bar{x}_n]$$

and ordering $\leq$ given by the usual component-wise order.

Lemma 5. *(Tarski's Fixed Point Theorem) Let $(X, \leq)$ be a complete lattice and F be a non-decreasing function on X. Then there are x^* and x_* such that $F(x^*) = x^*$, $F(x_*) = x_*$, and for any fixed point x, we have $x_* \leq x \leq x^*$.*

Proof. Define the set S as

$$S = \{x | x \leq F(x)\} \tag{3}$$

and define x^* as $x^* \equiv \sup S$. For any $x \in S$, $x \leq x^*$. Since F is non-decreasing, $x \leq F(x) \leq F(x^*)$. Thus, $F(x^*)$ is also an upper bound for S. But x^* is defined as the *least* upper bound of S. Thus

$$x^* \leq F(x^*) \tag{4}$$

Applying F to both sides of (4), we have $F(x^*) \leq F(F(x^*))$. But this implies that $F(x^*) \in S$, so that $F(x^*)$ is bounded by x^*. That is, $F(x^*) \leq x^*$. Taken together with (4), this means that $F(x^*) = x^*$. Any other fixed point of F must belong to S, and so x^* is the largest fixed point. The smallest fixed point x_* is defined as $\inf\{x | x \geq F(x)\}$, and the argument is exactly analogous.

Argument for Uniqueness

Suppose there are distinct solutions x, x'.

By Tarski, $x \leq x'$ and $x_i < x'_i$ for some i

Equity value of the system under x is strictly lower than under x'

Equity value of the system is value of fundamental assets

Contradiction.

Solvency Constraints

Value of all assets and liabilities determined by

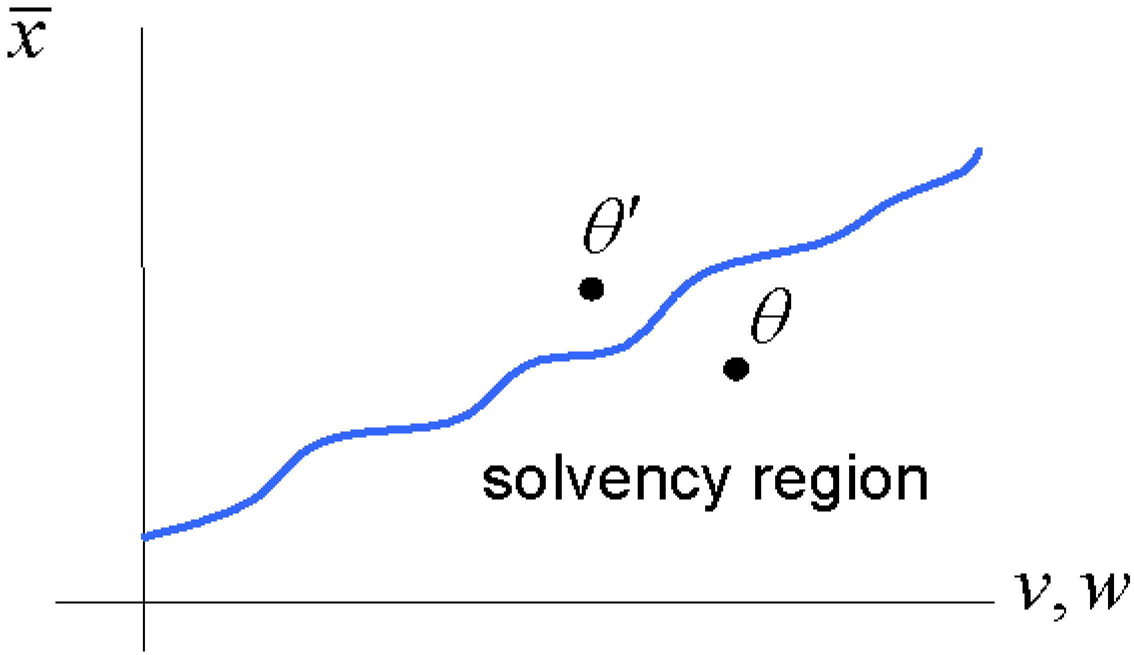
$$\theta = (v, w, \bar{x})$$

Constraints on equity/debt ratio

$$\frac{a_i - x_i}{x_i} \geq r^*$$

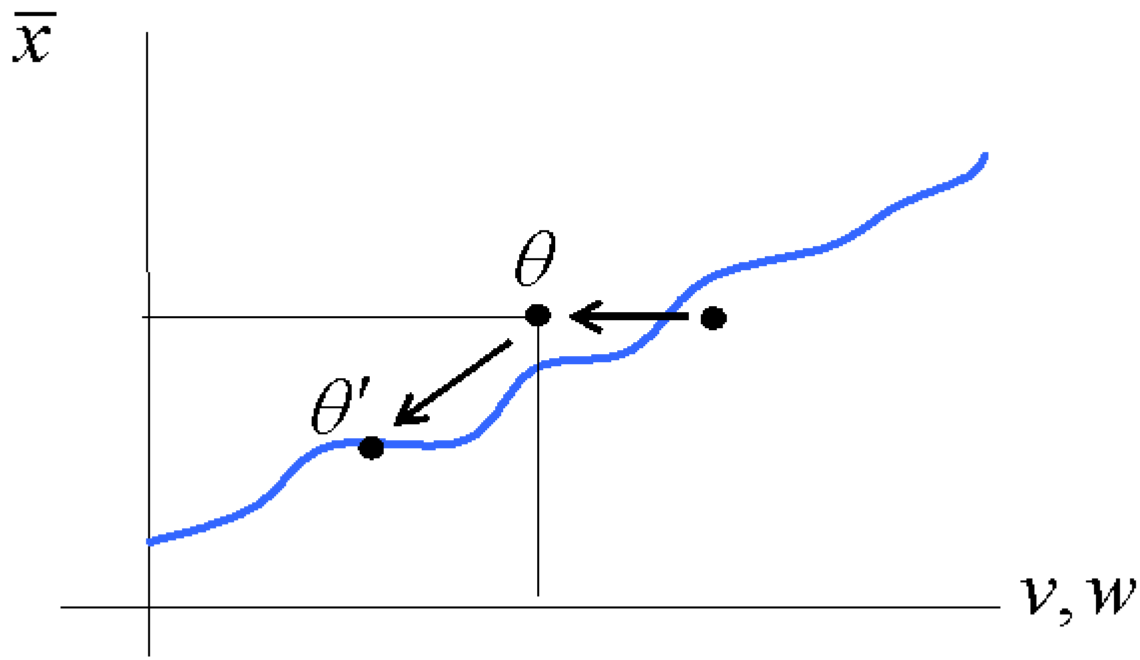
Spreads

$$1 - \frac{x_i}{\bar{x}_i}$$

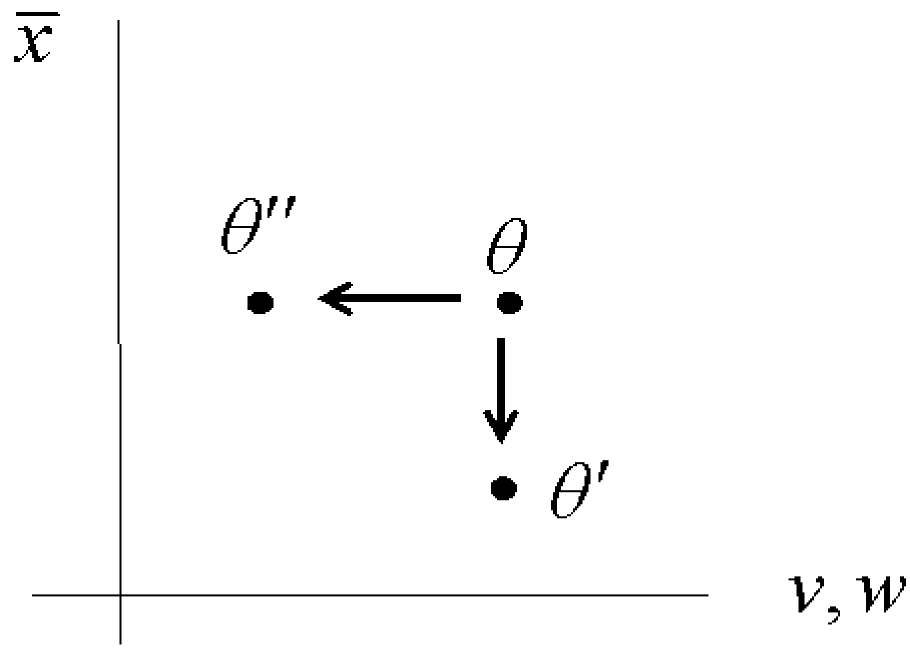


$\overline{x}$
θ'
θ
solvency region
v, w

Restoring Solvency



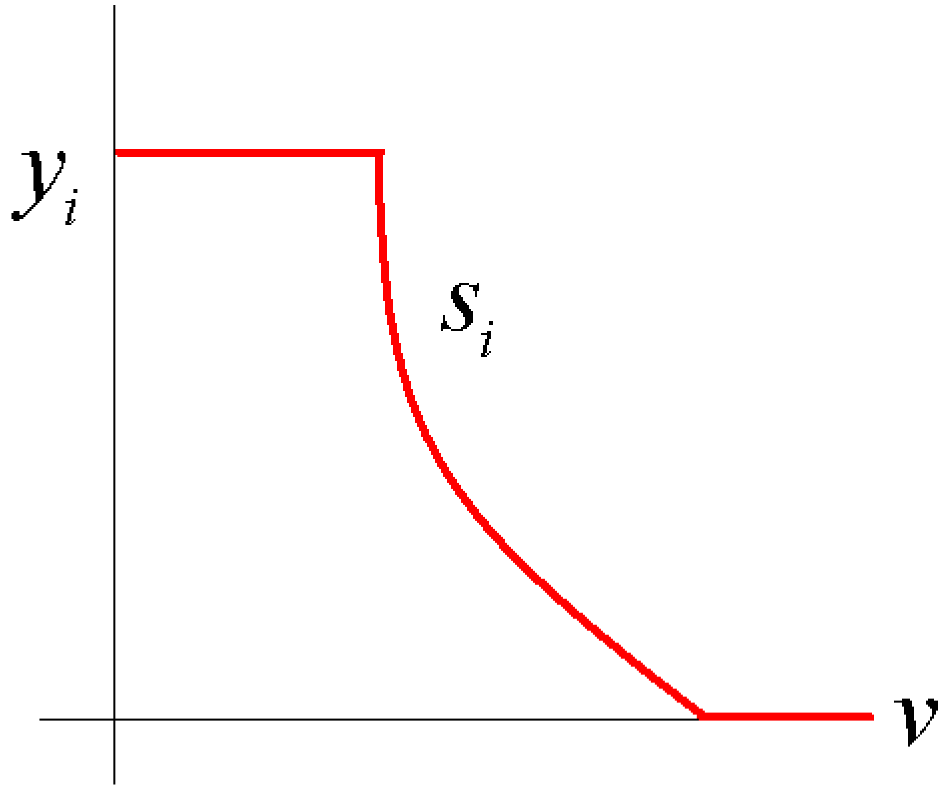
Two Scenarios for Spreads



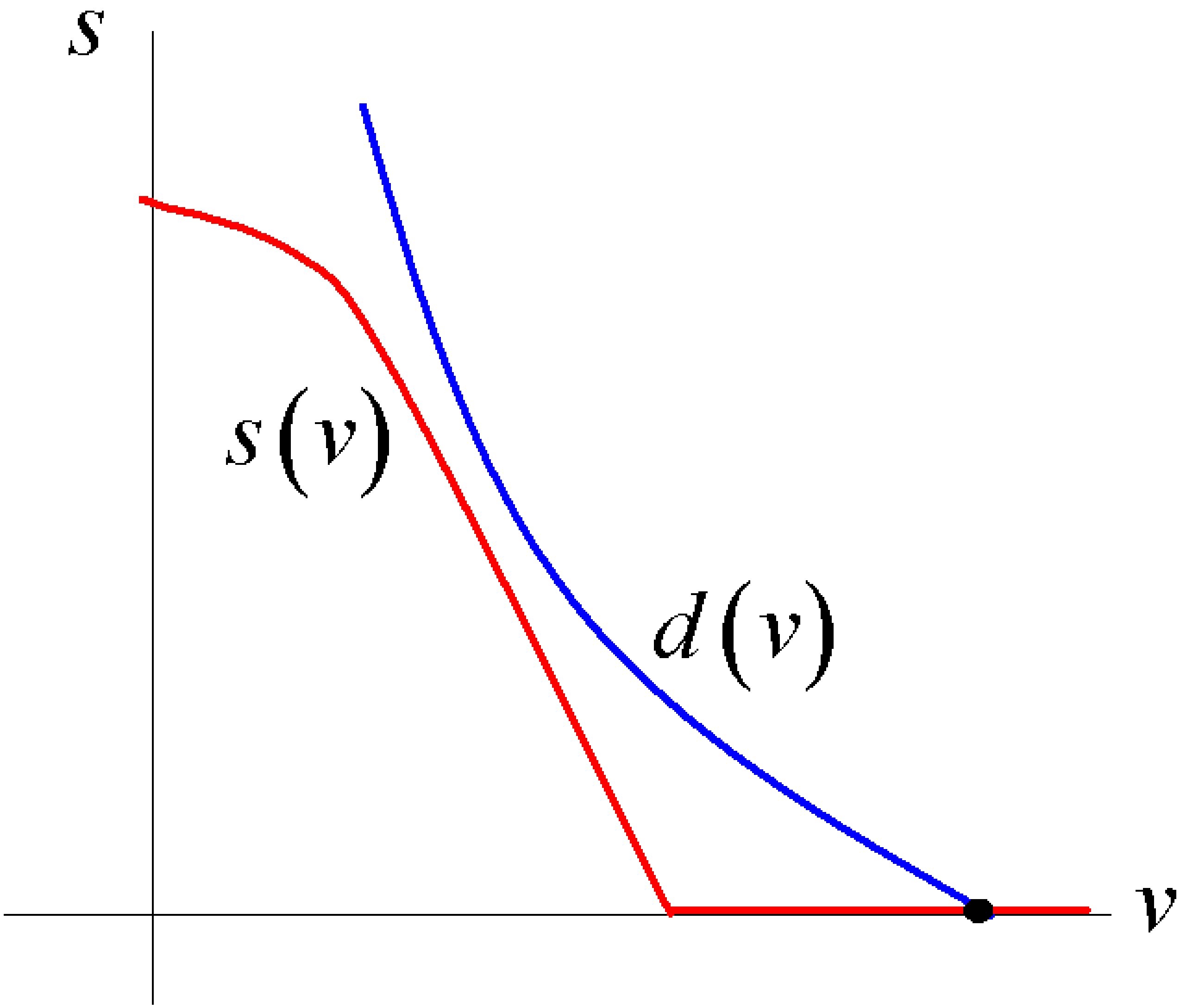
Sale s_i to restore solvency

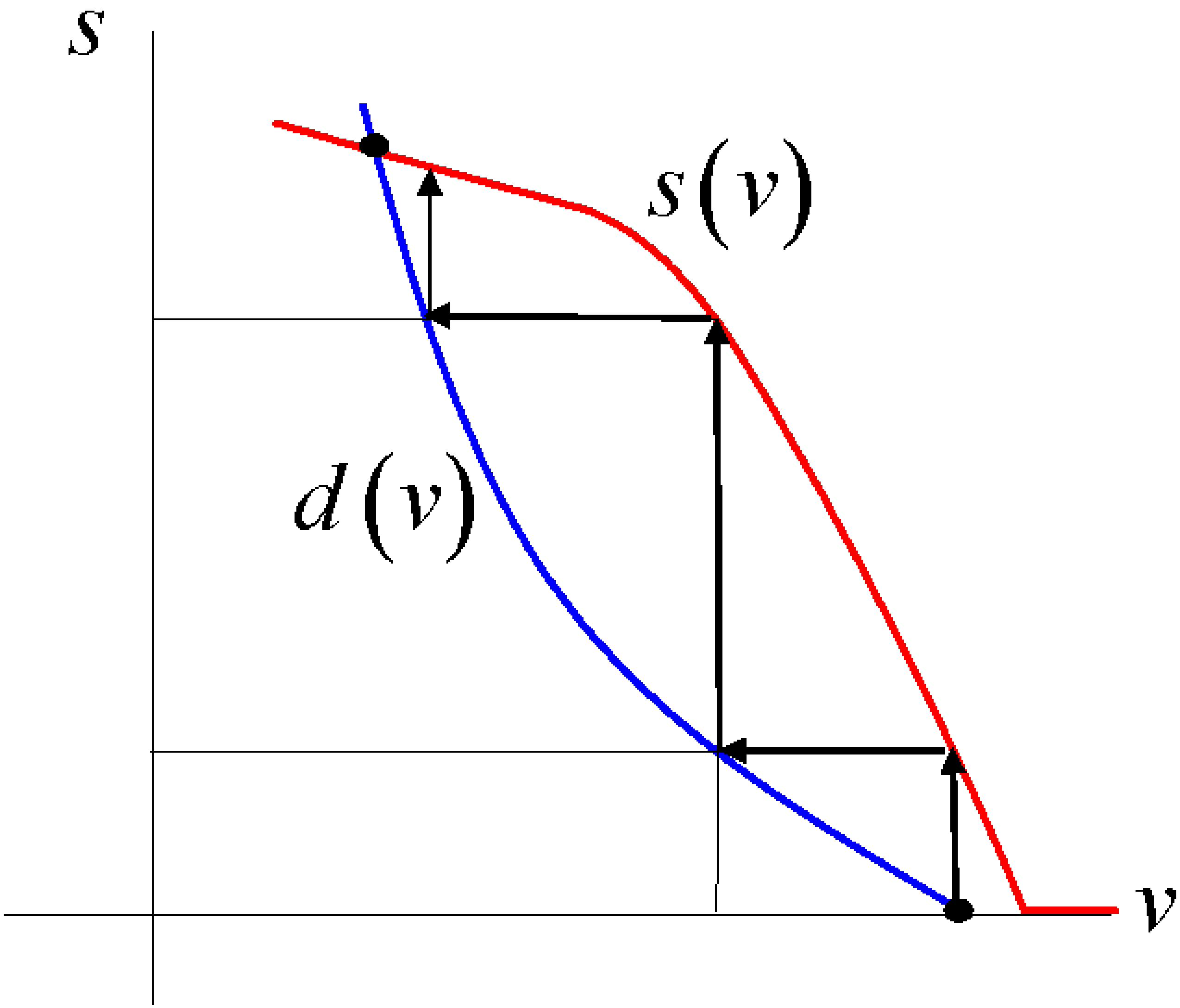
$$\frac{w_i + v\left(y_i - s_i\right) + b_i - \left(x_i^0 - vs_i\right)}{x_i^0 - vs_i} \geq r^* \tag{5}$$

$$s_i = \min\left\{y_i, \max\left\{0, \frac{\left(1 + r^*\right) x_i^0 - w_i - vy_i - b_i}{r^* v}\right\}\right\} \tag{6}$$



y_i
s_i
v

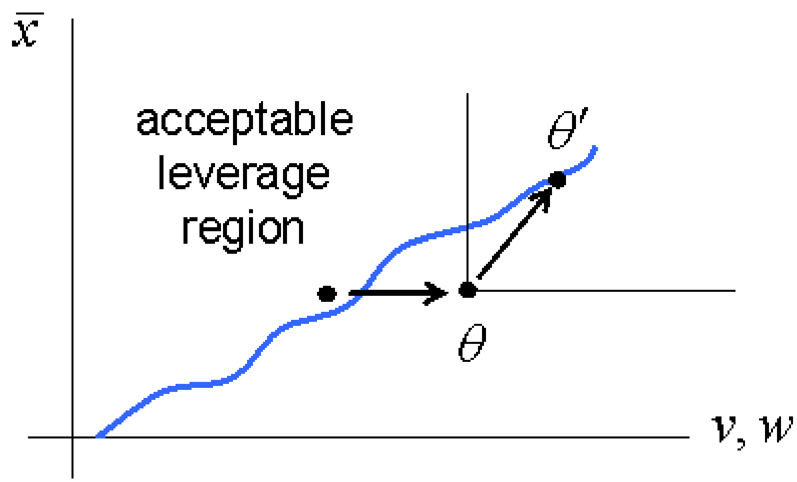

s
s(v)
d(v)
v


s
s(v)
d(v)
v

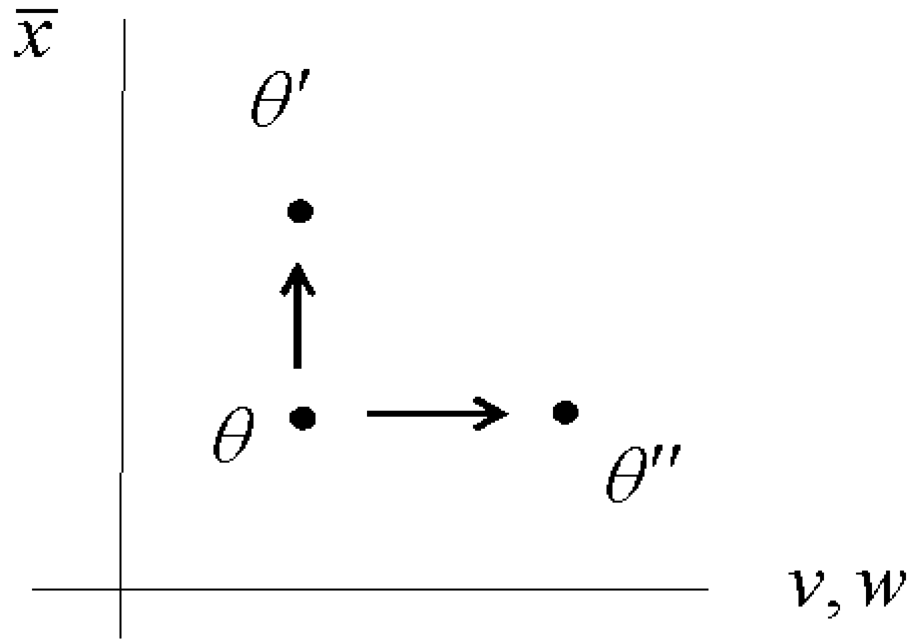
Minimum Leverage Constraint

Constraints on equity/debt ratio

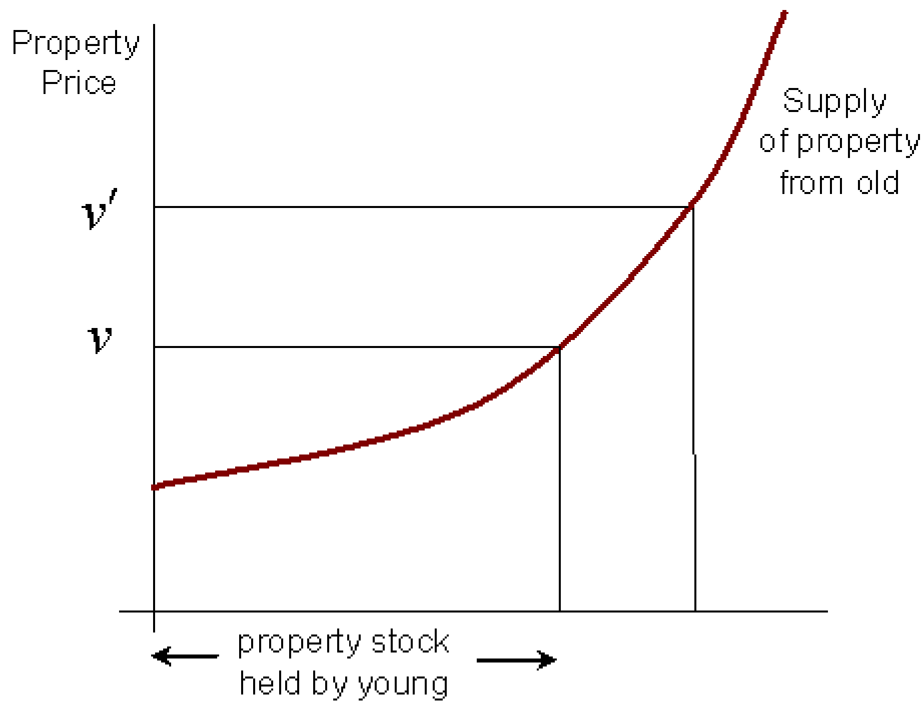
$$\frac{a_i - x_i}{x_i} \leq r^{**}$$



Scenarios for Spreads and Reaching for Yield



Example of Housing



Applications and Extensions

- Value at risk
- Financial stability
- Correlations in downturns
- "Risk appetite"
- Seniority structure
- Pricing of credit derivatives