Comparative Analyses of Expected Shortfall and Value-at-Risk under Market Stress[*]

Yasuhiro Yamai
Bank of Japan
Toshinao Yoshiba
Bank of Japan

ABSTRACT

In this paper, we compare Value-at-Risk (VaR) and expected shortfall under market stress. Assuming that the multivariate extreme value distribution represents asset returns under market stress, we simulate asset returns with this distribution. With these simulated asset returns, we examine whether market stress affects the properties of VaR and expected shortfall.

Our findings are as follows. First, VaR and expected shortfall may underestimate the risk of securities with fat-tailed properties and a high potential for large losses. Second, VaR and expected shortfall may both disregard the tail dependence of asset returns. Third, expected shortfall has less of a problem in disregarding the fat tails and the tail dependence than VaR does.

Key Words: Value-at-Risk, Expected shortfall, Tail risk, Market stress, Multivariate extreme value theory, Tail dependence

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[*] The views expressed here are those of the authors and do not reflect those of the Bank of Japan. (E-mail: yasuhiro.yamai@boj.or.jp; toshinao.yoshiba@boj.or.jp)

I. Introduction

It is a well-known fact that Value-at-Risk[1] (VaR) models do not work under market stress. VaR models are usually based on normal asset returns and do not work under extreme price fluctuations. The case in point is the financial market crisis of the fall of 1998. Concerning this crisis, the BIS Committee on the Global Financial System [1999] notes that "a large majority of interviewees admitted that last autumn's events were in the 'tails' of distributions and that VaR models were useless for measuring and monitoring market risk." Our question is this: Is this a problem of the estimation methods, or of VaR as a risk measure?

The estimation methods used for standard VaR models have problems for measuring extreme price movements. They assume that the asset returns follow a normal distribution. So they disregard the fat-tailed properties of actual returns, and underestimate the likelihood of extreme price movements.

On the other hand, the concept of VaR as a risk measure has problems for measuring extreme price movements. By definition, VaR only measures the distribution quantile, and disregards extreme loss beyond the VaR level. Thus, VaR may ignore important information regarding the tails of the underlying distributions. The BIS Committee on the Global Financial System [2000] identifies this problem as *tail risk*.

To alleviate the problems inherent in VaR, Artzner *et al.* [1997, 1999] propose the use of expected shortfall. Expected shortfall is the conditional expectation of loss given that the loss is beyond the VaR level.[2] Thus, by definition, expected shortfall considers loss beyond the VaR level. Yamai and Yoshiba [2002c] show that expected shortfall has no tail risk under more lenient

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[1] VaR at the 100(1-α)% confidence level is the upper 100α percentile of the loss distribution. We denote the VaR at the 100(1-α)% confidence level as $VaR_\alpha(Z)$, where Z is the random variable of loss.

[2] When the distributions of loss Z are continuous, expected shortfall at the 100(1-α)% confidence level ($ES_\alpha(Z)$) is defined by the following equation.

$$ES_\alpha(Z) = E[Z \mid Z \geq VaR_\alpha(Z)].$$

When the underlying distributions are discontinuous, see Definition 2 of Acerbi and Tasche [2001].

conditions than VaR.

The existing research implies that the tail risk of VaR and expected shortfall may be more significant under market stress than under normal market conditions. The loss under market stress is larger and less frequent than that under normal conditions. According to Yamai and Yoshiba [2002a], the tail risk is significant when asset losses are infrequent and large.[3]

In this paper, we examine whether the tail risk of VaR and expected shortfall is actually significant under market stress. We assume that the multivariate extreme value distributions represent the asset returns under market stress. With this assumption, we simulate asset returns with those distributions, and compare VaR and expected shortfall.[4,5]

Our assumption of the multivariate extreme value distributions is based on the theoretical results of extreme value theory. This theory states that the multivariate exceedances over a high threshold asymptotically follow the multivariate extreme value distributions. As extremely large fluctuations characterize asset returns under market stress, we assume that the asset returns under market stress follow the multivariate extreme value distributions.

Following this Introduction, Chapter 2 introduces the concepts and definitions of the tail risk of VaR and expected shortfall based on Yamai and Yoshiba [2002a,2002c]. Chapter 3 provides a general introduction to multivariate

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[3] Jorion [2000] makes the following comment in analyzing the failure of Long-Term Capital Management (LTCM): "The payoff patterns of the investment strategy [of LTCM] were akin to short positions in options. Even if it had measured its risk correctly, the firm failed to manage its risk properly."

[4] Prior comparative analyses of VaR and expected shortfall focus on their sub-additivity. For example, Artzner *et al*. [1997, 1999] show that expected shortfall is sub-additive, while VaR is not. Acerbi, Nordio, and Sirtori [2001] prove that expected shortfall is sub-additive, including the cases where the underlying profit/loss distributions are discontinuous. Rockafeller and Uryasev [2000] utilize the sub-additivity of the expected shortfall to find an efficient algorithm for optimizing expected shortfall.

[5] The other important aspect of the comparative analyses of VaR and expected shortfall is their estimation errors. Yamai and Yoshiba [2002b] show that expected shortfall needs a larger size sample than VaR for the same level of accuracy.

extreme value theory. Chapter 4 adopts univariate extreme value distributions to examine how the fat-tailed properties of these distributions result in the problems of VaR and expected shortfall. Chapter 5 adopts simulations with multivariate extreme value distributions[6] to examine how tail dependence results in the tail risk of VaR and expected shortfall. Chapter 6 presents empirical analyses to examine whether past financial crisis have resulted in the tail risk of VaR and expected shortfall. Finally, Chapter 7 presents the conclusions and implications of this paper.

II. Tail Risk of VaR and Expected Shortfall

A. The Definition and Concept of the Tail Risk of VaR

In this paper, we say that *VaR has tail risk* when VaR fails to summarize the relative choice between portfolios as a result of its underestimation of the risk of portfolios with fat-tailed properties and a high potential for large losses.[7,8] The tail risk of VaR emerges since it measures only a single quantile of the profit/loss distributions and disregards any loss beyond the VaR level. This may lead one to think that securities with a higher potential for large losses are less risky than securities with a lower potential for large losses.

For example, suppose that the VaR at the 99% confidence level of portfolio A is 10 million and that of portfolio B is 15 million. Given these numbers, one may conclude that portfolio B is more risky than portfolio A. However, the investor does not know how much may be lost outside of the confidence interval. When the

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[6] For other financial applications of multivariate extreme value theory, see Longin and Solnik [2001], Embrechts, de Haan and Huang [2000], and Hartmann, Straetmans and de Vries [2000].

[7] We only consider whether VaR and expected shortfall are effective for the relative choice of portfolios. We do not consider the issue of the absolute level of risk, such as whether VaR is appropriate as a benchmark of risk capital.

[8] For details regarding the general concept and definition of the tail risk of risk measures, see Yamai and Yoshiba [2002c].

maximum loss of portfolio A is 1 trillion and that of B is 16 million, portfolio A should be considered more risky since it loses much more than portfolio B under the worst case. In this case, VaR has tail risk since VaR fails to summarize the choice between portfolios A and B as a result of its disregard of the tail of profit/loss distributions.

We further illustrate the concept of the tail risk of VaR with two examples.

(Example 1) Option Portfolio (Danielsson [2001])

Danielsson [2001] shows that VaR is conducive to manipulation since it measures only a single quantile. We introduce his illustration as a typical example of the tail risk of VaR.

The solid line in Figure 1 depicts the distribution function of the profit/loss of a given security. The VaR of this security is VaR_0 as it is the lower quantile of the profit/loss distribution.

One is able to decrease this VaR to an arbitrary level by selling and buying options of this security. Suppose the desired VaR level is VaR_D. One way to achieve this is to write a put with a strike price right below VaR_0 and buy a put with a strike price just above VaR_D. The dotted line in Figure 1 depicts the distribution function of the profit/loss after buying and selling the options. The VaR is decreased from VaR_0 to VaR_D. This trading strategy increases the potential for large loss. The right end of Figure 1 shows that the probability of large loss is increased.

This example shows that the tail risk of VaR can be significant with simple option trading. One is able to manipulate VaR by buying and selling options. As a result of this manipulation, the potential for large loss is increased. VaR fails to consider this perverse effect since it disregards any loss beyond the confidence level.

(Example 2) Credit Portfolio (Lucas *et al.* [2001])

The next example demonstrates the tail risk of VaR in a credit portfolio, using the result of Lucas *et al.* [2001].

Lucas *et al.* [2001] derive an analytic approximation to the credit loss distribution of large portfolios. To illustrate their general result, Lucas *et al.* [2001] provide a simple example of credit loss calculation.[9] They consider a bond portfolio where the amount of credit exposure for individual bonds is identical and the default is triggered by a single factor. For simplicity, they assume that the loss is recognized in the default mode and that the factor sensitivities of the latent variables and default probabilities are homogeneous.[10] They show that the credit loss of the bond portfolio converges almost surely to *C*, as defined in the following equation, when the number of bonds approaches infinity (Lucas *et al.* [2001], p. 1643, Equation (14)).

$$C \approx \Phi\left(\frac{s-\rho Y}{\sqrt{1-\rho^2}}\right). \tag{1}$$

Φ :The distribution function of the standard normal distribution

Y :Random variable following the standard normal distribution

s :The value of $\Phi^{-1}(p)$ when the default rate is p, and Φ^{-1} is the inverse of Φ.

ρ :Correlation coefficient among the latent variables

Based on this result, we calculate the distribution functions of the limiting credit loss C for ρ =0.7 and 0.9, and plot them in Figure 2.

The results show that VaR has tail risk. The bond portfolio is more concentrated when $\rho = 0.9$ than when $\rho = 0.7$. The tail of the credit loss distribution is fatter when $\rho = 0.9$ than when $\rho = 0.7$. Thus, the bond portfolio is more risky when $\rho = 0.9$ than when $\rho = 0.7$. However, the VaR at the 95% confidence interval is higher when $\rho = 0.7$ than when $\rho = 0.9$. This shows that VaR fails to consider credit concentration since it disregards the loss beyond the confidence level.

The preceding examples show that VaR has tail risk when the loss distributions intersect beyond the confidence level. In such cases, one is able to

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9 Lucas *et al.* [2001] also develop more general analyses in their paper.

10 The total exposure of the bond portfolio is 1.

decrease VaR by manipulating the tails of the loss distributions. This manipulation of the distribution tails increases the potential for extreme losses, and may lead to a failure of risk management. This problem is significant when the portfolio profit/loss is non-linear and the distribution function of the profit/loss is discontinuous.[11]

B. The Tail Risk of Expected Shortfall

We define the tail risk of expected shortfall in the same way as the tail risk of VaR. In this paper, we say that *expected shortfall has tail risk* when expected shortfall fails to summarize the relative choice between portfolios as a result of its underestimation of the risk of portfolios with fat-tailed properties and a high potential for large losses.

To illustrate our definition of the tail risk of expected shortfall, we present an example from Yamai and Yoshiba [2002c]. Table 1 shows the payoff and profit/loss of two sample portfolios A and B. The expected payoff and the initial investment amount of both portfolios are equal at 97.05.

In most of the cases, both portfolios A and B do not incur large losses. The probability that the loss is less than 10 is about 99% for both portfolios.

The magnitude of extreme loss is different. Portfolio A never loses more than half of its value while Portfolio B may lose three quarters of its value. Thus, portfolio B is more risky than Portfolio A when one is worried about extreme loss.

Table 2 shows the VaR and expected shortfall of the two portfolios at the 99% confidence level. Both VaR and expected shortfall are higher for Portfolio A, which has a lower magnitude of extreme loss. Thus, expected shortfall has tail risk since it chooses the more risky portfolio as a result of its disregard of extreme losses.

The example above shows that expected shortfall may have tail risk. However, the tail risk of expected shortfall is less significant than that of VaR. Yamai and Yoshiba [2002c] show that expected shortfall has no tail risk under

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[11] Yamai and Yoshiba [2002c] show that VaR has no tail risk when the loss distributions are of the same type of an elliptical distribution.

more lenient conditions than VaR. This is because VaR completely disregards any loss beyond the confidence level while expected shortfall takes this into account as a conditional expectation.

III. Multivariate Extreme Value Theory

In this chapter, we give a brief introduction to multivariate extreme value theory.[12] We use this theory to represent asset returns under market stress in the following chapters.

Multivariate extreme value theory consists of two modeling aspects: the tails of the marginal distributions and the dependence structure among extreme values.

We restrict our attention to the bivariate case in this paper.

A. Univariate Extreme Value Theory

Let Z denote a random variable and F the distribution function of Z. We consider extreme values in terms of exceedances with a threshold $\theta\,(\theta > 0)$. The exceedances are defined as $m_\theta(Z) = \max(Z, \theta)$. Z is larger than θ with probability p, and smaller than θ with probability $1-p$. Then, by the definition of exceedances, $p = 1 - F(\theta)$. We call p tail probability.

The conditional distribution F_θ defined below gives the stochastic behavior of extreme values.

$$F_\theta(x) = \Pr\{Z - \theta \le x | Z > \theta\} = \frac{F(x) - F(\theta)}{1 - F(\theta)},\ \theta \le x. \tag{2}$$

This is the distribution function of $(Z - \theta)$ given that Z exceeds θ. F_θ is not known precisely unless F is known.

The extreme value theory tells us the approximation to F_θ that is applicable for high values of threshold θ. The Pickands-Balkema-de Haan theorem shows that as the value of θ tends to the right end point of F, F_θ

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12 For detailed explanations of extreme value theory, see Coles [2001], Embrechts, Klüppelberg, and Mikosch [1997], Kotz and Nadarajah [2000], Resnick [1987].

converges to a generalized Pareto distribution. The generalized Pareto distribution is represented as follows.[13, 14]

$$G_{\xi,\sigma}(x) = 1-(1+\xi\cdot\frac{x}{\sigma})^{-1/\xi},\ x \ge 0. \tag{3}$$

With equations (2) and (3), when the value of θ is sufficiently large, the distribution function of exceedances $m_\theta(Z)$, denoted by $F_m(x)$, is approximated as follows.

$$F_m(x) \approx (1-F(\theta))G_{\xi,\sigma}(x-\theta)+F(\theta) = 1-p(1+\xi\cdot\frac{x-\theta}{\sigma})^{-1/\xi},\ x \ge \theta. \tag{4}$$

In this paper, we call $F_m(x)$ the *distribution of exceedances*.

The distribution of exceedances is described by three parameters: the tail index ξ, the scale parameter σ, and the tail probability p. The tail index ξ represents how fat the tail of the distribution is, so the tail is fat when ξ is large (see Figure 3). The scale parameter σ represents how dispersed the distribution is, so the distribution is dispersed when σ is large (see Figure 4). The tail probability p determines the threshold θ as $F_m(\theta) \approx 1-p$.

When the confidence level of VaR and expected shortfall is less than p, the distribution of exceedances is used to calculate VaR and expected shortfall. (See Chapter 4 for the specific calculations).

B. Copula

As a preliminary to the dependence modeling of extreme values, we provide a simple explanation of copula.[15]

Suppose we have two-dimensional random variables (Z_1, Z_2). Their joint distribution function $F(x_1, x_2) = P[Z_1 \le x_1, Z_2 \le x_2]$ fully describes their marginal behavior and dependence structure. The main idea of copula is that we separate

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13 See Coles [2001] and Embrechts, Klüppelberg, and Mikosch [1997] for a detailed explanation of this theorem.

14 In this paper we assume that $\xi \ne 0$.

15 For the precise definition of copula and proofs of the theorems adopted here, see Embrechts, McNeil, and Straumann [2002], Joe [1997], Nelsen [1999], Frees and Valdez [1998], etc.

this joint distribution into the part that describes the dependence structure and the part that describes the marginal behavior.

Let $(F_1(x_1), F_2(x_2))$ denote the marginal distribution functions of (Z_1, Z_2). Suppose we transform (Z_1, Z_2) to have standard uniform marginal distributions.[16] This is done by $(Z_1, Z_2) \mapsto (F_1(Z_1), F_2(Z_2))$. The joint distribution function C of the random variable $(F_1(Z_1), F_2(Z_2))$ is called the copula of the random vector (Z_1, Z_2). It follows that

$$F(x_1, x_2) = P[Z_1 \le x_1, Z_2 \le x_2] = C(F_1(x_1), F_2(x_2)). \tag{5}$$

Sklar's theorem shows that (5) holds with any F for some copula C and that C is unique when $F_1(x_1)$ and $F_2(x_2)$ are continuous.

In general, the copula is defined as the distribution function of a random vector with standard uniform marginal distributions. In other words, the distribution function C is a copula function for the two random variables U_1, U_2 that follow the standard uniform distribution.

$$C(u_1, u_2) = \Pr[U_1 \le u_1, U_2 \le u_2]. \tag{6}$$

One of the most important properties of the copula is its invariance property. This property says that a copula is invariant under increasing and continuous transformations of the marginals. That is, when the copula of (Z_1, Z_2) is $C(u_1, u_2)$ and $h_1(\bullet), h_2(\bullet)$ are increasing continuous functions, the copula of $(h_1(Z_1), h_2(Z_2))$ is also $C(u_1, u_2)$.

The invariance property and Sklar's theorem show that a copula is interpreted as the dependence structure of random variables. The copula represents the part that is not described by the marginals, and is invariant under the transformation of the marginals.

C. Multivariate Extreme Value Theory

We give a brief illustration of the bivariate exceedances approach as a model for the dependence structure of extreme values.[17]

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16 The standard uniform distribution is the uniform distribution over the interval [0,1].

17 For more detailed explanations of multivariate extreme value theory, see Coles [2001] Ch.8 , Kotz and Nadarajah [2000] Ch.3, McNeil [2000], Resnick [1987] Ch.5, etc.

Let $Z = (Z_1, Z_2)$ denote the two-dimensional vector of random variables and $F(Z_1, Z_2)$ the distribution function of Z. The bivariate exceedances of Z correspond to the vector of univariate exceedances defined with a two dimensional vector of threshold $\theta = (\theta_1, \theta_2)$ (see Figure 5). These exceedances are defined as follows.

$$m_{(\theta_1,\theta_2)}(Z_1, Z_2) = (\max(Z_1, \theta_1), \max(Z_2, \theta_2)) . \tag{7}$$

The marginal distributions of the bivariate exceedances defined in (7) converge to the distribution of exceedances introduced in section A when the thresholds tend to the right end points of the marginal distributions. This is because the bivariate exceedance is the vector of univariate exceedances whose distribution converges to a generalized Pareto distribution.

The copula of bivariate exceedances also converges to a class of copula that satisfies several conditions. Ledford and Tawn [1996] show that this class is represented by the following equation (see Appendix A for details).

$$C(u_1, u_2) = \exp\{-V(-\frac{1}{\log u_1}, -\frac{1}{\log u_2})\}, \tag{8}$$

where

$$V(z_1, z_2) = \int_0^1 \max\{s z_1^{-1}, (1-s) z_2^{-1}\} dH(s) , \tag{9}$$

and H is a non-negative measure on [0,1] satisfying the following condition.

$$\int_0^1 s dH(s) = \int_0^1 (1-s) dH(s) = 1 . \tag{10}$$

Following Hefferman [2000], we call this type of copula the *bivariate extreme value copula* or the *extreme value copula*.

The class of the extreme value copula is wide, being constrained only by (10). We have an infinite number of parameterized extreme value copula. In practice, we choose a parametric family of copula that satisfies (10), and use the copula for the analysis of bivariate extreme values.

One standard type of bivariate extreme value copula is the Gumbel copula. The Gumbel copula is the most frequently used extreme value copula for applied statistics, engineering, and finance (Gumbel [1960], Tawn [1988], Embrechts, McNeil, and Straumann [2002], McNeil [2000], Longin and Solnik [2001]). The

Gumbel copula is expressed by:

$$C(u_1,u_2) = \exp\{-[(-\log u_1)^{\alpha} + (-\log u_2)^{\alpha}]^{1/\alpha}\}, \tag{11}$$

for a parameter $\alpha \in [1,\infty]$. We obtain (11) by defining V in (9) as follows.

$$V(z_1,z_2) = (z_1^{-\alpha} + z_2^{-\alpha})^{1/\alpha}. \tag{12}$$

The dependence parameter α controls the level of dependence between random variables. $\alpha = 1$ corresponds to full dependence and $\alpha = \infty$ corresponds to independence.

The Gumbel copula has several advantages over other parameterized extreme value copulas.[18] It includes the special cases of independence and full dependence, and only one parameter is needed to model the dependence structure. The Gumbel copula is tractable, which facilitates simulations and maximum likelihood estimations. Given these advantages, we adopt the Gumbel copula as the extreme value copula.

To summarize, extreme value theory shows that the bivariate exceedances asymptotically follow a joint distribution whose marginals are the distributions of exceedances and whose copula is the extreme value copula.

D. Tail Dependence

We introduce the concept of tail dependence between random variables. Suppose that a random vector (Z_1,Z_2) has a joint distribution function $F(Z_1,Z_2)$ with marginals $F_1(x_1), F_2(x_2)$.

Assume that marginals are equal. We define a dependence measure χ as follows.

$$\chi \equiv \lim_{z \to z^+} \Pr\{Z_1 > z \mid Z_2 > z\}, \tag{13}$$

where z^+ is the right end point of F

χ measures the asymptotic survival probability over one value to be large given that the other is also large. When $\chi = 0$, we say Z_1 and Z_2 are *asymptotically*

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[18] For other parameterized extreme value copulas, see, for example, Joe [1997] and Kotz and Nadarajah [2000].

independent. When $\chi > 0$, we say Z_1 and Z_2 are *asymptotically dependent*. χ increases with the strength of dependence within the class of asymptotically dependent variables.

When F has different marginals F_{Z_1} and F_{Z_2}, χ is defined as follows.

$$\chi \equiv \lim_{u \to 1} \Pr\{F_{Z_1}(Z_1) > u \mid F_{Z_2}(Z_2) > u\}, \tag{14}$$

Further defining the other dependence measure $\chi(u)$ as in (15), the relationship $\chi = \lim_{u \to 1} \chi(u)$ holds (Coles, Hefferman, and Tawn [1999]).

$$\chi(u) \equiv 2 - \frac{\log \Pr\{F_{Z_1}(Z_1) < u, F_{Z_2}(Z_2) < u\}}{\log \Pr\{F_{Z_1}(Z_1) < u\}}, \text{ for } 0 \le u \le 1. \tag{15}$$

Although χ measures dependence when random variables are asymptotically dependent, it fails to do so when random variables are asymptotically independent. When random variables are asymptotically independent, $\chi = 0$ by definition and χ is unable to provide dependence information.

The class of asymptotically independent copulas includes important copulas such as the Gaussian copula and the Frank copula, which are introduced in the next section. Ledford and Tawn [1996, 1997] and Coles, Hefferman, and Tawn [1999] say that the asymptotically independent case is important in the analysis of multivariate extreme values.

To counter this shortcoming of the dependence measure χ, Coles, Hefferman, and Tawn [1999] propose a new dependence measure $\bar{\chi}$ as defined below.

$$\bar{\chi} \equiv \lim_{u \to 1} \bar{\chi}(u) \tag{16}$$

$$\text{where } \bar{\chi}(u) \equiv \frac{2 \log \Pr\{F_{Z_1}(Z_1) > u\}}{\log \Pr\{F_{Z_1}(Z_1) > u, F_{Z_2}(Z_2) > u\}} - 1 \tag{17}$$

$\bar{\chi}$ measures dependence within the class of asymptotically independent variables. For asymptotically independent random variables, $-1 < \bar{\chi} < 1$. For asymptotically dependent random variables, $\bar{\chi} = 1$.

Thus, the combination $(\chi, \bar{\chi})$ measures tail dependence for both asymptotically dependent and independent case (see Table 3). For asymptotically dependent random variables, $\bar{\chi}=1$ and χ measures tail dependence. For asymptotically independent random variables, $\chi=0$ and $\bar{\chi}$ measures tail dependence.

E. Copula and Tail Dependence

With some calculations, it is shown that $\chi(u)$ is constant for the bivariate extreme value copula as follows.

$$\chi(u)=\chi=2-V(1,1)\text{. for all } 0\le u\le 1 \tag{18}$$

For the Gumbel copula, this becomes $\chi=2-2^{1/\alpha}$ $(\alpha\ge 1)$ (see Table 4). Thus, for the bivariate extreme value copula, random variables are either independent or asymptotically dependent. In other words, the bivariate extreme copula is unable to represent the dependence structure when random variables are asymptotically independent.

Ledford and Tawn [1996, 1997] and Coles [2001] say that multivariate exceedances may be asymptotically independent and that modeling multivariate exceedances with the extreme value copula is likely to lead to misleading results in this case. They say that the use of asymptotically independent copulas is effective when the multivariate exceedances are asymptotically independent. Hefferman [2000] provides a list of asymptotically independent copulas that are useful for modeling multivariate extreme values.

In this paper, we adopt the Gaussian copula and the Frank copula as asymptotically independent copulas. These are defined as follows (See Table 4).

• Gaussian Copula

$$C(u,v)=\Phi_{\rho}(\Phi^{-1}(u),\Phi^{-1}(v)) \tag{19}$$

where Φ_{ρ} is the distribution function of a bivariate standard normal distribution with a correlation coefficient ρ, and Φ^{-1} is the inverse function of the distribution function for the univariate standard normal distribution.

• Frank Copula[19]

$$C(u,v) = -\frac{1}{\delta}\ln\left(\frac{1-e^{-\delta}-(1-e^{-\delta u})(1-e^{-\delta v})}{1-e^{-\delta}}\right) \quad (20)$$

The dependence parameters ρ and δ control the level of dependence between random variables. For the Gaussian copula, $\rho = \pm 1$ corresponds to full dependence and $\rho = 0$ corresponds to independence. For the Frank copula, $\delta = \pm\infty$ corresponds to full dependence and $\delta = 0$ corresponds to independence.

For both of these copulas, random variables are asymptotically independent. For the Gaussian copula with $-1 < \rho < 1$, $\chi = 0$ and $\bar{\chi} = \rho$. For the Frank copula, $\chi = \bar{\chi} = 0$.[20] The latter shows that the Frank copula has very weak tail dependence.

The use of asymptotically independent copula for modeling multivariate exceedances may bring some doubt since extreme value theory shows that the asymptotic copula of exceedances is the extreme value copula. However, the rate of convergence of marginals may be higher than that of the copula. In this case, the generalized Pareto distribution well approximates the marginals of exceedances while the extreme value copula does not approximate the dependence structure of exceedances. Thus, in some cases, it is valid to assume that marginals are modeled by the generalized Pareto distribution while dependence is modeled by asymptotically independent copula.

IV. The Tail Risk under Univariate Extreme Value Distributions

In this chapter, we examine whether VaR and expected shortfall have tail risk when asset returns are described by the univariate extreme value distribution. We use (4) to calculate the VaR and expected shortfall of two securities with different tail fatness, and examine whether VaR and expected shortfall underestimate the

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[19] This definition of the Frank copula follows Joe [1997].

[20] See Ledford and Tawn [1996, 1997], Coles, Hefferman, and Tawn [1999], and Hefferman [2000] for the definition and concepts of tail dependence, including the derivations of χ and $\bar{\chi}$ for each copula.

risk of securities with fat-tailed properties and a high potential for large loss.

Suppose Z_1 and Z_2 are random variables denoting the loss of two securities. Using the univariate extreme value theory introduced in III.A, with high thresholds, the exceedances of Z_1 and Z_2 follow the distributions below.

$$F_{m(Z_1)}(x) = 1 - p_1(1 + \xi_1 \cdot \frac{x - \theta_1}{\sigma_1})^{-1/\xi_1}. \tag{21}$$

$$F_{m(Z_2)}(x) = 1 - p_2(1 + \xi_2 \cdot \frac{x - \theta_2}{\sigma_2})^{-1/\xi_2}. \tag{22}$$

As an example of the tail risk of VaR, we set the parameter values as follows: the tail probability is $p_1 = p_2 = 0.1$; the threshold value is $\theta_1 = \theta_2 = 0.05$; the tail indices are $\xi_1 = 0.1$ and $\xi_2 = 0.5$; and the scale parameters are $\sigma_1 = 0.05$ and $\sigma_2 = 0.035$. Figure 6 plots (21) and (22) with this parameter setting.

Figure 6 shows that VaR has tail risk in this example. Given $\xi_2 > \xi_1$, Z_2 has a fatter tail than Z_1 (see Chapter 3(1)). Thus, Z_2 has a higher potential for large loss than Z_1. However, Figure 6 shows that the VaR at the 95% confidence level is higher for Z_1 than for Z_2. Thus, VaR indicates that Z_1 is more risky than Z_2. As in the two examples in Chapter 2(1), VaR has tail risk as the distribution functions intersect beyond the VaR confidence level.

We derive the conditions for the tail risk of VaR. Following McNeil [2000], we calculate the VaR from (21) and (22). Let $VaR_\alpha(Z)$ denote the VaR of Z at the $(1-\alpha)$ confidence level. Since VaR is the upper $(1-\alpha)$ quantile of the loss distribution, the following holds.

$$1 - \alpha \approx 1 - p(1 + \xi \cdot \frac{VaR_\alpha(Z) - \theta}{\sigma})^{-1/\xi}. \tag{23}$$

We then solve (23) to obtain the following.

$$VaR_\alpha(Z) \approx \theta + \frac{\sigma}{\xi}\left(\left(\frac{p}{\alpha}\right)^{\xi} - 1\right). \tag{24}$$

With (24), we derive the condition of the tail risk of VaR as follows. Without the loss of generality, we assume $\xi_2 > \xi_1$, or that the tail of Z_2 is fatter than the tail of Z_1. In other words, Z_2 has higher potential for extreme loss than Z_1. VaR has tail risk when the VaR of Z_2 is smaller than that of Z_1, or when the following inequality holds.

$$VaR_\alpha(Z_1) > VaR_\alpha(Z_2)\,. \tag{25}$$

Assuming $\theta_1 = \theta_2$ and $p_1 = p_2 = p$ for simplification, we obtain the following condition from (24) and (25).

$$\frac{\sigma_1}{\sigma_2} > \bar{\kappa}_{VaR}\,,\ \text{where}\ \bar{\kappa}_{VaR} = \frac{\xi_1}{\xi_2}\left(\frac{(p/\alpha)^{\xi_2} - 1}{(p/\alpha)^{\xi_1} - 1}\right). \tag{26}$$

The value $\bar{\kappa}_{VaR}$ indicates how strict the condition for the tail risk of VaR is. When $\bar{\kappa}_{VaR}$ is small, a small difference between the scale parameters σ_1 and σ_2 brings about tail risk of VaR. When $\bar{\kappa}_{VaR}$ is large, a large difference between σ_1 and σ_2 is needed to bring about tail risk of VaR.

Table 5 shows the value of $\bar{\kappa}_{VaR}$ with varying (ξ_1, ξ_2) for VaR at the 95% and 99% confidence levels, when p is 0.05 and 0.1.[21] This table shows two aspects of this condition.

First, the scale parameter of the thin-tailed distribution σ_1 must be larger than the scale parameter of the fat-tailed distribution σ_2. This is because $\bar{\kappa}_{VaR} > 1$ for all combinations of (ξ_1, ξ_2).

Figure 7 illustrates this point. The figure plots the distribution of exceedances with parameter values $\xi_1 = 0.5,\ \sigma_1 = 1$. The figure also plots the distribution of exceedances with parameter values $\xi_2 = 0.1$ and $\sigma_2 = 1$, 1.5 and 2. Here, we denote the VaR for $\xi_1 = 0.5,\ \sigma_1 = 1$ as $VaR(\xi_1 = 0.5, \sigma_1 = 1)$ and that for $\xi_2 = 0.1,\ \sigma_2 = \sigma$ as $VaR(\xi_2 = 0.1, \sigma_2 = \sigma)$. The distribution with $\xi_1 = 0.5$ has a fatter tail and higher potential for large loss than the distribution with $\xi_2 = 0.1$. Thus, VaR has tail risk if $VaR(\xi_1 = 0.5, \sigma_1 = 1) < VaR(\xi_2 = 0.1, \sigma_2 = \sigma)$. From the figure, we find $VaR(\xi_1 = 0.5, \sigma_1 = 1) < VaR(\xi_2 = 0.1, \sigma_2 = 2)$ with a confidence level below 99%, and $VaR(\xi_1 = 0.5, \sigma_1 = 1) < VaR(\xi_2 = 0.1, \sigma_2 = 1.5)$ with a confidence level below 98%. On the other hand, $VaR(\xi_1 = 0.5, \sigma_1 = 1) > VaR(\xi_2 = 0.1, \sigma_2 = 1)$ with a confidence level above 95%. Therefore, VaR has tail risk with a high confidence level when the difference between the scale parameters is large.

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[21] When the tail probability is $p = 0.05$, the VaR at the confidence level of 95% is not beyond the threshold, so we do not calculate VaR at the confidence level of 95% when $p = 0.05$.

Second, the smaller the difference between the tail indices ξ_1 and ξ_2, the more lenient the conditions for the tail risk of VaR. This is because $\overline{\kappa}_{VaR}$ is small when the difference between the tail indices is small.

Figure 8 illustrates this point. The figure plots the distribution of exceedances with parameter values $\xi_1 = 0.1,\ \sigma_1 = 1$. The figure also plots the distribution of exceedances with parameter values $\sigma_2 = 0.75$ and $\xi_2 = 0.3,\ 0.5,\ 0.9$. Here, we denote the VaR for $\xi_1 = 0.1,\ \sigma_1 = 1$ as $VaR(\xi_1 = 0.1, \sigma_1 = 1)$ and that for $\xi_2 = \xi,\ \sigma_2 = 0.75$ as $VaR(\xi_2 = \xi, \sigma_2 = 0.75)$. As the distribution tail is fatter with $\xi_2 = \xi,\ \sigma_2 = 0.75$ than with $\xi_1 = 0.1,\ \sigma_1 = 1$, VaR has tail risk if $VaR(\xi_1 = 0.1, \sigma_1 = 1) > VaR(\xi_2 = \xi, \sigma_2 = 0.75)$. We find $VaR(\xi_1 = 0.1, \sigma_1 = 1)$ $> VaR(\xi_2 = 0.3, \sigma_2 = 0.75)$ with a confidence level below 99%, and $VaR(\xi_1 = 0.1, \sigma_1 = 1) > VaR(\xi_2 = 0.5, \sigma_2 = 0.75)$ with a confidence level below 97%. On the other hand, $VaR(\xi_1 = 0.1, \sigma_1 = 1) < VaR(\xi_2 = 0.9, \sigma_2 = 0.75)$ with a confidence level above 95%. Therefore, VaR has tail risk with a high confidence level when the difference between the tail indices is small.

We analyze the condition for the tail risk of expected shortfall as we analyzed that of VaR. Following McNeil [2000], we can calculate the expected shortfall of Z at the $(1-\alpha)$ confidence level (denoted by $ES_\alpha(Z)$) from (24).[22]

$$\begin{aligned} ES_\alpha(Z) &= E[Z \mid Z \geq VaR_\alpha(Z)] \\ &= VaR_\alpha(Z) + E[Z-\theta) - (VaR_\alpha(Z)-\theta) \mid Z-\theta \geq VaR_\alpha(Z)-\theta] \\ &= VaR_\alpha(Z) + \frac{\sigma + \xi \cdot (VaR_\alpha(Z)-\theta)}{1-\xi} \\ &= \frac{\sigma - \xi\theta}{1-\xi} + \frac{VaR_\alpha(Z)}{1-\xi} \approx \theta + \frac{\sigma}{1-\xi}\left\{1 + \frac{1}{\xi}\left(\left(\frac{p}{\alpha}\right)^{\xi} - 1\right)\right\}, \end{aligned} \tag{27}$$

Given $\xi_2 > \xi_1$, expected shortfall has tail risk when the following inequality holds.

$$ES_\alpha(Z_1) > ES_\alpha(Z_2). \tag{28}$$

Assuming $\theta_1 = \theta_2$ and $p_1 = p_2 = p$ for simplification, we obtain the following condition from (27) and (28).

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22 The third equality is based on Embrechts, Klüppelberg, and Mikosch [1997], Theorem 3.4.13 (e).

$$\frac{\sigma_1}{\sigma_2} > \overline{\kappa}_{ES}, \text{ where } \overline{\kappa}_{ES} = \frac{1-\xi_1}{1-\xi_2}\left(\frac{1+\left((p/\alpha)^{\xi_2}-1\right)/\xi_2}{1+\left((p/\alpha)^{\xi_1}-1\right)/\xi_1}\right) \tag{29}$$

Table 6 shows the value of $\overline{\kappa}_{ES}$ with varying (ξ_1,ξ_2) for expected shortfall at the 95% and 99% confidence levels, when p is 0.05 and 0.1.[23] This table shows that the conditions for the tail risk of expected shortfall are stricter than those for the tail risk of VaR. This confirms the result of Yamai and Yoshiba [2002c] that expected shortfall has no tail risk under more lenient conditions than VaR.

To summarize, VaR and expected shortfall may underestimate the risk of securities with fat-tailed properties and a high potential for large loss. The condition for tail risk to emerge depends on the parameters of loss distribution and the confidence level.

V. The Tail Risk under Multivariate Extreme Value Distribution

The use of risk measures may lead to a failure of risk management when they fail to consider the change in dependence between asset returns. The credit portfolio example in II.A shows that VaR disregards the increase in default correlation and thus fails to note the high potential for extreme loss in concentrated credit portfolios. In this case, the use of VaR for credit portfolios may lead to credit concentration.

In this chapter, we examine whether VaR and expected shortfall disregard the changes in dependence under a multivariate extreme value distribution. As the multivariate extreme value distribution, we use the joint distribution of exceedances introduced in III.C. The marginal of this distribution is the generalized Pareto and its copula is the Gumbel copula. We also use the Gaussian and Frank copulas for the copulas of exceedances for the case where the exceedances are asymptotically independent.

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[23] We do not calculate expected shortfall at the confidence level of 95% when $p = 0.05$ (see Footnote 21).

A. The Difficulty of Applying Multivariate Extreme Value Distribution to Risk Measurement

The application of multivariate extreme value distribution to financial risk measurement has some problems that the univariate application does not. In the univariate case, the model for exceedances enables us to calculate VaR and expected shortfall as in chapter IV. This is because the VaR and expected shortfall of exceedances are equal to the VaR and expected shortfall of the original loss data. However, in the multivariate case, the model for exceedances is not sufficient to calculate VaR and expected shortfall. This is because, in the multivariate case, the sums of exceedances is not necessarily equal to the exceedances of the sums. To calculate VaR and expected shortfall, we need the exceedances of the sums, which is not available only with the model for exceedances.[24, 25, 26]

A simple example illustrates this point (Figure 9). Let (U_1, U_2) denote a vector of independent standard uniform random variables. With a threshold value of $(\theta_1, \theta_2) = (0.9, 0.9)$, the exceedances of (U_1, U_2) is $(m_{0.9}(U_1), m_{0.9}(U_2))$ $= (\max(U_1, 0.9), \max(U_2, 0.9))$. With the convolution theorem, the 95% upper quantile of $U_1 + U_2$ is calculated to be 1.68, while that of $m_{\theta_1}(U_1) + m_{\theta_2}(U_2)$ is calculated to be 1.88.[27] Thus, the sum of exceedances is larger than the

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24 This is also a problem when the model for maxima is used for calculating VaR and expected shortfall. This is because the sums of maxima are not necessarily equal to the maxima of sums. Hauksson *et al.* [2000] and Bouyé [2001] propose the use of multivariate generalized extreme value distributions for financial risk measurement, but they do not address this problem.

25 The quantile of the sums of exceedances is equal to that of the original data when the underlying random variables are fully dependent.

26 McNeil [2000] says that multivariate extreme value modeling has the problem of "the curse of dimensionality". He notes that, when the number of dimension is more than two, the estimation of copula is not tractable.

27 The upper 95% quantile of $U_1 + U_2$ is calculated as follows. Denote the distribution function of $U_1 + U_2$ as $G(x)$. Clearly, the upper 95% quantile of $U_1 + U_2$ is greater than 1. So assuming $x > 1$, $G(x)$ is calculated by the convolution theorem as follows.

$$G(x) = \int_0^1 \Pr[U_1 \le x - u] du = -\frac{1}{2}(x-2)^2 + 1$$

exceedances of the sum.

This example shows that, to calculate VaR and expected shortfall in the multivariate case, we need a model for non-exceedances as well as one for exceedances.

In this paper, we assume that the marginal distribution of the non-exceedances is the standard normal distribution as we interpret the non-exceedances as asset loss under normal market conditions. That is, we assume that the marginal distribution is expressed by (30) below (Figure 10). [28]

$$F(x)=\begin{cases}\Phi(x) & (x<\Phi^{-1}(1-p)),\\ 1-p(1+\xi\cdot\dfrac{x-\Phi^{-1}(1-p)}{\sigma})^{-1/\xi} & (x\geq\Phi^{-1}(1-p)).\end{cases} \tag{30}$$

Φ :the distribution function of the standard normal

Φ^{-1} :the inverse function of Φ

In the following analysis, we simulate two dependent asset losses to analyze the tail risk of VaR and expected shortfall.[29] In the simulation, we assume

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The upper 95% quantile is x that satisfies $G(x)=0.95$, which is calculated as $x\approx 1.6838$.

The upper 95% quantile of the sum of the exceedances is calculated as follows. Define $H(x)\equiv \Pr[\max(U_1,0.9)+\max(U_2,0.9)\leq x]$. Using the convolution theorem, this is restated as follows.

$$H(x)=\int_0^1 \Pr[\max(U_1,0.9)\leq x-u]\cdot\Pr[\max(U_2,0.9)=u]du=\begin{cases} x^2/2-0.81 & (x\leq 1.9)\\ -(x-2)^2/2+1 & (x>1.9)\end{cases}$$

The upper 95% quantile is x that satisfies $G(x)=0.95$, which is calculated as $x\approx 1.8761$.

[28] A different assumption might be that the marginal distribution of exceedances is a non-standard normal distribution, a *t*-distribution, a generalized Pareto distribution, or an empirical distribution produced from actual data. Assuming a non-standard normal distribution, a *t*-distribution, and a generalized Pareto distribution, we simulated asset loss as in sections B and C of this chapter, and found the same result as in those sections. Furthermore, under the assumption of a generalized Pareto distribution, the convolution theorem is applied to obtain the analytics of the tail risk of VaR (see Appendix B for the details).

[29] We use the Mersenne Twister for generating uniform random numbers, and the Box-Müller method for transforming the uniform random numbers into normal random numbers. We follow Frees and Valdez [1998] in simulating the Gumbel copula, and Joe

that the marginal distribution of asset loss is (30). We also assume that the copula of asset loss is one of three copulas introduced in section III.E: Gumbel, Gaussian, and Frank. We set the marginal distribution of each asset loss as identical so that we can examine the pure effect of dependence on the tail risk of VaR and expected shortfall. We limit our attention to the cases where the tail index is $0<\xi<1$.[30]

B. One Specific Copula Case

In this section, we assume that the change in the dependence structure of asset loss is represented by the change in the dependence parameters within one specific copula. Under this assumption, we examine whether VaR and expected shortfall consider the change in dependence by taking the following steps. First, we take one of the three copulas introduced in III.E: Gumbel, Gaussian or Frank. Second, we simulate asset losses under the one copula for varied dependence parameter levels (Gumbel: α, Gaussian: ρ, and Frank: δ). Third, we calculate VaR and expected shortfall with the simulated asset losses for each dependence parameter level.

If VaR and expected shortfall do not increase with the rise in the level of dependence, VaR and expected shortfall disregard dependence and thus have tail risk.

Figure 11 shows an example of this analysis. The figure plots the empirical distribution of the sum of two simulated asset losses. These losses are simulated adopting (30) as the marginals and the Gumbel copula as the copula. The parameters of the marginal are set at $\xi=0.5$, $\sigma=1$, $p=0.1$, and the dependence parameter α of the Gumbel copula is set at 1.0, 1.1, 1.5, 2.0 and ∞.[31] For each dependence parameter, we conduct one million simulations

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[1997] for simulating the Gaussian and Frank copulas.

30 The generalized Pareto distribution with $\xi>1$ is so fat-tailed that its mean is infinite (Embrechts, Klüppelberg, and Mikosch [1997], Theorem 3.4.13 (a)).

The generalized Pareto distribution with $\xi>1$ has several interesting properties. However, it is not considered in this paper because such a fat-tailed distribution is rarely observed in financial data. For details, see Appendix B, Footnote 37.

31 Under the Gumbel copula $\chi=2-2^{1/\alpha}$, so the corresponding values of χ become $\chi=0,\ 0.12,\ 0.41,\ 0.59,\ 1$.

The result shows that the distribution tail gets fatter as the value of the dependence parameter α increases, or the asset losses are more dependent. Furthermore, the empirical distributions do not intersect with each other. This shows that the portfolio diversification effect works to decrease the risk of the portfolio and that VaR has no tail risk regardless of its confidence level.

Table 7 provides a more general analysis. The figure gives the VaR and expected shortfall under one million simulations for each copula with various dependence parameter levels. Two of the three marginal distribution parameters $(\xi,\ \sigma,\ p)$ are set at $\sigma = 1,\ \ p = 0.1$, and the tail index ξ is set at 0.1, 0.25, 0.5 and 0.75. One of the copulas (Gumbel, Gaussian and Frank) is adopted. With these marginals and copulas, asset losses are simulated. VaR and expected shortfall are calculated for varied dependence parameter levels (Gumbel: α, Gaussian: ρ, and Frank: δ).

Table 7 shows that VaR and expected shortfall consider the change in dependence and have no tail risk in most of the cases. VaR and expected shortfall increase as the value of the dependence parameter rises, except for the Frank copula with extremely high dependence parameter levels.[32]

To summarize, VaR and expected shortfall have no tail risk when the change in dependence is represented by the change in parameters using one specific copula. Thus, if we select portfolios whose dependence structure is nested in one of the three copulas above, we can depend on VaR and expected shortfall for measuring dependent risks.

C. Different Copulas Case

In the previous section, we assume that the change in the dependence of asset losses is represented by the change in the parameters using one specific copula. However, this assumption has a problem. One specific copula does not represent both asymptotic dependence and asymptotic independence.

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32 In the case of the Frank copula, the VaR at the 95% confidence level when $\delta = \infty$ (full dependence) is smaller than the VaR when $\delta = 9$.

This might be because the Frank copula has low tail dependence ($\chi = \overline{\chi} = 0$) and does not represent tail dependence when δ is large.

Let us consider an example of this problem. Suppose we have two portfolios both composed of two securities. Also suppose that the security returns of one portfolio are asymptotically dependent while those of the other are asymptotically independent. Adopting one specific copula and changing the dependence parameters to describe the change in dependence does not work in this case. This is because one specific copula does not represent the change from asymptotic dependence to asymptotic independence. We need different types of copulas to compare asymptotic dependence with asymptotic independence.

In this section, we assume that the change in dependence is represented by the change in copula. We adopt the Gumbel, Gaussian, and Frank copulas introduced in III.E since the Gumbel copula corresponds to asymptotic dependence and the Gaussian and Frank copulas correspond to asymptotic independence. By changing copula from Gumbel to Gaussian and Frank, we can change the dependence structure from asymptotic dependence to asymptotic independence.

In comparing the results with three copulas, we set the values of the dependence parameters of those copulas (Gumbel: α, Gaussian: ρ, and Frank: δ) so that the Spearman's rho (ρ_S) is equal across those copulas.[33,34] By setting the Spearman's rho equal, we can eliminate the effect of global dependence and examine the pure effect of tail dependence since the Spearman's rho is a measure of

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33 The Spearman's rho is the linear correlation of the marginals, and is defined by the following equation.

$$\rho_S(Z_1, Z_2) \equiv \frac{Cov(F_{Z_1}(Z_1), F_{Z_2}(Z_2))}{\sqrt{V[F_{Z_1}(Z_1)]V[F_{Z_2}(Z_2)]}}$$

The Spearman's rho differs from χ and $\overline{\chi}$ in that it measures global dependence while χ and $\overline{\chi}$ measures tail dependence.

The Spearman's rho does not fully represent the dependence structures since the combination of the Spearman's rho and the marginal distribution does not uniquely define the joint distribution. In particular, it does not represent the asymptotic dependence measured by χ and $\overline{\chi}$. Nevertheless, the Spearman's rho is a relatively superior measure as a single measure of global dependence (see Embrechts, McNeil, and Straumann [2002]).

34 We use the calculation in Joe [1997] (p. 147, Table 5.2) for the values of the dependence parameters that equate the Spearman's rho.

global dependence.

The upper half of Figure 12 shows the empirical distributions of the sums of two simulated asset losses for the Gumbel, Gaussian, and Frank copulas. This is generated from one million simulations for each copula where the parameters are fixed at $\xi = 0.5$, $\sigma = 1$, $\rho_S = 0.5$, $p = 0.1$. The range of the horizontal axis (cumulative probability) is above 99.5%.

The tail shape of the loss distribution for each copula is consistent with the tail dependence of each copula. The empirical loss distribution for the Gumbel copula, which is asymptotically dependent ($\chi > 0, \overline{\chi} = 1$), has the fattest tail. The empirical loss distribution for the Frank copula, which has the weakest tail dependence ($\chi = 0, \overline{\chi} = 0$), has the thinnest tail.[35]

This shows that the potential for extreme loss is high when the tail dependence is high. Thus, if we are worried about extreme loss, portfolios with higher tail dependence should be considered more risky than those with lower tail dependence. As for the three copulas adopted here, we should consider the Gumbel copula as the most risky and the Frank copula the least risky in terms of tail risk. In this context, VaR and expected shortfall have tail risk when they do not increase in the order of Frank, Gaussian, and Gumbel copulas.

The lower half of Figure 12 shows that VaR has tail risk in this example. The figure shows that the VaR at the 95% confidence level increases in the order of Gumbel, Gaussian, and Frank. VaR says that the Gumbel copula is the least risky while the Frank copula is the most risky. This contradicts our observation of the upper tail described above.

Table 8 provides a more general analysis. The table shows the results of VaR and expected shortfall calculations for one million simulations for each copula with the tail index of the marginal distribution of $\xi =$ 0.1, 0.25, 0.5, and 0.75, and Spearman's rho of $\rho_s =$ 0.2, 0.5 and 0.8.

The findings of the analysis are threefold. First, VaR and expected shortfall vary depending on the copula adopted. This means that the type of copula affects the level of VaR and expected shortfall. The difference is large when the tail

[35] See Figure 7 for the values of χ and $\overline{\chi}$ for each copula.

index and the Spearman's rho are large.

Second, VaR at the 95% confidence level has tail risk when the tail index ξ is 0.25 or higher. For example, when $\xi = 0.5$ and $\rho_S = 0.8$, the VaR at the 95% confidence level is largest for the Frank copula and smallest for the Gumbel copula. On the other hand, VaR at the 99% and 99.9% confidence level has no tail risk, except when the tail is as fat as $\xi = 0.75$.

Third, expected shortfall has no tail risk at the 95, 99, or 99.9% confidence level, except when the tail is as fat as $\xi = 0.75$. This confirms the result of Yamai and Yoshiba [2002c] that expected shortfall has no tail risk under more lenient conditions than VaR.

D. Different Marginals Case

In sections B and C, the marginal distributions are assumed to be identical. In financial data, however, the distributions of asset returns are rarely identical. In this section, we extend our analysis to the different marginals case. We examine whether the conclusions in sections B and C are still valid when the marginal distributions are different.

1. Independence vs. Full Dependence Case

We examine whether the results in section B (the specific copula case) are still valid when the marginal distributions are different. We compare independence and full dependence, noting the fact that independence and full dependence is nested in the Gumbel, Gaussian, and Frank copulas. When the VaR for independence is higher than the VaR for full dependence, VaR has tail risk.

We simulate independent and fully dependent asset losses with all combinations of parameters of the marginal distributions from $\xi_1 = 0.1$, 0.25, 0.5, 0.75, $\xi_2 = 0.1$, 0.25, 0.5, 0.75, $\sigma_1 = 1$, $\sigma_2 = 1.00$, 1.25, 1.5,..., 9.5, 9.75, 10. We set the number of simulations at one million for each parameter combination. We calculate VaR and expected shortfall for both independence and full dependence, and compare them to see whether they have tail risk. We adopt the tail probability of $p = 0.1$.

We found that the VaR for full dependence is never smaller than the VaR for independence.[36] Thus, at least within this framework, VaR captures full dependence and independence when the marginal distributions are different.

2. Different Copulas Case

We next examine whether the results in section C (the different copulas case) are still valid when the marginal distributions are different. We follow the same steps as in section C except that we set different parameter levels for two marginal distributions.

Under each one of the three copulas, as in section C, we simulate asset losses following the same method used in the previous subsection.

We find that VaR at the 95% confidence level may have tail risk even when the distribution tail is not so fat as $\xi = 0.25$.[37] This means that the conditions of the tail risk of VaR are more lenient when the marginals are different than when they are identical. Table 9 shows that, with a tail index of $\xi = 0.1$, the VaR at the 95% confidence level has tail risk. The VaR is larger for the Gaussian copula than for the Gumbel copula.[38]

On the other hand, at the confidence level of 99%, we find that VaR has tail risk only when the tail is so fat as $\xi = 0.75$.

VI. Empirical Analyses

In chapters IV and V, we examine the tail risk of VaR and expected shortfall under extreme value distributions. We summarize the results as follows.

In the univariate case, VaR and expected shortfall may underestimate the risk of securities with fat-tailed properties and a high potential for large losses. The conditions for this to happen are expressed by a simple analytical inequality.

In the multivariate case, VaR and expected shortfall may both disregard the tail

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36 The results of this simulation are omitted here due to space restrictions.

37 See Footnote 36.

38 This finding was confirmed by running 10 million simulations.

dependence when the tails of the marginal distributions are fat.

In this chapter, we conduct empirical analyses with foreign exchange rate data to confirm whether VaR and expected shortfall have tail risk in actual financial data. We focus on the following questions.

Do VaR and expected shortfall underestimate the risk of currencies with fat-tailed properties and a high potential for large losses in the univariate case?

Is there asymptotic dependence that may bring the tail risk of VaR and expected shortfall in the multivariate case?

A. Data

The data used for the analyses are the daily logarithmic changes of exchange rates of three industrialized countries and 18 emerging economies.[39,40,41] The raw historical data are the exchange rates per one U.S. dollar from November 1, 1993 to October 29, 2001.

B. Univariate Analyses

We estimate the parameters of the generalized Pareto distribution on the daily exchange rate data. We use the maximum likelihood method described in Embrechts, Klüppelberg, and Mikosch [1997], and Coles [2001]. We vary the tail probability as 1%, 2%, ..., 10%, and estimate the parameters ξ, σ, and θ for each.

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39 The data is sourced from Bloomberg.

40 We set the exchange rate as constant over holidays at the levels of the previous business day. This treatment does not affect our results as we estimate only the tails of distributions.

41 The currencies of developed countries are as follows: the Japanese yen, the German mark, and the UK pound. The currencies of emerging economies are as follows: the Hong Kong dollar, the Indonesian rupiah, the Malaysian ringgit, the Philippine peso, the Singapore dollar, the South Korean won, the new Taiwan dollar, the Thai baht, the Czech koruna, the Hungarian forint, the Polish zloty, the Slovakian koruna, the Brazilian real, the Chilean peso, the Colombian peso, the Mexican new peso, the Peruvian new sol, and the Venezuelan bolivar.

We then calculate the VaR and expected shortfall at the confidence levels of 95% and 99% using the estimated parameter values.

Table 10 shows the estimation results, and these findings may be summarized as follows. First, the tail indices are higher for the emerging economies (especially those in Asia and South America) than for the developed countries. In other words, the distribution tails are fatter in the emerging economies than in the developed countries.

Second, the scale parameter (σ) is smaller in the emerging economies than in the developed countries. This suggests that the condition for tail risk derived in chapter IV may hold.

Third, VaR has tail risk in comparing the risk of some emerging economies and some developed countries. For example, let us compare the VaR for Japan and those for emerging economies.[42] The VaR at the 95% confidence level for all the emerging economies except for Indonesia and Brazil is smaller than that for Japan. Even the VaR at the 99% confidence level is smaller for ten emerging economies (Hong Kong, Singapore, Taiwan, Hungary, Poland, Slovakia, Chile, Columbia, Peru, and Venezuela) than that for Japan.

Fourth, expected shortfall also has tail risk in comparing the risk of some emerging economies and some developed countries. For example, the expected shortfall at the 99% confidence level is smaller for six emerging economies (Hong Kong, Singapore, Taiwan, Chile, Columbia, and Peru) than for Japan.[43]

Fifth, expected shortfall has tail risk in fewer cases than VaR. This is consistent with our findings in chapter IV.

C. Bivariate Analyses (An Example)

We provide an example where VaR has tail risk in actual exchange rate data in the bivariate case. We pick five currencies in Southeast Asian countries: the

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[42] In the comparison here, we use the averages of the VaRs at the 95% confidence level in the right tail with the tail probabilities from 5% to 10%, and the average of VaRs at the 99% confidence level in the right tail with the tail probabilities from 1% to 10%.

[43] In the comparison here, we use the average of the expected shortfalls at the 99% confidence level in the right tail with the tail probabilities from 1% to 10%.

Indonesian rupiah, the Malaysian ringgit, the Philippine peso, the Singapore dollar, and the Thai baht.

First, we estimate the parameters of the bivariate extreme value distribution introduced in chapter III. We adopt the same method as Longin and Solnik [2001]. As in the analyses in chapters IV and V, we assume that the marginal distributions of bivariate exceedances are approximated by the generalized Pareto distribution (the distribution of exceedance as in (4), to be exact) and that their copula is approximated by the Gumbel copula. Given tail probabilities p_1 and p_2, the joint bivariate distribution of exceedances is described by the following parameters: the tail indices of the marginals (ξ_1 and ξ_2), the scale parameters of the marginals (σ_1 and σ_2), the thresholds (θ_1 and θ_2), and the dependence parameter of the Gumbel copula (α).

We estimate those parameters on the right tails of each pair from Southeast Asian currencies by the maximum likelihood method[44] for the tail probability of 10%. Table 11 shows the results of the estimation.

After the estimation, we examine whether VaR and expected shortfall disregard tail dependence with the estimated parameter levels. We take the same step as in section V.C. First, we simulate the logarithm changes in exchange rates with the distribution of exceedances and the Gumbel copula, using the parameter levels estimated here. Second, we also simulate the logarithm changes in exchange rates with the Gaussian and Frank copulas. The dependence parameters for the Gaussian and Frank copulas are set so that the Spearman's rho (ρ_S) is equal to that of Gumbel copula with the dependence parameter α at the estimated level. Third, we calculate the VaR and expected shortfall of the sums of the logarithm changes in two exchange rates. We run ten million simulations for each case.

Table 12 shows the result of those simulations. We find that the VaR at the 95% confidence level has tail risk for each pair of Southeast Asian currencies since the VaRs are larger for the Gaussian copula than for the Gumbel copula. Thus, VaR may disregard tail dependence in actual financial data. On the other hand, the VaR at the 99% confidence level and the expected shortfall at the 95%

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[44] See Longin and Solnik [2001] and Ledford and Tawn [1996] for the construction of the maximum likelihood function.

and 99% confidence levels have no tail risk in this example.

VII. Conclusions and Implications

This paper shows that VaR and expected shortfall have tail risk under extreme value distributions. In the univariate case, VaR and expected shortfall may underestimate the risk of securities with fat-tailed properties and a high potential for large losses. In the multivariate case, VaR and expected shortfall may disregard the tail dependence.

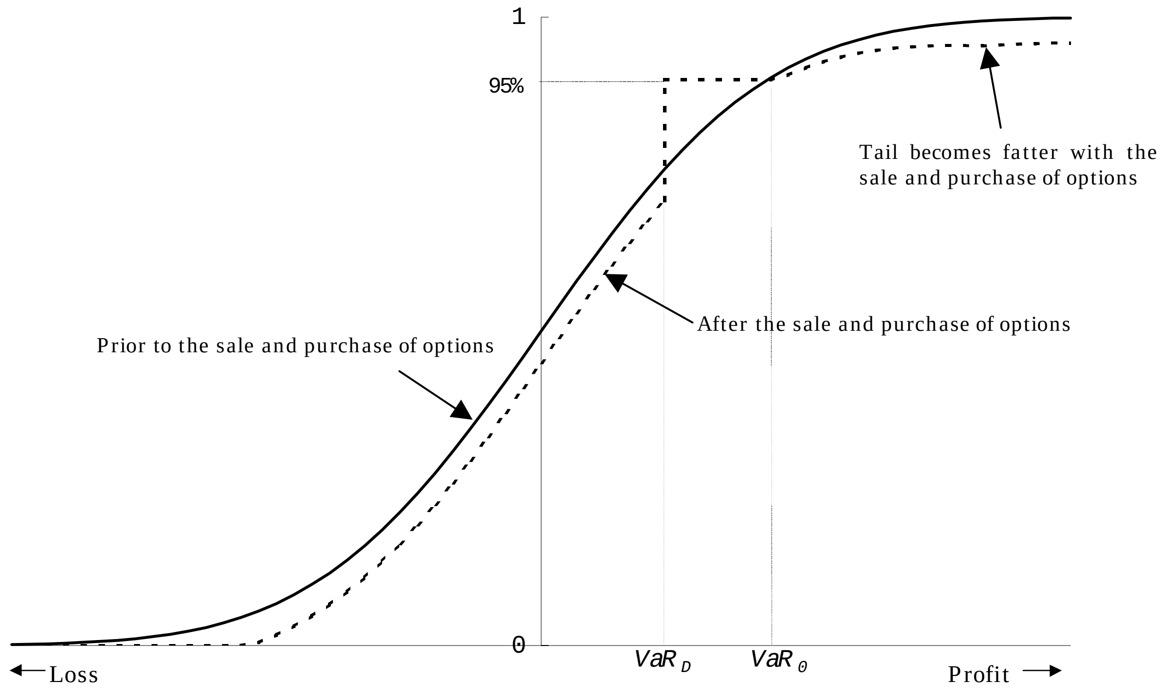
The tail risk is the result of the interaction among various factors. These include the tail index, the scale parameter, the tail probability, the confidence level, and the dependence structure.

These findings imply that the use of VaR and expected shortfall should not dominate financial risk management. Dependence on a single risk measure has a problem in disregarding important information of the risk of portfolios. To capture the information disregarded by VaR and expected shortfall, it is essential to monitor diverse aspects of the profit/loss distribution, such as tail fatness and asymptotic dependence.

The findings also imply that the widespread use of VaR for risk management could lead to market instability.[45] Basak and Shapiro [2001] show that when investors use VaR for their risk management, their optimizing behavior may result in market positions that are subject to extreme loss because VaR provides misleading information regarding the distribution tail. They also note that such investor behavior could result in higher volatility in equilibrium security prices. This paper shows that, under extreme value distribution, VaR may provide misleading information regarding the distribution tail.

[45] See Dunbar [2001] for the practitioners' view for this argument.

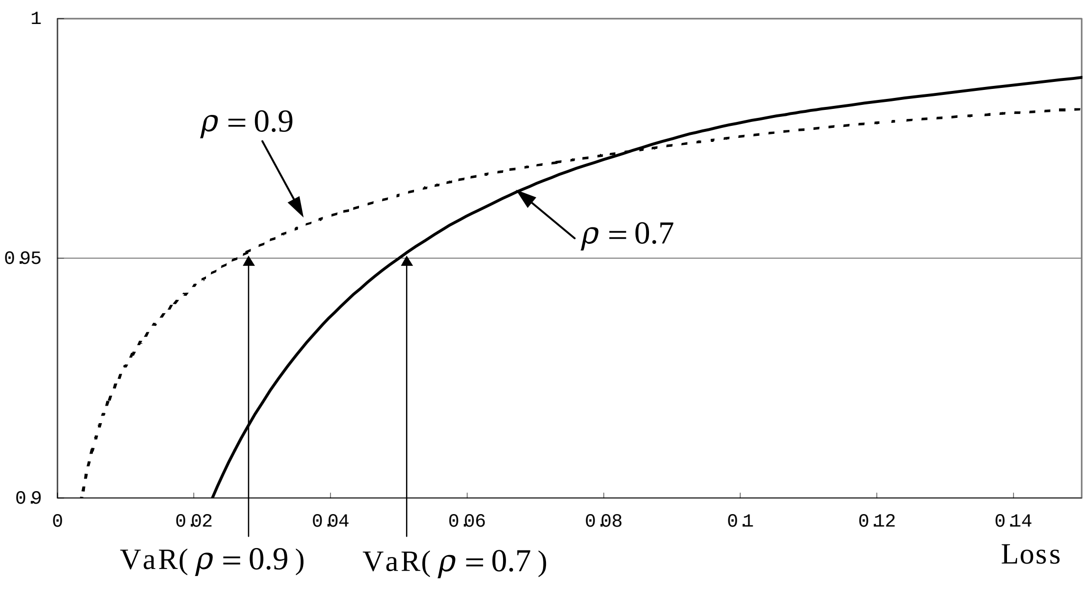
Figure 1 Tail Risk of VaR with Option Trading



Source: Based on Danielsson [2001], Figure 2.

Figure 2 Tail Risk of VaR in a Credit Portfolio

(Loss Distribution of a Uniform Portfolio with a Default Rate of 1%)



Source: Calculated from Equation (14) in Lucas *et al*. [2001].

Table 1 Sample Portfolio Payoff

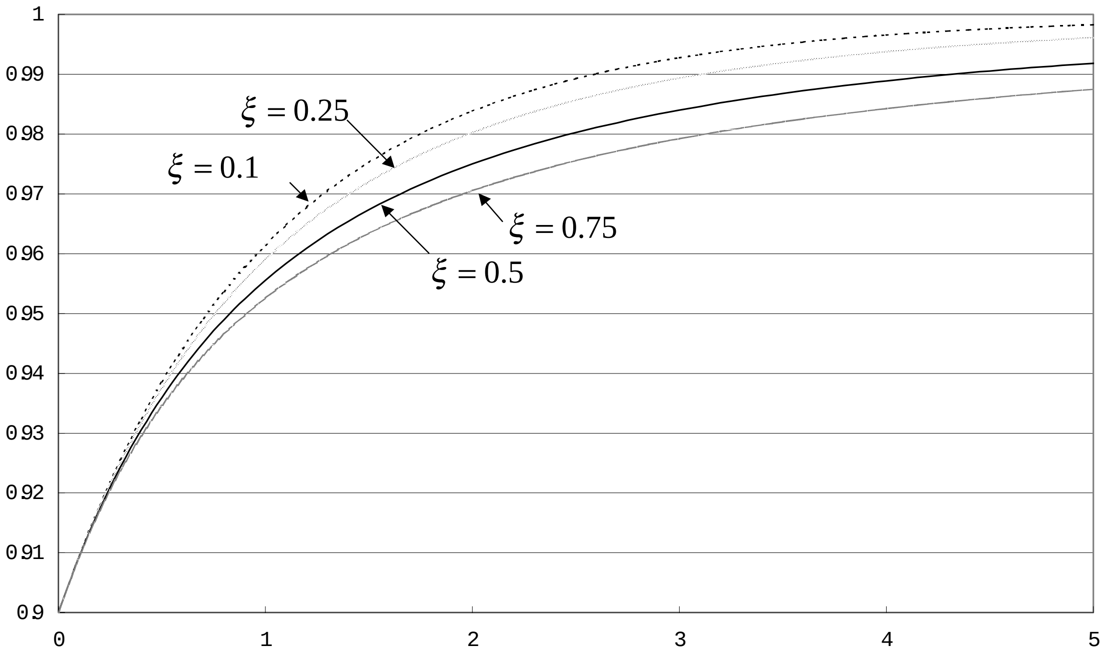
Portfolio A			Portfolio B		
Payoff	Loss	Probability	Payoff	Loss	Probability
100	- 2.95	50.000%	98	- 0.95	50.000%
95	2.05	49.000%	97	0.05	49.000%
50	47.05	1.000%	90	7.05	0.457%
			20	77.05	0.543%

Note: The probability that Portfolio B has a payoff of 90 or 20 is rounded off, and not precisely expressed. The model is set so that the sum of the probabilities of these payoffs is 1% and the expected payoff is 97.05.

Table 2 Sample Portfolio VaR and Expected Shortfall

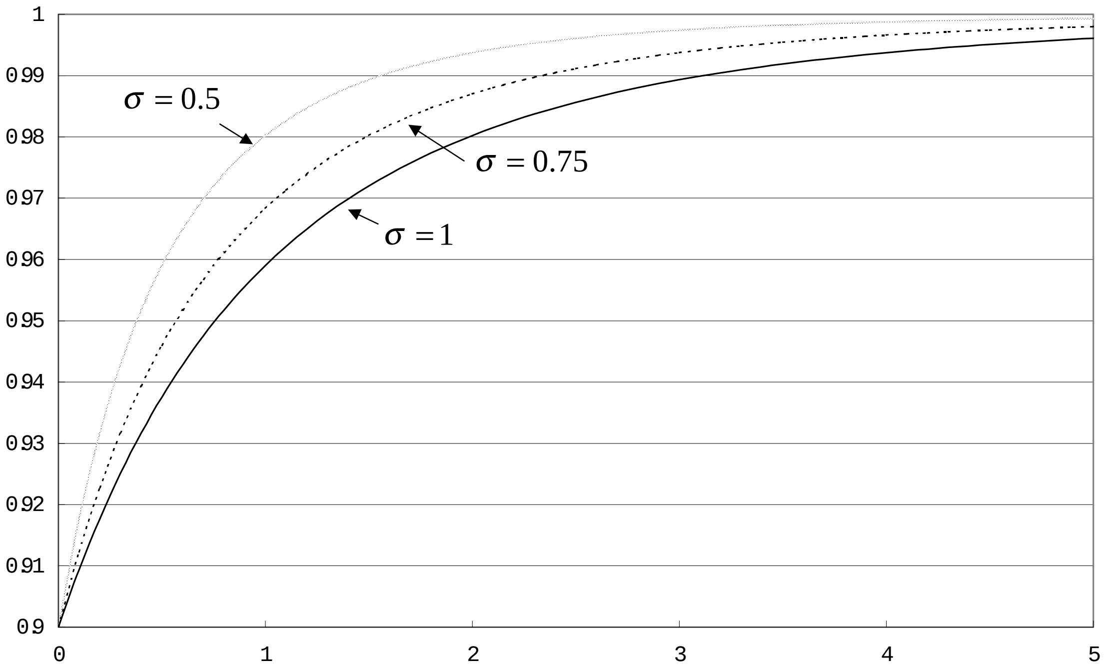
	Portfolio A	Portfolio B
Expected Payoff	97.05	97.05
VaR (confidence level: 99%)	47.05	7.05
Expected Shortfall (confidence level: 99%)	47.05	45.05

Figure 3 Distribution of Exceedances with Varied Tail Indices



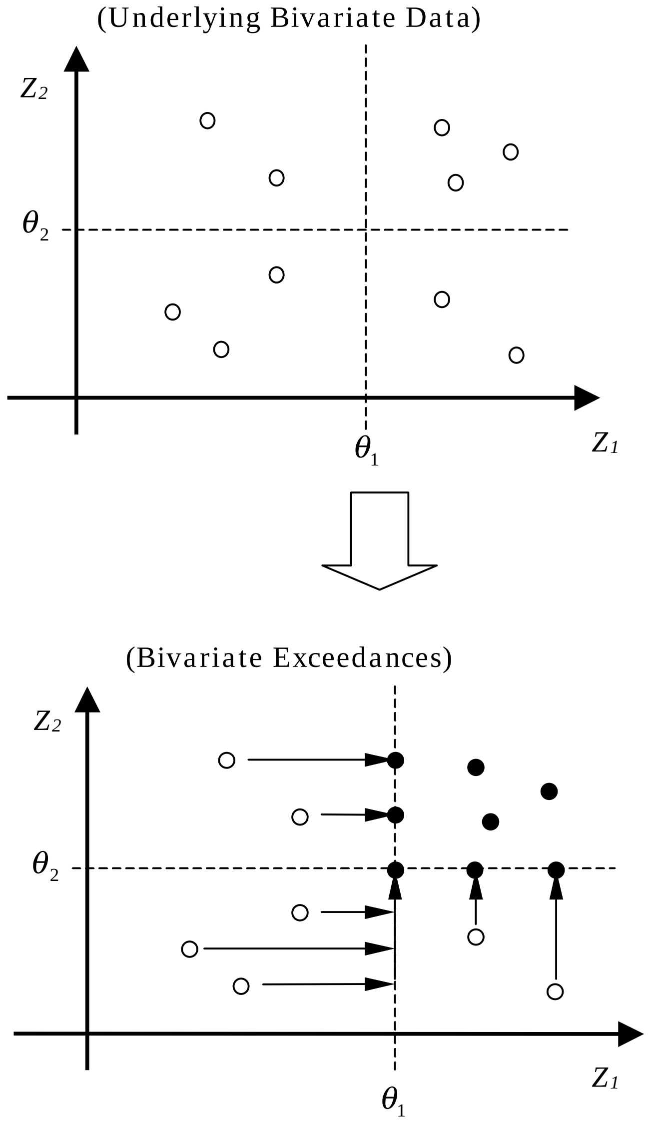
Note: Where the tail probability is $p = 0.1$, the threshold value is $\theta = 0$, and the scale parameter is $\sigma = 1$.

Figure 4 Distribution of Exceedances with Varied Scale Parameters



Note: Where the tail probability is $p = 0.1$, the threshold value is $\theta = 0$, and the tail index is $\xi = 0.25$.

Figure 5 Image Diagram of Bivariate Exceedances



Source: Based on Reiss and Thomas [2000], Figure 10.1.
Note: The white circles represent the values of the underlying bivariate data and the black circles represent their exceedances.

Table 3 Asymptotic Dependence and Dependence Measures χ and $\bar{\chi}$

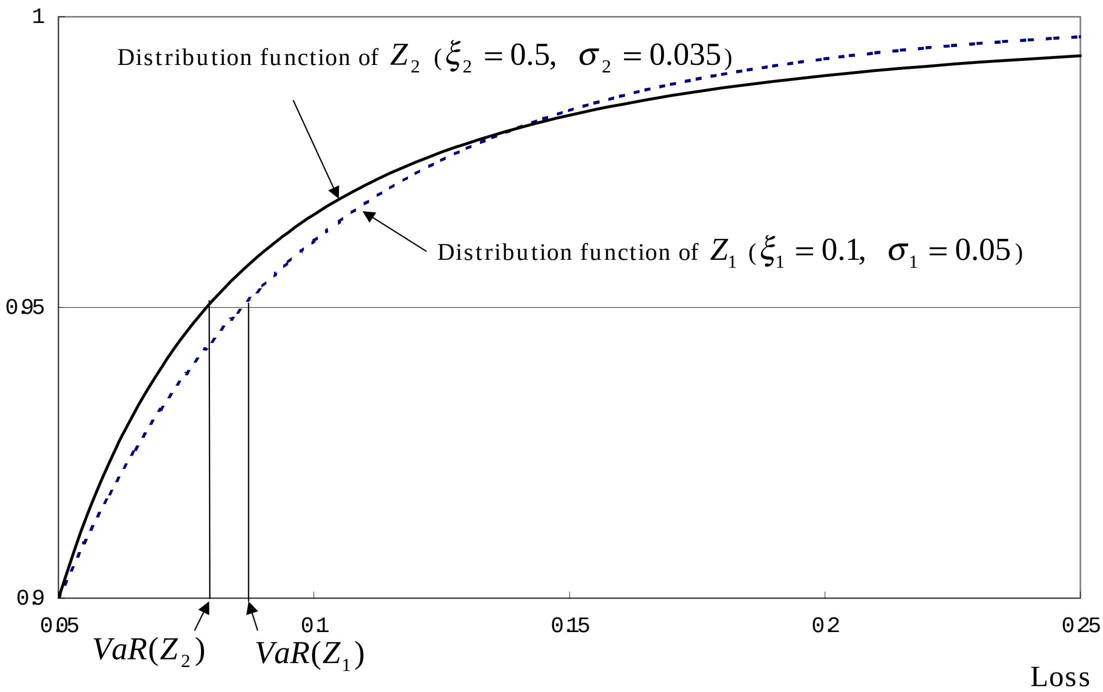
	Independent	Asymptotically Independent	Asymptotically Dependent
χ	$\chi = 0$	$\chi = 0$	$0 < \chi \leq 1$
$\bar{\chi}$	$\bar{\chi} = 0$	$-1 < \bar{\chi} < 1$	$\bar{\chi} = 1$
Reference	Represented by the extreme value copula	Not represented by the extreme value copula	Represented by the extreme value copula

Note: When independent, $\bar{\chi} = 0$. But the reverse is not necessarily true.

Table 4 Properties of the Copulas used in this Paper

	Equation	Dependence Structure	χ	$\bar{\chi}$
Gumbel	$C(u,v)=\exp\{-[(-\log u)^{\alpha}+(-\log v)^{\alpha}]^{1/\alpha}\}$	Independent when $\alpha=1$ Fully dependent when $\alpha=\infty$	$\chi=2-2^{1/\alpha}$ $(\alpha\geq 1)$	$\bar{\chi}=1$
Gaussian	$C(u,v)=\Phi_{\rho}(\Phi^{-1}(u),\Phi^{-1}(v))$	Independent when $\rho=0$ Fully dependent when $\rho=\pm 1$	$\chi=0$ $(-1<\rho<1)$	$\bar{\chi}=\rho$
Frank	$C(u,v)=-\frac{1}{\delta}\ln\left(\frac{1-e^{-\delta}-(1-e^{-\delta u})(1-e^{-\delta v})}{1-e^{-\delta}}\right)$	Independent when $\delta=0$ Fully dependent when $\delta=\pm\infty$	$\chi=0$	$\bar{\chi}=0$

Figure 6 Example Plot of the Distribution of Exceedances



Note: The tail probability is $p_1=p_2=0.1$ and the threshold value is $\theta_1=\theta_2=0.05$.

Table 5 Threshold Value $\bar{\kappa}_{VaR}$ for the Tail Risk of VaR

(Tail Probability: $p = 0.1$, Confidence Level: 95%)

		ξ_1									
		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ξ_2	0.10	-	-	-	-	-	-	-	-	-	-
	0.20	1.036	-	-	-	-	-	-	-	-	-
	0.30	1.073	1.036	-	-	-	-	-	-	-	-
	0.40	1.113	1.074	1.037	-	-	-	-	-	-	-
	0.50	1.154	1.114	1.075	1.037	-	-	-	-	-	-
	0.60	1.198	1.156	1.116	1.076	1.038	-	-	-	-	-
	0.70	1.243	1.200	1.158	1.117	1.077	1.038	-	-	-	-
	0.80	1.291	1.246	1.202	1.160	1.118	1.078	1.038	-	-	-
	0.90	1.341	1.294	1.249	1.205	1.162	1.120	1.079	1.039	-	-
	1.00	1.393	1.345	1.298	1.252	1.207	1.163	1.121	1.079	1.039	-

(Tail Probability: $p = 0.1$, Confidence Level: 99%

		ξ_1									
		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ξ_2	0.10	-	-	-	-	-	-	-	-	-	-
	0.20	1.129	-	-	-	-	-	-	-	-	-
	0.30	1.281	1.134	-	-	-	-	-	-	-	-
	0.40	1.460	1.292	1.139	-	-	-	-	-	-	-
	0.50	1.670	1.479	1.304	1.144	-	-	-	-	-	-
	0.60	1.919	1.699	1.498	1.315	1.149	-	-	-	-	-
	0.70	2.213	1.960	1.728	1.516	1.325	1.154	-	-	-	-
	0.80	2.563	2.269	2.001	1.756	1.535	1.336	1.158	-	-	-
	0.90	2.980	2.638	2.325	2.041	1.784	1.553	1.346	1.162	-	-
	1.00	3.476	3.077	2.713	2.381	2.081	1.811	1.570	1.356	1.167	-

(Tail Probability: $p = 0.05$, Confidence Level: 99%

		ξ_1									
		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ξ_2	0.10	-	-	-	-	-	-	-	-	-	-
	0.20	1.087	-	-	-	-	-	-	-	-	-
	0.30	1.185	1.090	-	-	-	-	-	-	-	-
	0.40	1.294	1.190	1.092	-	-	-	-	-	-	-
	0.50	1.416	1.302	1.195	1.094	-	-	-	-	-	-
	0.60	1.552	1.428	1.310	1.200	1.097	-	-	-	-	-
	0.70	1.706	1.569	1.440	1.319	1.205	1.099	-	-	-	-
	0.80	1.878	1.727	1.585	1.452	1.327	1.210	1.101	-	-	-
	0.90	2.072	1.906	1.749	1.602	1.464	1.335	1.215	1.103	-	-
	1.00	2.291	2.107	1.933	1.771	1.618	1.476	1.343	1.220	1.105	-

Note: VaR has tail risk when σ_1/σ_2 is more than $\bar{\kappa}_{VaR}$.

Figure 7 Varied Scale Parameters and the Tail Risk of VaR

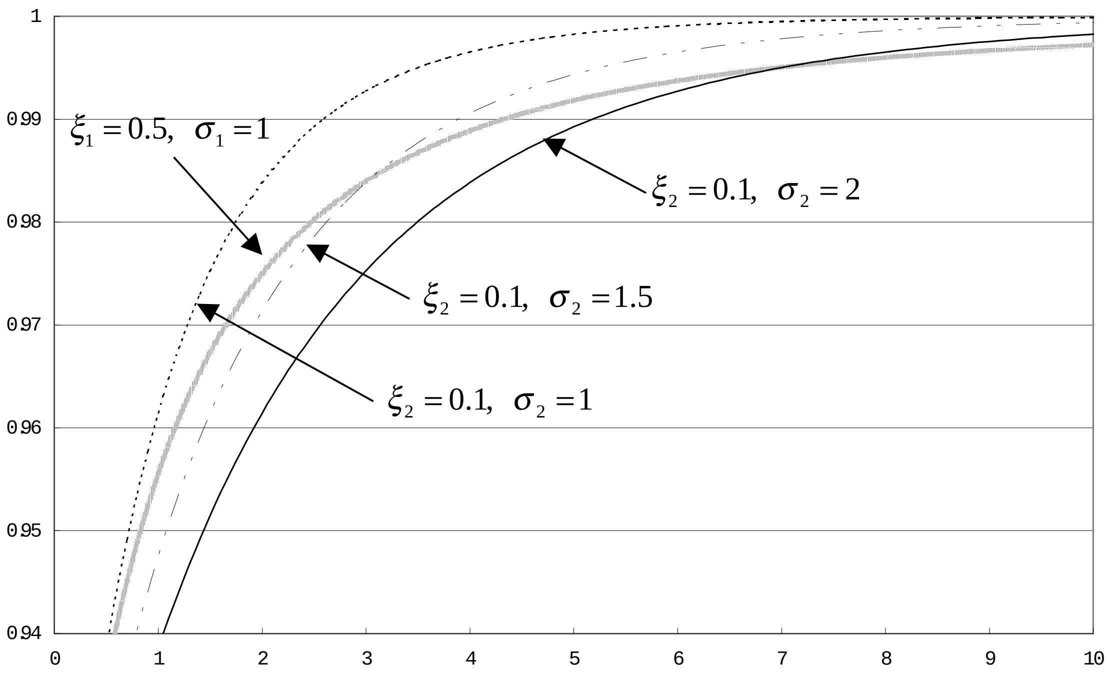


Figure 8 Varied Tail Indices and the Tail Risk of VaR

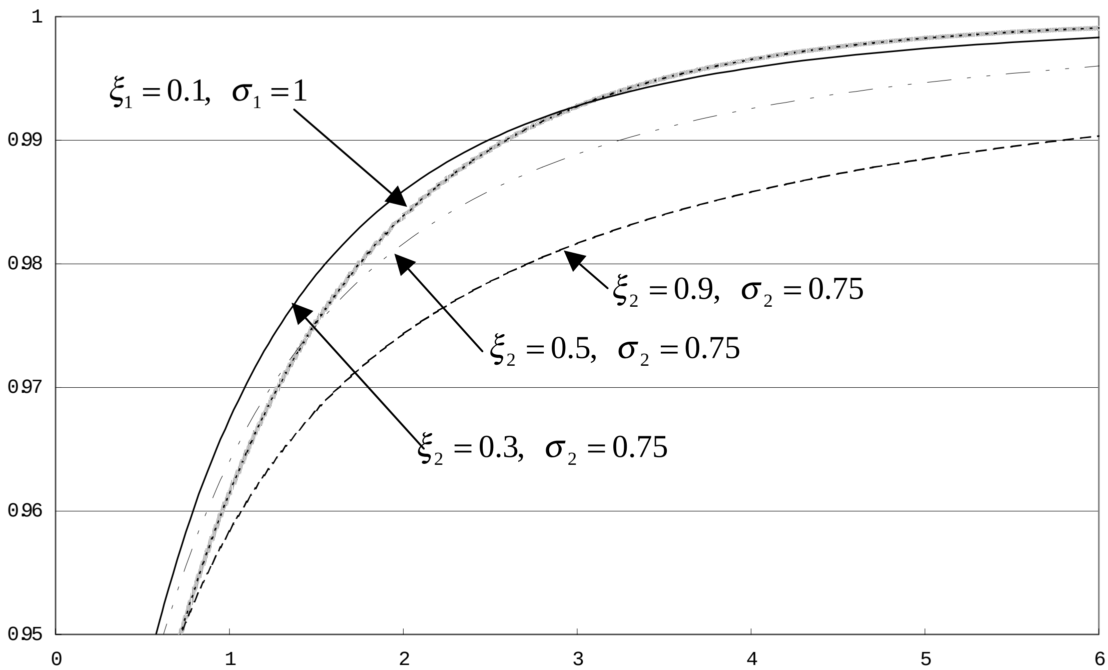


Table 6 Threshold Value $\bar{\kappa}_{ES}$ for the Tail Risk of Expected Shortfall

(Tail Probability: $p=0.1$, Confidence Level: 95%)

		ξ_1									
		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ξ_2	0.10	-	-	-	-	-	-	-	-	-	-
	0.20	1.142	-	-	-	-	-	-	-	-	-
	0.30	1.325	1.161	-	-	-	-	-	-	-	-
	0.40	1.571	1.376	1.185	-	-	-	-	-	-	-
	0.50	1.916	1.678	1.446	1.220	-	-	-	-	-	-
	0.60	2.436	2.133	1.838	1.551	1.271	-	-	-	-	-
	0.70	3.305	2.894	2.494	2.104	1.725	1.357	-	-	-	-
	0.80	5.047	4.420	3.808	3.213	2.634	2.072	1.527	-	-	-
	0.90	10.281	9.004	7.758	6.545	5.366	4.221	3.111	2.037	-	-
	1.00	-	-	-	-	-	-	-	-	-	-

(Tail Probability: $p=0.1$, Confidence Level: 99%)

		ξ_1									
		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ξ_2	0.10	-	-	-	-	-	-	-	-	-	-
	0.20	1.230	-	-	-	-	-	-	-	-	-
	0.30	1.547	1.257	-	-	-	-	-	-	-	-
	0.40	1.998	1.624	1.292	-	-	-	-	-	-	-
	0.50	2.670	2.171	1.727	1.337	-	-	-	-	-	-
	0.60	3.741	3.042	2.419	1.873	1.401	-	-	-	-	-
	0.70	5.626	4.574	3.638	2.817	2.107	1.504	-	-	-	-
	0.80	9.575	7.784	6.191	4.793	3.586	2.559	1.702	-	-	-
	0.90	21.852	17.765	14.129	10.940	8.184	5.841	3.884	2.282	-	-
	1.00	-	-	-	-	-	-	-	-	-	-

(Tail Probability: $p=0.05$, Confidence Level: 99%)

		ξ_1									
		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ξ_2	0.10	-	-	-	-	-	-	-	-	-	-
	0.20	1.187	-	-	-	-	-	-	-	-	-
	0.30	1.437	1.210	-	-	-	-	-	-	-	-
	0.40	1.780	1.499	1.239	-	-	-	-	-	-	-
	0.50	2.276	1.917	1.584	1.278	-	-	-	-	-	-
	0.60	3.040	2.560	2.116	1.708	1.336	-	-	-	-	-
	0.70	4.347	3.660	3.025	2.442	1.910	1.430	-	-	-	-
	0.80	7.013	5.906	4.881	3.940	3.082	2.307	1.613	-	-	-
	0.90	15.136	12.747	10.535	8.503	6.651	4.978	3.482	2.158	-	-
	1.00	-	-	-	-	-	-	-	-	-	-

Note: Expected shortfall has tail risk when σ_1/σ_2 is more than $\bar{\kappa}_{ES}$. When $\xi=1$, we are unable to calculate expected shortfall as the first moment diverges.

Figure 9 Upward Bias when Using Exceedances for Risk Measurement

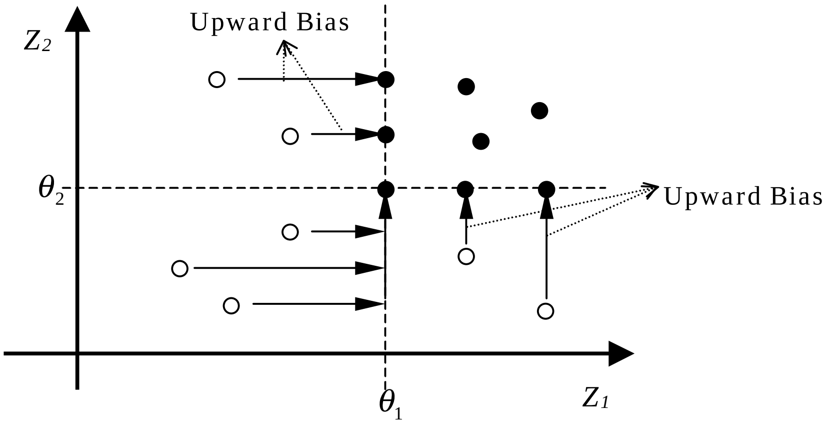


Figure 10 The Marginal Distribution Function Assumed in This Paper

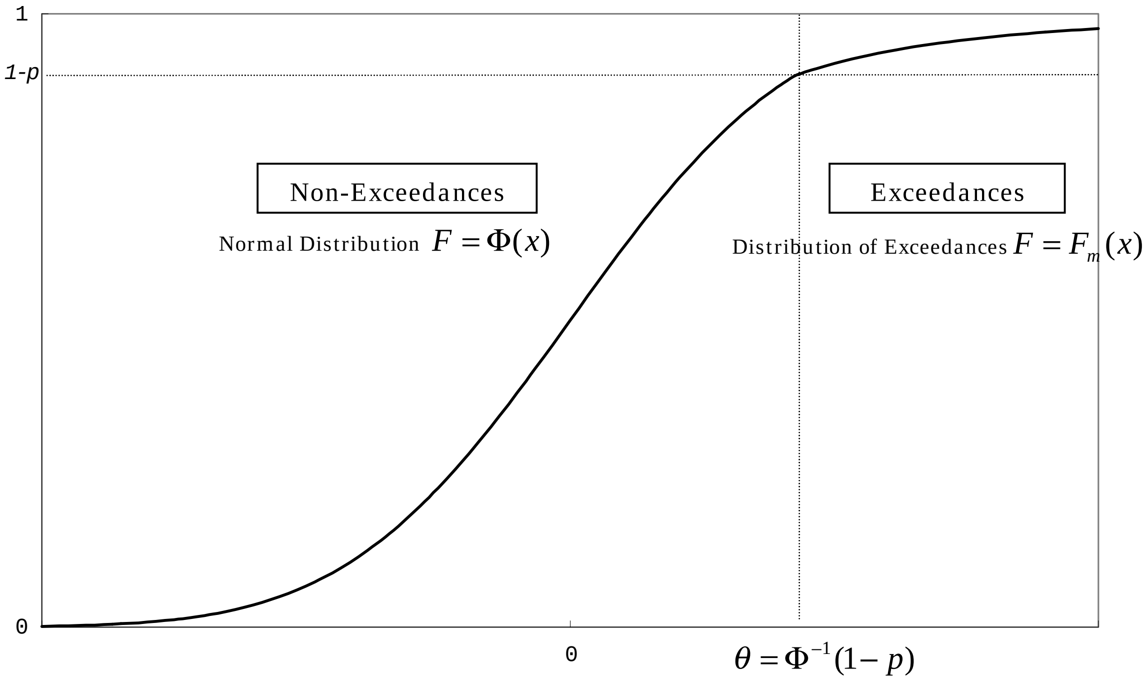
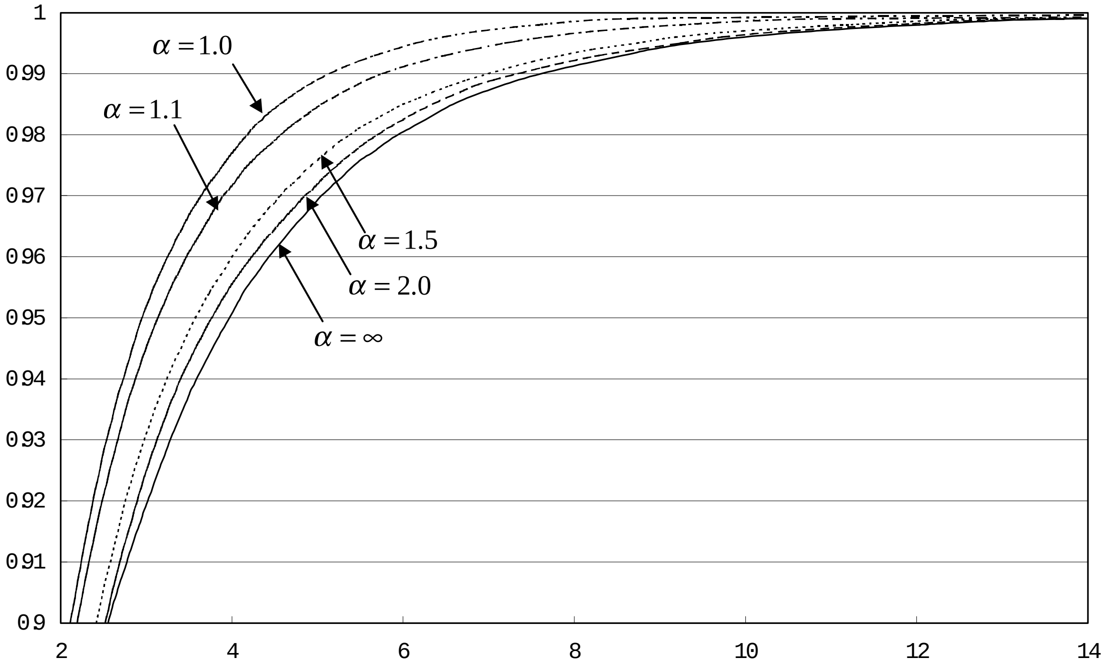


Figure 11 Empirical Distribution Functions of the Sums under the Gumbel Copula



Note: Empirical distributions are plotted from one million simulations with the marginal distribution parameters set at $\xi = 0.5$, $\sigma = 1$, $p = 0.1$.

Table 7 VaR and Expected Shortfall under Changes in the Dependence Parameter Using a Specific Copula

Gumbel $\xi = 0.1$

α	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
1.0	2.971	5.165	8.748	4.357	6.715	10.670
1.1	3.150	5.777	10.724	4.852	7.915	13.702
1.2	3.299	6.252	11.822	5.189	8.623	14.974
1.3	3.412	6.563	12.429	5.425	9.071	15.676
1.4	3.505	6.798	12.861	5.597	9.374	16.117
1.5	3.577	6.980	13.111	5.725	9.586	16.410
1.6	3.634	7.087	13.295	5.822	9.740	16.615
1.7	3.682	7.178	13.417	5.898	9.857	16.767
1.8	3.718	7.247	13.485	5.958	9.948	16.886
1.9	3.748	7.307	13.547	6.007	10.020	16.983
2.0	3.772	7.357	13.602	6.048	10.078	17.060
5.0	3.957	7.672	13.966	6.311	10.417	17.561
10.0	3.981	7.694	14.033	6.342	10.456	17.595
∞	3.993	7.703	14.219	6.352	10.502	17.613

Gaussian $\xi = 0.1$

ρ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	2.971	5.165	8.748	4.357	6.715	10.670
0.1	3.124	5.435	9.275	4.585	7.086	11.257
0.2	3.250	5.687	9.747	4.786	7.423	11.842
0.3	3.366	5.932	10.262	4.986	7.770	12.473
0.4	3.476	6.180	10.798	5.183	8.129	13.159
0.5	3.576	6.424	11.324	5.380	8.505	13.891
0.6	3.671	6.671	11.939	5.577	8.898	14.663
0.7	3.761	6.923	12.507	5.775	9.309	15.464
0.8	3.842	7.198	13.132	5.978	9.736	16.288
0.9	3.921	7.501	13.727	6.189	10.172	17.149
1	3.993	7.703	14.219	6.352	10.502	17.613

Frank $\xi = 0.1$

δ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	2.971	5.165	8.748	4.357	6.715	10.670
1	3.171	5.438	9.071	4.600	7.017	11.025
2	3.348	5.687	9.392	4.817	7.290	11.344
3	3.492	5.901	9.656	5.000	7.524	11.618
4	3.607	6.074	9.875	5.153	7.720	11.852
5	3.699	6.226	10.056	5.278	7.884	12.049
6	3.770	6.349	10.217	5.380	8.022	12.218
7	3.828	6.451	10.362	5.466	8.141	12.363
8	3.874	6.539	10.484	5.538	8.245	12.489
9	3.914	6.614	10.599	5.600	8.337	12.601
∞	3.993	7.703	14.219	6.352	10.502	17.613

Note: VaR and expected shortfall are calculated from one million simulations for each copula with the marginal distribution parameters set at $\sigma = 1$, $p = 0.1$. The tail index values are shown in the upper left of each table.

Gumbel $\xi = 0.25$

α	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
1.0	3.125	6.065	12.465	5.083	8.858	17.463
1.1	3.302	6.694	14.824	5.595	10.170	21.106
1.2	3.437	7.162	16.085	5.949	10.994	23.018
1.3	3.543	7.501	16.986	6.200	11.538	24.174
1.4	3.628	7.745	17.557	6.384	11.920	24.944
1.5	3.696	7.920	18.004	6.521	12.195	25.479
1.6	3.750	8.049	18.214	6.626	12.398	25.863
1.7	3.792	8.152	18.429	6.708	12.554	26.154
1.8	3.827	8.231	18.594	6.773	12.675	26.383
1.9	3.852	8.284	18.652	6.827	12.773	26.568
2.0	3.874	8.339	18.732	6.871	12.852	26.718
5.0	4.036	8.699	19.286	7.159	13.330	27.802
10.0	4.059	8.726	19.414	7.194	13.388	27.911
∞	4.071	8.735	19.778	7.206	13.454	27.837

Gaussian $\xi = 0.25$

ρ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	3.125	6.065	12.465	5.083	8.858	17.463
0.1	3.288	6.354	13.068	5.330	9.284	18.200
0.2	3.412	6.618	13.669	5.542	9.657	18.876
0.3	3.529	6.886	14.259	5.753	10.051	19.682
0.4	3.635	7.152	14.947	5.964	10.468	20.593
0.5	3.730	7.412	15.689	6.176	10.914	21.629
0.6	3.819	7.667	16.531	6.388	11.395	22.804
0.7	3.900	7.938	17.371	6.602	11.913	24.111
0.8	3.967	8.218	18.229	6.822	12.469	25.539
0.9	4.027	8.541	19.083	7.052	13.058	27.123
1	4.071	8.735	19.778	7.206	13.454	27.837

Frank $\xi = 0.25$

δ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	3.125	6.065	12.465	5.083	8.858	17.463
1	3.328	6.345	12.869	5.335	9.180	17.847
2	3.506	6.608	13.170	5.561	9.478	18.210
3	3.654	6.847	13.453	5.755	9.739	18.531
4	3.770	7.034	13.740	5.916	9.960	18.803
5	3.863	7.202	14.000	6.050	10.145	19.037
6	3.935	7.340	14.168	6.159	10.302	19.237
7	3.991	7.451	14.308	6.250	10.437	19.409
8	4.035	7.554	14.468	6.328	10.556	19.566
9	4.071	7.641	14.598	6.394	10.662	19.705
∞	4.071	8.735	19.778	7.206	13.454	27.837

Note: VaR and expected shortfall are calculated from one million simulations for each copula with the marginal distribution parameters set at $\sigma = 1,\ \ p = 0.1$. The tail index values are shown in the upper left of each table.

Gumbel $\xi = 0.5$

α	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
1.0	3.442	8.441	27.131	7.419	17.092	53.729
1.1	3.595	9.024	30.316	7.929	18.507	57.999
1.2	3.715	9.501	31.524	8.310	19.550	61.639
1.3	3.800	9.850	32.876	8.585	20.293	64.136
1.4	3.873	10.078	34.013	8.789	20.839	65.995
1.5	3.927	10.268	34.691	8.942	21.249	67.384
1.6	3.972	10.398	35.156	9.060	21.563	68.453
1.7	4.005	10.501	35.501	9.153	21.811	69.273
1.8	4.033	10.576	35.800	9.229	22.007	69.936
1.9	4.051	10.632	35.911	9.290	22.168	70.512
2.0	4.068	10.682	36.003	9.341	22.301	70.991
5.0	4.186	11.084	36.846	9.701	23.260	75.714
10.0	4.203	11.106	37.187	9.759	23.447	76.842
∞	4.213	11.115	38.301	9.755	23.448	75.100

Gaussian $\xi = 0.5$

ρ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	3.442	8.441	27.131	7.419	17.092	53.729
0.1	3.615	8.803	28.052	7.728	17.747	55.729
0.2	3.739	9.077	28.675	7.968	18.231	57.090
0.3	3.851	9.379	29.438	8.209	18.763	58.693
0.4	3.949	9.679	30.552	8.451	19.337	60.525
0.5	4.037	9.943	31.695	8.693	19.947	62.477
0.6	4.106	10.216	32.864	8.934	20.614	64.683
0.7	4.167	10.481	34.683	9.176	21.358	67.279
0.8	4.207	10.753	36.224	9.425	22.204	70.588
0.9	4.230	11.062	37.467	9.691	23.159	74.816
1	4.213	11.115	38.301	9.755	23.448	75.100

Frank $\xi = 0.5$

δ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	3.442	8.441	27.131	7.419	17.092	53.729
1	3.643	8.751	27.474	7.686	17.449	54.247
2	3.821	9.042	27.927	7.930	17.793	54.692
3	3.973	9.299	28.258	8.141	18.105	55.133
4	4.093	9.521	28.649	8.318	18.375	55.491
5	4.185	9.691	29.054	8.465	18.601	55.791
6	4.255	9.861	29.387	8.587	18.792	56.074
7	4.308	10.004	29.730	8.688	18.955	56.312
8	4.351	10.110	29.853	8.774	19.101	56.522
9	4.382	10.212	29.870	8.847	19.233	56.723
∞	4.213	11.115	38.301	9.755	23.448	75.100

Note: VaR and expected shortfall are calculated from one million simulations for each copula with the marginal distribution parameters set at $\sigma = 1,\ \ p = 0.1$. The tail index values are shown in the upper left of each table.

Gumbel $\xi = 0.75$

α	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
1.0	3.847	12.654	68.724	14.106	45.232	232.931
1.1	3.961	13.076	73.107	14.193	44.817	220.165
1.2	4.054	13.468	74.485	14.574	46.065	226.977
1.3	4.117	13.752	74.705	14.878	47.107	232.902
1.4	4.167	13.980	75.957	15.110	47.934	237.781
1.5	4.209	14.130	78.154	15.288	48.578	241.740
1.6	4.243	14.277	77.773	15.427	49.087	244.924
1.7	4.263	14.314	78.758	15.540	49.504	247.554
1.8	4.278	14.362	79.165	15.633	49.861	249.744
1.9	4.291	14.373	79.241	15.713	50.164	251.761
2.0	4.302	14.380	78.839	15.781	50.431	253.590
5.0	4.355	14.716	78.988	16.542	53.710	282.245
10.0	4.364	14.714	80.040	16.844	55.155	295.725
∞	4.373	14.720	83.395	16.517	53.579	275.707

Gaussian $\xi = 0.75$

ρ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	3.847	12.654	68.724	14.106	45.232	232.931
0.1	4.026	13.186	70.836	14.668	47.050	243.941
0.2	4.145	13.434	71.028	15.092	48.451	254.063
0.3	4.254	13.751	72.412	15.531	49.982	265.049
0.4	4.344	14.094	74.657	15.921	51.324	273.791
0.5	4.411	14.387	77.344	16.217	52.268	278.229
0.6	4.468	14.556	78.944	16.429	52.907	279.463
0.7	4.493	14.736	81.197	16.610	53.526	280.122
0.8	4.500	14.931	83.456	16.802	54.359	283.397
0.9	4.468	15.092	84.647	17.040	55.548	291.229
1	4.373	14.720	83.395	16.517	53.579	275.707

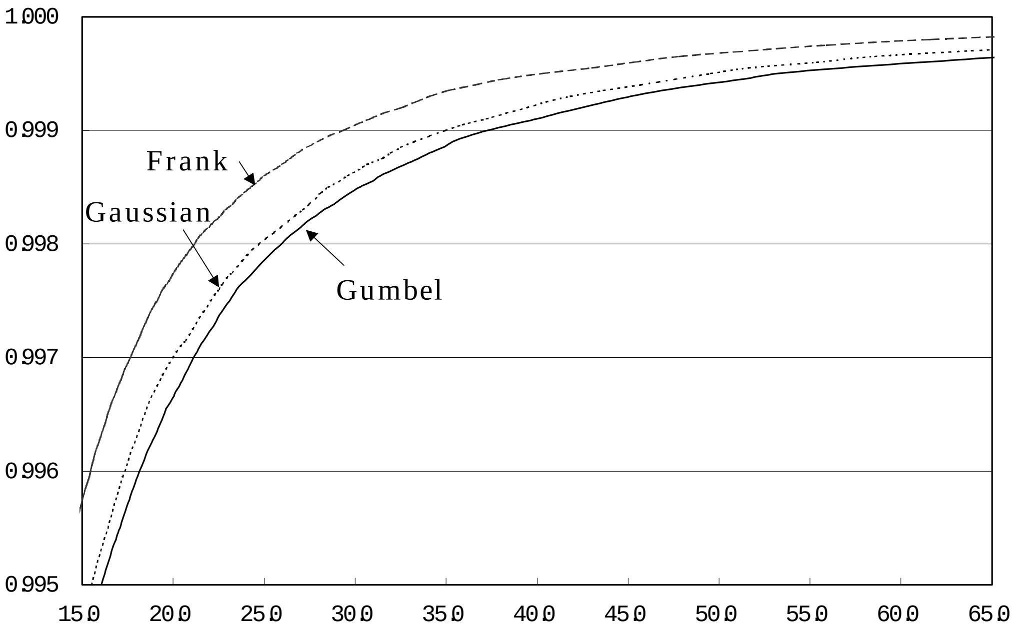
Frank $\xi = 0.75$

δ	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
0	3.847	12.654	68.724	14.106	45.232	232.931
1	4.051	12.988	68.816	14.397	45.680	234.046
2	4.229	13.318	69.598	14.665	46.116	234.846
3	4.376	13.620	70.071	14.897	46.507	235.493
4	4.494	13.879	70.484	15.091	46.843	235.963
5	4.580	14.069	70.999	15.251	47.117	236.298
6	4.650	14.258	71.637	15.383	47.344	236.603
7	4.703	14.398	73.037	15.493	47.537	236.907
8	4.739	14.515	72.559	15.587	47.708	237.163
9	4.767	14.634	72.669	15.669	47.873	237.456
∞	4.373	14.720	83.395	16.517	53.579	275.707

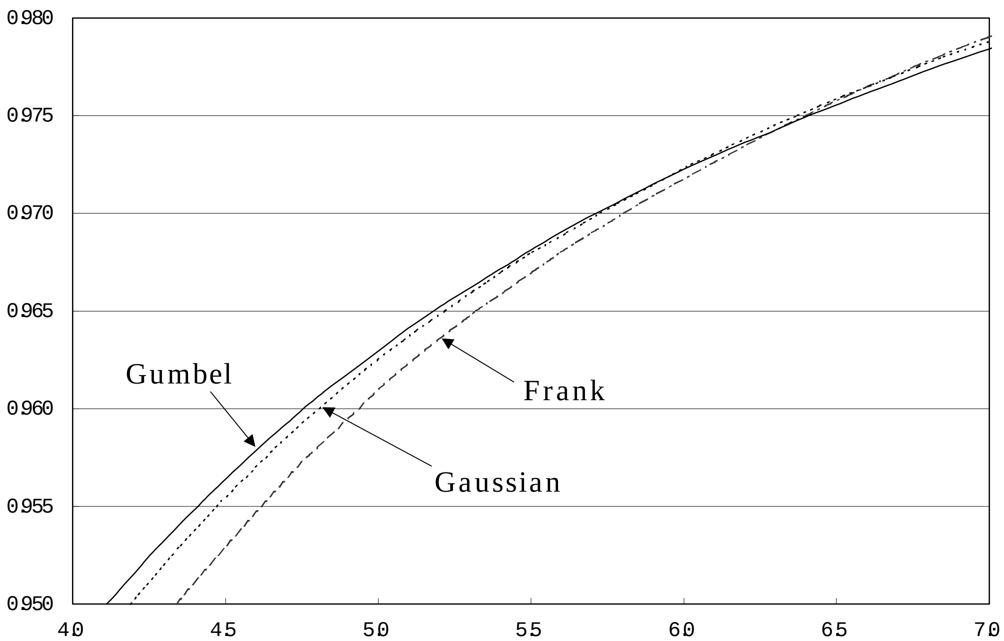
Note: VaR and expected shortfall are calculated from one million simulations for each copula with the marginal distribution parameters set at $\sigma = 1$, $p = 0.1$. The tail index values are shown in the upper left of each table.

Figure 12 Empirical Distributions under Gumbel, Gaussian, and Frank Copulas

(Portion with a Cumulative Probability of at Least 99.5%)



(Portion with a Cumulative Probability of 95 – 98%)



Note: The marginal distribution parameters are set at $\xi = 0.5,\ \sigma = 1,\ p = 0.1$. The empirical distributions are generated by conducting one million simulations for each copula. For all of the copula parameters, the Spearman's rho is set at $\rho_S = 0.5$.

Table 8 VaR and Expected Shortfall under Different Copulas

$\xi = 0.1$

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Independent	2.971	5.165	8.748	4.357	6.715	10.670
Fully Dependent	3.993	7.703	14.219	6.352	10.502	17.613

Spearman's rho=0.2

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.212	5.493	9.152	4.651	7.080	11.098
Gaussian	3.261	5.709	9.784	4.804	7.454	11.897
Gumbel	3.245	6.080	11.426	5.069	8.381	14.566

Spearman's rho=0.5

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.547	5.982	9.770	5.073	7.617	11.728
Gaussian	3.594	6.463	11.425	5.416	8.575	14.027
Gumbel	3.601	7.024	13.184	5.766	9.653	16.500

Spearman's rho=0.8

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.869	6.529	10.478	5.531	8.235	12.477
Gaussian	3.851	7.236	13.207	6.005	9.792	16.399
Gumbel	3.858	7.526	13.836	6.185	10.261	17.312

$\xi = 0.25$

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Independent	3.125	6.065	12.465	5.083	8.858	17.463
Fully Dependent	4.071	8.735	19.778	7.206	13.454	27.837

Spearman's rho=0.2

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.369	6.403	12.914	5.387	9.248	17.927
Gaussian	3.422	6.643	13.728	5.561	9.691	18.944
Gumbel	3.389	6.988	15.598	5.822	10.707	22.383

Spearman's rho=0.5

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.711	6.934	13.600	5.831	9.843	18.659
Gaussian	3.747	7.455	15.861	6.214	10.998	21.830
Gumbel	3.720	7.979	18.086	6.566	12.284	25.647

Spearman's rho=0.8

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	4.031	7.544	14.456	6.320	10.544	19.551
Gaussian	3.974	8.263	18.334	6.851	12.544	25.735
Gumbel	3.949	8.526	19.090	7.020	13.106	27.229

Note: VaR and expected shortfall are calculated by conducting one million simulations for each copula. The marginal distribution parameters are set at $\sigma = 1, \quad p = 0.1$.

$\xi = 0.5$

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Independent	3.442	8.441	27.131	7.419	17.092	53.729
Fully Dependent	4.213	11.115	38.301	9.755	23.448	75.100

Spearman's rho=0.2

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.684	8.812	27.657	7.742	17.527	54.353
Gaussian	3.748	9.105	28.750	7.989	18.277	57.226
Gumbel	3.672	9.325	31.444	8.172	19.177	60.376

Spearman's rho=0.5

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	4.031	9.407	28.422	8.224	18.232	55.300
Gaussian	4.052	9.988	31.896	8.736	20.062	62.854
Gumbel	3.947	10.332	34.770	8.993	21.384	67.848

Spearman's rho=0.8

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	4.347	10.100	29.790	8.766	19.087	56.502
Gaussian	4.211	10.798	36.249	9.459	22.322	71.084
Gumbel	4.119	10.888	36.572	9.518	22.757	72.817

$\xi = 0.75$

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Independent	3.847	12.654	68.724	14.106	45.232	232.931
Fully Dependent	4.373	14.720	83.395	16.517	53.579	275.707

Spearman's rho=0.2

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	4.092	13.022	69.291	14.459	45.778	234.268
Gaussian	4.157	13.465	71.011	15.131	48.589	255.050
Gumbel	4.028	13.288	73.602	14.429	45.581	224.180

Spearman's rho=0.5

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	4.433	13.722	70.411	14.989	46.666	235.724
Gaussian	4.424	14.411	77.312	16.260	52.397	278.633
Gumbel	4.222	14.188	79.041	15.348	48.795	243.099

Spearman's rho=0.8

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	4.737	14.512	72.461	15.578	47.691	237.134
Gaussian	4.496	14.932	83.944	16.830	54.489	284.102
Gumbel	4.326	14.549	80.106	16.057	51.537	261.953

Note: VaR and expected shortfall are calculated by conducting one million simulations for each copula. The marginal distribution parameters are set at $\sigma = 1, \quad p = 0.1$.

Table 9 VaR and Expected Shortfall under Different Copula for Different Marginal Distributions (Example)

($\xi_1 = 0.1,\ \ \xi_2 = 0.1,\ \ \sigma_1 = 1,\ \ \sigma_2 = 2,\ \ \rho_S = 0.2,\ \ p = 0.1$)

	VaR(95%)	VaR(99%)	VaR(99.9%)	ES(95%)	ES(99%)	ES(99.9%)
Frank	3.8542	7.4521	13.7438	6.1475	10.1535	17.1047
Gaussian	3.8806	7.7226	14.3812	6.3062	10.5660	17.7964
Gumbel	3.8569	8.1702	16.3774	6.6234	11.7285	21.3039

Table 10 Estimation of the Parameters of the Distribution of Exceedances of Daily Log Changes of the Foreign Exchange Rates (per one US Dollar)

• Developed Countries

Japan (Yen)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.3988	0.0085	0.0169	-0.0024	0.0092	0.0169	0.0230
2%	0.0485	0.0048	0.0141	0.0097	0.0146	0.0174	0.0226
3%	-0.0169	0.0054	0.0117	0.0090	0.0143	0.0176	0.0228
4%	0.1482	0.0040	0.0110	0.0101	0.0146	0.0171	0.0228
5%	0.1126	0.0039	0.0102	0.0102	0.0145	0.0170	0.0223
6%	0.0767	0.0042	0.0092	0.0100	0.0146	0.0173	0.0225
7%	0.0767	0.0042	0.0086	0.0100	0.0146	0.0173	0.0225
8%	0.0950	0.0039	0.0081	0.0100	0.0146	0.0172	0.0224
9%	0.0761	0.0041	0.0076	0.0100	0.0146	0.0173	0.0225
10%	0.0796	0.0040	0.0072	0.0100	0.0146	0.0173	0.0225

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.0162	0.0094	0.0199	0.0045	0.0140	0.0198	0.0290
2%	0.0875	0.0071	0.0157	0.0093	0.0166	0.0207	0.0290
3%	0.0996	0.0067	0.0128	0.0095	0.0166	0.0206	0.0289
4%	0.1083	0.0064	0.0111	0.0097	0.0166	0.0206	0.0289
5%	0.1880	0.0054	0.0101	0.0101	0.0167	0.0202	0.0292
6%	0.1647	0.0054	0.0091	0.0101	0.0168	0.0204	0.0291
7%	0.1484	0.0053	0.0083	0.0101	0.0166	0.0202	0.0285
8%	0.1603	0.0052	0.0075	0.0101	0.0167	0.0204	0.0290
9%	0.2138	0.0046	0.0072	0.0101	0.0167	0.0201	0.0295
10%	0.1848	0.0048	0.0066	0.0102	0.0168	0.0203	0.0293

Germany (Mark)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0484	0.0039	0.0146	0.0085	0.0123	0.0146	0.0187
2%	0.0642	0.0036	0.0122	0.0089	0.0126	0.0147	0.0187
3%	0.0633	0.0035	0.0107	0.0090	0.0126	0.0147	0.0187
4%	0.0596	0.0034	0.0097	0.0090	0.0126	0.0147	0.0187
5%	0.0741	0.0033	0.0091	0.0091	0.0126	0.0147	0.0187
6%	0.0443	0.0035	0.0083	0.0090	0.0126	0.0148	0.0187
7%	-0.0326	0.0040	0.0075	0.0088	0.0127	0.0151	0.0187
8%	-0.0876	0.0045	0.0067	0.0088	0.0128	0.0153	0.0188
9%	-0.0482	0.0042	0.0064	0.0088	0.0127	0.0151	0.0188
10%	-0.0496	0.0042	0.0059	0.0088	0.0127	0.0151	0.0188

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.0958	0.0045	0.0153	0.0073	0.0122	0.0152	0.0194
2%	0.0365	0.0038	0.0127	0.0093	0.0130	0.0153	0.0193
3%	-0.0024	0.0040	0.0109	0.0088	0.0129	0.0153	0.0194
4%	-0.0721	0.0046	0.0094	0.0084	0.0128	0.0156	0.0195
5%	-0.0334	0.0044	0.0086	0.0086	0.0128	0.0154	0.0194
6%	-0.0045	0.0041	0.0080	0.0087	0.0128	0.0153	0.0194
7%	0.0137	0.0040	0.0074	0.0088	0.0128	0.0153	0.0194
8%	0.0029	0.0040	0.0069	0.0088	0.0128	0.0153	0.0194
9%	-0.0275	0.0043	0.0063	0.0088	0.0129	0.0154	0.0193
10%	-0.0226	0.0043	0.0058	0.0088	0.0129	0.0154	0.0194

UK (Pound)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.0461	0.0029	0.0114	0.0066	0.0096	0.0114	0.0142
2%	0.0777	0.0022	0.0102	0.0082	0.0104	0.0117	0.0142
3%	-0.0782	0.0030	0.0087	0.0071	0.0100	0.0118	0.0144
4%	-0.1037	0.0032	0.0077	0.0070	0.0100	0.0119	0.0144
5%	-0.1188	0.0035	0.0068	0.0068	0.0100	0.0120	0.0146
6%	-0.1120	0.0035	0.0062	0.0068	0.0100	0.0119	0.0145
7%	-0.1160	0.0036	0.0056	0.0068	0.0100	0.0120	0.0146
8%	-0.1120	0.0036	0.0052	0.0069	0.0099	0.0119	0.0145
9%	-0.0967	0.0035	0.0048	0.0069	0.0099	0.0119	0.0145
10%	-0.0770	0.0034	0.0045	0.0069	0.0099	0.0118	0.0145

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.3167	0.0023	0.0119	0.0091	0.0110	0.0119	0.0152
2%	0.1368	0.0026	0.0099	0.0076	0.0103	0.0118	0.0151
3%	0.0112	0.0033	0.0082	0.0065	0.0099	0.0119	0.0153
4%	0.0688	0.0029	0.0075	0.0069	0.0100	0.0118	0.0152
5%	0.0654	0.0029	0.0069	0.0069	0.0100	0.0118	0.0152
6%	0.0505	0.0029	0.0063	0.0068	0.0100	0.0118	0.0152
7%	0.0521	0.0029	0.0059	0.0068	0.0100	0.0118	0.0152
8%	0.0284	0.0030	0.0054	0.0068	0.0100	0.0119	0.0152
9%	0.0127	0.0031	0.0050	0.0068	0.0100	0.0120	0.0152
10%	-0.0314	0.0034	0.0045	0.0068	0.0101	0.0121	0.0152

- Asia

Hong Kong (Dollar)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.7191	0.0002	0.0006	0.0004	0.0006	0.0006	0.0012
2%	0.5968	0.0001	0.0005	0.0004	0.0006	0.0006	0.0011
3%	0.2707	0.0002	0.0003	0.0002	0.0005	0.0006	0.0010
4%	0.2644	0.0002	0.0003	0.0002	0.0005	0.0006	0.0010
5%	0.2691	0.0002	0.0002	0.0002	0.0005	0.0006	0.0010
6%	0.2847	0.0002	0.0002	0.0002	0.0005	0.0006	0.0010
7%	0.3002	0.0001	0.0002	0.0002	0.0005	0.0006	0.0009
8%	0.2942	0.0001	0.0002	0.0002	0.0005	0.0006	0.0010
9%	0.2776	0.0002	0.0001	0.0002	0.0005	0.0006	0.0010
10%	0.3097	0.0001	0.0001	0.0002	0.0005	0.0006	0.0009

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0653	0.0004	0.0006	0.0000	0.0004	0.0006	0.0011
2%	0.1254	0.0003	0.0004	0.0001	0.0004	0.0006	0.0011
3%	0.2045	0.0003	0.0003	0.0002	0.0005	0.0006	0.0010
4%	0.2474	0.0002	0.0003	0.0002	0.0005	0.0006	0.0010
5%	0.2558	0.0002	0.0002	0.0002	0.0005	0.0006	0.0010
6%	0.2757	0.0002	0.0002	0.0002	0.0005	0.0006	0.0009
7%	0.2966	0.0001	0.0002	0.0002	0.0004	0.0005	0.0009
8%	0.2875	0.0001	0.0002	0.0002	0.0005	0.0006	0.0009
9%	0.2767	0.0001	0.0001	0.0002	0.0005	0.0006	0.0009
10%	0.3071	0.0001	0.0001	0.0002	0.0004	0.0005	0.0009

Indonesia (Rupiah)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.2216	0.0419	0.0673	-0.0142	0.0349	0.0670	0.1014
2%	-0.1642	0.0436	0.0380	-0.0055	0.0381	0.0663	0.0998
3%	0.1050	0.0291	0.0279	0.0135	0.0444	0.0620	0.0986
4%	0.3070	0.0208	0.0228	0.0183	0.0463	0.0587	0.1046
5%	0.3138	0.0193	0.0183	0.0183	0.0463	0.0586	0.1051
6%	0.3457	0.0173	0.0153	0.0186	0.0466	0.0581	0.1071
7%	0.4031	0.0149	0.0134	0.0188	0.0473	0.0574	0.1121
8%	0.4179	0.0138	0.0116	0.0187	0.0476	0.0573	0.1137
9%	0.3819	0.0139	0.0097	0.0188	0.0470	0.0576	0.1097
10%	0.4088	0.0128	0.0084	0.0187	0.0475	0.0574	0.1130

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.4270	0.0187	0.0569	0.0350	0.0514	0.0567	0.0894
2%	0.1896	0.0250	0.0354	0.0142	0.0401	0.0537	0.0889
3%	0.2215	0.0213	0.0269	0.0167	0.0412	0.0536	0.0886
4%	0.2307	0.0195	0.0216	0.0174	0.0415	0.0535	0.0884
5%	0.2198	0.0189	0.0172	0.0172	0.0414	0.0537	0.0883
6%	0.2280	0.0178	0.0141	0.0174	0.0415	0.0536	0.0883
7%	0.2114	0.0178	0.0111	0.0173	0.0415	0.0539	0.0879
8%	0.2377	0.0164	0.0092	0.0174	0.0415	0.0534	0.0886
9%	0.2277	0.0163	0.0071	0.0174	0.0416	0.0537	0.0886
10%	0.2743	0.0146	0.0062	0.0173	0.0416	0.0529	0.0907

Malaysia (Ringgit)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0026	0.0121	0.0205	0.0010	0.0130	0.0204	0.0325
2%	0.0473	0.0106	0.0139	0.0044	0.0150	0.0213	0.0328
3%	-0.0210	0.0119	0.0089	0.0028	0.0146	0.0218	0.0332
4%	0.0130	0.0111	0.0060	0.0035	0.0147	0.0215	0.0330
5%	0.0581	0.0100	0.0041	0.0041	0.0148	0.0211	0.0328
6%	0.1713	0.0082	0.0031	0.0046	0.0148	0.0203	0.0337
7%	0.3473	0.0061	0.0025	0.0047	0.0152	0.0195	0.0378
8%	0.5202	0.0046	0.0022	0.0046	0.0167	0.0193	0.0473
9%	0.6630	0.0035	0.0019	0.0044	0.0199	0.0194	0.0644
10%	0.7371	0.0030	0.0016	0.0043	0.0232	0.0196	0.0815

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.1915	0.0152	0.0200	-0.0087	0.0086	0.0199	0.0326
2%	0.0294	0.0117	0.0113	0.0006	0.0124	0.0194	0.0318
3%	0.0059	0.0121	0.0063	0.0002	0.0123	0.0197	0.0320
4%	0.1299	0.0096	0.0043	0.0022	0.0129	0.0188	0.0320
5%	0.3236	0.0068	0.0033	0.0033	0.0133	0.0176	0.0344
6%	0.5093	0.0048	0.0027	0.0037	0.0144	0.0169	0.0414
7%	0.6266	0.0038	0.0023	0.0037	0.0161	0.0166	0.0507
8%	0.7174	0.0030	0.0019	0.0036	0.0187	0.0165	0.0643
9%	0.7713	0.0026	0.0017	0.0036	0.0213	0.0165	0.0780
10%	0.7626	0.0024	0.0014	0.0036	0.0208	0.0165	0.0754

Philippines (Peso)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.1785	0.0126	0.0224	0.0048	0.0163	0.0224	0.0376
2%	0.2242	0.0102	0.0146	0.0060	0.0168	0.0222	0.0376
3%	0.2923	0.0082	0.0112	0.0074	0.0173	0.0219	0.0378
4%	0.2938	0.0076	0.0088	0.0072	0.0173	0.0218	0.0379
5%	0.2805	0.0073	0.0071	0.0071	0.0172	0.0219	0.0378
6%	0.3479	0.0062	0.0062	0.0073	0.0174	0.0215	0.0391
7%	0.3059	0.0063	0.0050	0.0073	0.0173	0.0217	0.0381
8%	0.3465	0.0056	0.0044	0.0073	0.0174	0.0215	0.0391
9%	0.3839	0.0051	0.0039	0.0072	0.0175	0.0214	0.0404
10%	0.4156	0.0046	0.0035	0.0072	0.0177	0.0213	0.0418

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.4173	0.0083	0.0185	0.0088	0.0160	0.0185	0.0326
2%	0.3344	0.0075	0.0125	0.0065	0.0148	0.0182	0.0323
3%	0.2872	0.0072	0.0091	0.0057	0.0144	0.0184	0.0322
4%	0.3775	0.0056	0.0077	0.0065	0.0148	0.0179	0.0331
5%	0.5049	0.0042	0.0069	0.0069	0.0154	0.0173	0.0363
6%	0.4317	0.0043	0.0060	0.0068	0.0150	0.0176	0.0340
7%	0.3333	0.0048	0.0050	0.0067	0.0147	0.0181	0.0319
8%	0.3170	0.0047	0.0043	0.0067	0.0147	0.0182	0.0316
9%	0.2980	0.0047	0.0036	0.0067	0.0147	0.0183	0.0313
10%	0.2915	0.0046	0.0031	0.0067	0.0147	0.0183	0.0311

Singapore (Singapore Dollar)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.1834	0.0059	0.0103	-0.0008	0.0059	0.0103	0.0153
2%	-0.0190	0.0048	0.0070	0.0025	0.0074	0.0104	0.0151
3%	0.1158	0.0038	0.0057	0.0038	0.0078	0.0101	0.0150
4%	0.2528	0.0029	0.0049	0.0043	0.0080	0.0098	0.0153
5%	0.2665	0.0027	0.0043	0.0043	0.0080	0.0098	0.0154
6%	0.2867	0.0025	0.0039	0.0044	0.0080	0.0097	0.0155
7%	0.2710	0.0024	0.0035	0.0044	0.0080	0.0098	0.0154
8%	0.3463	0.0021	0.0033	0.0044	0.0081	0.0096	0.0161
9%	0.3118	0.0021	0.0030	0.0044	0.0081	0.0097	0.0157
10%	0.3256	0.0020	0.0028	0.0044	0.0081	0.0096	0.0159

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.1481	0.0057	0.0103	0.0021	0.0074	0.0103	0.0169
2%	0.2112	0.0045	0.0070	0.0033	0.0079	0.0103	0.0169
3%	0.2843	0.0036	0.0056	0.0039	0.0082	0.0102	0.0170
4%	0.3276	0.0030	0.0048	0.0041	0.0083	0.0101	0.0172
5%	0.3626	0.0026	0.0043	0.0043	0.0084	0.0100	0.0174
6%	0.2964	0.0028	0.0036	0.0042	0.0083	0.0102	0.0169
7%	0.3076	0.0026	0.0033	0.0042	0.0083	0.0102	0.0170
8%	0.3191	0.0024	0.0030	0.0042	0.0083	0.0101	0.0170
9%	0.3253	0.0023	0.0027	0.0042	0.0083	0.0101	0.0171
10%	0.3368	0.0022	0.0025	0.0042	0.0083	0.0101	0.0173

South Korea (Won)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.2322	0.0491	0.0222	-0.0742	-0.0162	0.0218	0.0618
2%	0.6963	0.0118	0.0136	0.0056	0.0260	0.0240	0.0867
3%	0.8231	0.0071	0.0104	0.0075	0.0341	0.0233	0.1235
4%	0.8915	0.0050	0.0090	0.0080	0.0461	0.0228	0.1830
5%	0.7355	0.0053	0.0074	0.0074	0.0276	0.0239	0.0899
6%	0.6621	0.0053	0.0063	0.0073	0.0249	0.0244	0.0756
7%	0.6627	0.0048	0.0055	0.0073	0.0249	0.0244	0.0757
8%	0.7024	0.0041	0.0050	0.0073	0.0264	0.0242	0.0833
9%	0.6871	0.0039	0.0045	0.0073	0.0257	0.0243	0.0800
10%	0.6852	0.0036	0.0041	0.0073	0.0257	0.0243	0.0797

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0755	0.0280	0.0224	-0.0203	0.0066	0.0222	0.0525
2%	0.3220	0.0156	0.0122	-0.0003	0.0168	0.0242	0.0529
3%	0.5929	0.0087	0.0089	0.0051	0.0208	0.0224	0.0633
4%	0.7563	0.0056	0.0075	0.0063	0.0258	0.0213	0.0871
5%	0.8596	0.0041	0.0066	0.0066	0.0355	0.0207	0.1361
6%	0.8818	0.0034	0.0060	0.0066	0.0399	0.0206	0.1583
7%	0.7022	0.0039	0.0051	0.0065	0.0232	0.0214	0.0730
8%	0.6696	0.0038	0.0045	0.0066	0.0222	0.0215	0.0674
9%	0.6921	0.0034	0.0041	0.0065	0.0229	0.0214	0.0712
10%	0.7417	0.0029	0.0039	0.0065	0.0251	0.0214	0.0828

Taiwan (New Taiwan Dollar)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0930	0.0059	0.0086	-0.0003	0.0053	0.0086	0.0151
2%	0.2709	0.0041	0.0053	0.0020	0.0063	0.0084	0.0152
3%	0.3819	0.0030	0.0042	0.0028	0.0068	0.0083	0.0155
4%	0.3914	0.0026	0.0034	0.0029	0.0068	0.0082	0.0156
5%	0.4001	0.0024	0.0029	0.0029	0.0068	0.0082	0.0157
6%	0.3876	0.0022	0.0025	0.0029	0.0068	0.0082	0.0155
7%	0.4118	0.0020	0.0022	0.0029	0.0068	0.0082	0.0158
8%	0.4509	0.0018	0.0020	0.0029	0.0069	0.0081	0.0164
9%	0.4265	0.0018	0.0017	0.0029	0.0069	0.0082	0.0160
10%	0.4155	0.0017	0.0015	0.0029	0.0068	0.0082	0.0158

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.2632	0.0069	0.0071	-0.0069	0.0015	0.0070	0.0125
2%	-0.0737	0.0051	0.0044	-0.0004	0.0047	0.0079	0.0124
3%	0.0507	0.0040	0.0032	0.0012	0.0053	0.0077	0.0122
4%	0.4018	0.0022	0.0028	0.0024	0.0058	0.0070	0.0135
5%	0.5538	0.0016	0.0025	0.0025	0.0062	0.0068	0.0157
6%	0.5043	0.0016	0.0022	0.0025	0.0060	0.0068	0.0148
7%	0.4385	0.0016	0.0019	0.0025	0.0059	0.0069	0.0137
8%	0.4565	0.0015	0.0017	0.0025	0.0059	0.0069	0.0140
9%	0.4426	0.0014	0.0015	0.0025	0.0059	0.0069	0.0138
10%	0.3959	0.0015	0.0013	0.0025	0.0058	0.0070	0.0131

Thailand (Baht)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.3821	0.0131	0.0236	0.0078	0.0192	0.0235	0.0447
2%	0.3663	0.0102	0.0158	0.0078	0.0193	0.0238	0.0446
3%	0.4287	0.0078	0.0124	0.0088	0.0199	0.0235	0.0455
4%	0.3511	0.0080	0.0097	0.0080	0.0194	0.0240	0.0441
5%	0.2481	0.0090	0.0071	0.0071	0.0192	0.0250	0.0430
6%	0.3139	0.0076	0.0062	0.0076	0.0193	0.0244	0.0437
7%	0.3705	0.0065	0.0053	0.0077	0.0194	0.0239	0.0453
8%	0.4283	0.0056	0.0047	0.0077	0.0197	0.0237	0.0477
9%	0.4112	0.0055	0.0040	0.0077	0.0196	0.0237	0.0468
10%	0.4298	0.0051	0.0035	0.0077	0.0198	0.0237	0.0478

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.2904	0.0132	0.0229	0.0060	0.0176	0.0229	0.0414
2%	0.2742	0.0111	0.0143	0.0053	0.0172	0.0227	0.0413
3%	0.2957	0.0097	0.0099	0.0053	0.0172	0.0225	0.0415
4%	0.3043	0.0087	0.0074	0.0056	0.0173	0.0225	0.0416
5%	0.4270	0.0066	0.0063	0.0063	0.0178	0.0216	0.0445
6%	0.4958	0.0055	0.0054	0.0064	0.0183	0.0212	0.0477
7%	0.5661	0.0046	0.0048	0.0065	0.0192	0.0209	0.0526
8%	0.5245	0.0045	0.0041	0.0065	0.0186	0.0211	0.0493
9%	0.4422	0.0049	0.0033	0.0066	0.0179	0.0213	0.0444
10%	0.4430	0.0046	0.0028	0.0066	0.0179	0.0214	0.0444

• Eastern Europe

Czech (Czech Koruna)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.3606	0.0045	0.0169	0.0114	0.0153	0.0169	0.0239
2%	0.2503	0.0045	0.0135	0.0098	0.0145	0.0168	0.0240
3%	0.2221	0.0043	0.0116	0.0095	0.0145	0.0169	0.0240
4%	0.1999	0.0042	0.0103	0.0094	0.0144	0.0171	0.0240
5%	0.1833	0.0042	0.0094	0.0094	0.0145	0.0171	0.0240
6%	0.1495	0.0044	0.0084	0.0092	0.0145	0.0174	0.0241
7%	0.1319	0.0045	0.0076	0.0091	0.0145	0.0175	0.0241
8%	0.1260	0.0044	0.0070	0.0091	0.0145	0.0175	0.0241
9%	0.1209	0.0044	0.0064	0.0091	0.0145	0.0176	0.0241
10%	0.2034	0.0040	0.0061	0.0090	0.0148	0.0177	0.0257

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0091	0.0053	0.0160	0.0075	0.0127	0.0159	0.0213
2%	0.0485	0.0049	0.0125	0.0081	0.0130	0.0159	0.0212
3%	0.0786	0.0044	0.0108	0.0086	0.0132	0.0159	0.0211
4%	0.0950	0.0041	0.0098	0.0089	0.0133	0.0158	0.0210
5%	0.1036	0.0039	0.0089	0.0089	0.0133	0.0158	0.0210
6%	0.1910	0.0033	0.0084	0.0091	0.0133	0.0156	0.0214
7%	0.0911	0.0039	0.0076	0.0089	0.0134	0.0159	0.0210
8%	0.0920	0.0038	0.0071	0.0089	0.0133	0.0159	0.0210
9%	0.0832	0.0039	0.0066	0.0089	0.0134	0.0159	0.0210
10%	0.0669	0.0040	0.0061	0.0089	0.0134	0.0160	0.0210

Hungary (Forint)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.6080	0.0038	0.0151	0.0112	0.0148	0.0151	0.0248
2%	0.4775	0.0035	0.0118	0.0091	0.0135	0.0146	0.0240
3%	0.3707	0.0037	0.0098	0.0081	0.0129	0.0148	0.0237
4%	0.3213	0.0037	0.0085	0.0077	0.0128	0.0150	0.0236
5%	0.3517	0.0035	0.0076	0.0076	0.0130	0.0151	0.0246
6%	0.3412	0.0035	0.0068	0.0074	0.0131	0.0155	0.0253
7%	0.3412	0.0033	0.0063	0.0074	0.0131	0.0154	0.0252
8%	0.3397	0.0032	0.0058	0.0074	0.0131	0.0155	0.0253
9%	0.3416	0.0030	0.0055	0.0074	0.0130	0.0153	0.0251
10%	0.3373	0.0030	0.0051	0.0074	0.0131	0.0155	0.0253

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0688	0.0056	0.0129	0.0044	0.0097	0.0129	0.0189
2%	0.3084	0.0032	0.0109	0.0083	0.0118	0.0133	0.0191
3%	0.1789	0.0037	0.0090	0.0072	0.0113	0.0135	0.0190
4%	0.1508	0.0036	0.0079	0.0071	0.0113	0.0135	0.0188
5%	0.1605	0.0034	0.0072	0.0072	0.0113	0.0135	0.0188
6%	0.1296	0.0036	0.0064	0.0070	0.0113	0.0137	0.0190
7%	0.1317	0.0035	0.0058	0.0070	0.0113	0.0137	0.0190
8%	0.0990	0.0038	0.0051	0.0070	0.0114	0.0139	0.0191
9%	0.0781	0.0039	0.0046	0.0070	0.0114	0.0140	0.0191
10%	0.0497	0.0042	0.0040	0.0070	0.0115	0.0142	0.0191

Poland (Zloty)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.2344	0.0107	0.0163	-0.0047	0.0080	0.0163	0.0249
2%	-0.0221	0.0078	0.0119	0.0046	0.0125	0.0173	0.0248
3%	0.0484	0.0067	0.0095	0.0061	0.0130	0.0171	0.0246
4%	0.2413	0.0048	0.0084	0.0073	0.0134	0.0163	0.0253
5%	0.2813	0.0043	0.0074	0.0074	0.0134	0.0162	0.0256
6%	0.1926	0.0047	0.0064	0.0073	0.0134	0.0165	0.0248
7%	0.2054	0.0045	0.0057	0.0073	0.0134	0.0165	0.0249
8%	0.2809	0.0038	0.0054	0.0073	0.0134	0.0162	0.0258
9%	0.2698	0.0038	0.0049	0.0073	0.0134	0.0163	0.0256
10%	0.2999	0.0035	0.0046	0.0073	0.0134	0.0162	0.0262

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	-0.0731	0.0088	0.0137	-0.0014	0.0078	0.0136	0.0218
2%	0.2848	0.0046	0.0106	0.0068	0.0118	0.0141	0.0220
3%	0.4115	0.0033	0.0094	0.0079	0.0124	0.0139	0.0226
4%	0.3912	0.0030	0.0084	0.0077	0.0123	0.0139	0.0225
5%	0.2105	0.0039	0.0071	0.0071	0.0120	0.0146	0.0216
6%	0.1813	0.0039	0.0063	0.0071	0.0120	0.0146	0.0212
7%	0.1560	0.0041	0.0055	0.0069	0.0121	0.0149	0.0214
8%	0.1513	0.0041	0.0049	0.0069	0.0121	0.0149	0.0215
9%	0.1540	0.0040	0.0045	0.0069	0.0121	0.0149	0.0215
10%	0.2049	0.0036	0.0042	0.0069	0.0120	0.0147	0.0219

Slovakia (Slovakian Koruna)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.6071	0.0034	0.0150	0.0116	0.0148	0.0150	0.0235
2%	0.3394	0.0038	0.0121	0.0092	0.0133	0.0151	0.0222
3%	0.2923	0.0036	0.0104	0.0087	0.0131	0.0151	0.0222
4%	0.2068	0.0040	0.0090	0.0082	0.0130	0.0155	0.0222
5%	0.1710	0.0042	0.0079	0.0079	0.0130	0.0157	0.0223
6%	0.1572	0.0042	0.0071	0.0079	0.0130	0.0158	0.0224
7%	0.1134	0.0045	0.0062	0.0078	0.0131	0.0161	0.0225
8%	0.0972	0.0047	0.0055	0.0077	0.0131	0.0162	0.0226
9%	0.0885	0.0047	0.0049	0.0077	0.0132	0.0163	0.0226
10%	0.1554	0.0044	0.0044	0.0076	0.0135	0.0166	0.0241

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.1208	0.0050	0.0154	0.0081	0.0128	0.0154	0.0211
2%	0.0945	0.0050	0.0118	0.0074	0.0125	0.0154	0.0213
3%	0.0954	0.0048	0.0098	0.0074	0.0124	0.0154	0.0213
4%	0.1204	0.0044	0.0086	0.0077	0.0125	0.0152	0.0211
5%	0.0840	0.0047	0.0074	0.0074	0.0125	0.0155	0.0213
6%	0.0456	0.0050	0.0063	0.0072	0.0125	0.0157	0.0214
7%	0.0605	0.0048	0.0056	0.0073	0.0125	0.0156	0.0214
8%	0.0457	0.0049	0.0049	0.0073	0.0125	0.0157	0.0214
9%	0.0604	0.0048	0.0044	0.0073	0.0125	0.0156	0.0214
10%	0.0568	0.0048	0.0039	0.0073	0.0125	0.0156	0.0214

• Central and South America

Brazil (Real)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.3537	0.0122	0.0237	0.0087	0.0193	0.0236	0.0424
2%	1.3430	0.0025	0.0189	0.0176	0.0155	0.0217	0.0035
3%	0.9208	0.0026	0.0177	0.0166	0.0373	0.0227	0.1137
4%	0.7524	0.0027	0.0167	0.0162	0.0253	0.0232	0.0538
5%	0.6026	0.0030	0.0158	0.0158	0.0233	0.0239	0.0435
6%	0.5014	0.0032	0.0150	0.0156	0.0227	0.0244	0.0402
7%	0.3285	0.0044	0.0137	0.0152	0.0225	0.0256	0.0381
8%	0.1712	0.0065	0.0115	0.0147	0.0232	0.0277	0.0388
9%	0.0834	0.0084	0.0092	0.0143	0.0240	0.0296	0.0406
10%	0.0609	0.0090	0.0078	0.0142	0.0243	0.0302	0.0413

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.4644	0.0078	0.0174	0.0085	0.0154	0.0173	0.0318
2%	0.4784	0.0058	0.0121	0.0078	0.0149	0.0168	0.0321
3%	0.4397	0.0050	0.0100	0.0077	0.0148	0.0170	0.0315
4%	0.3022	0.0058	0.0077	0.0065	0.0143	0.0178	0.0304
5%	0.3351	0.0051	0.0067	0.0067	0.0143	0.0176	0.0307
6%	0.2791	0.0054	0.0055	0.0065	0.0143	0.0179	0.0302
7%	0.2658	0.0053	0.0046	0.0064	0.0143	0.0180	0.0301
8%	0.2595	0.0054	0.0036	0.0064	0.0146	0.0185	0.0310
9%	0.2758	0.0048	0.0033	0.0064	0.0143	0.0179	0.0302
10%	0.2698	0.0049	0.0027	0.0064	0.0144	0.0182	0.0306

Chile (Peso)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.1755	0.0038	0.0101	0.0047	0.0082	0.0101	0.0147
2%	0.1296	0.0035	0.0078	0.0047	0.0083	0.0103	0.0147
3%	0.1988	0.0029	0.0066	0.0052	0.0085	0.0102	0.0148
4%	0.2410	0.0026	0.0059	0.0053	0.0085	0.0101	0.0148
5%	0.1555	0.0028	0.0051	0.0051	0.0085	0.0103	0.0146
6%	0.1719	0.0026	0.0047	0.0052	0.0084	0.0102	0.0145
7%	0.2308	0.0023	0.0044	0.0052	0.0085	0.0101	0.0149
8%	0.2172	0.0023	0.0040	0.0052	0.0085	0.0102	0.0148
9%	0.1676	0.0025	0.0037	0.0052	0.0085	0.0102	0.0145
10%	0.1768	0.0024	0.0034	0.0052	0.0084	0.0102	0.0145

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.1465	0.0030	0.0090	0.0047	0.0075	0.0090	0.0125
2%	0.1423	0.0028	0.0068	0.0044	0.0073	0.0089	0.0124
3%	0.1466	0.0026	0.0058	0.0046	0.0074	0.0089	0.0124
4%	0.1605	0.0024	0.0051	0.0046	0.0074	0.0088	0.0124
5%	0.1223	0.0026	0.0044	0.0044	0.0073	0.0090	0.0125
6%	0.1050	0.0026	0.0039	0.0044	0.0074	0.0090	0.0126
7%	0.1225	0.0025	0.0036	0.0044	0.0073	0.0090	0.0125
8%	0.1314	0.0024	0.0033	0.0044	0.0073	0.0089	0.0125
9%	0.1109	0.0025	0.0029	0.0044	0.0074	0.0090	0.0126
10%	0.1078	0.0024	0.0027	0.0044	0.0074	0.0090	0.0125

Columbia (Peso)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.4404	0.0035	0.0136	0.0095	0.0126	0.0136	0.0199
2%	0.1930	0.0047	0.0097	0.0057	0.0106	0.0132	0.0198
3%	0.1832	0.0045	0.0078	0.0056	0.0106	0.0132	0.0199
4%	0.1950	0.0041	0.0067	0.0058	0.0106	0.0132	0.0198
5%	0.2351	0.0036	0.0060	0.0060	0.0107	0.0130	0.0199
6%	0.2376	0.0034	0.0054	0.0060	0.0107	0.0130	0.0199
7%	0.2091	0.0035	0.0048	0.0060	0.0107	0.0131	0.0196
8%	0.2156	0.0033	0.0044	0.0060	0.0106	0.0130	0.0195
9%	0.2120	0.0032	0.0040	0.0060	0.0107	0.0131	0.0196
10%	0.2197	0.0031	0.0037	0.0060	0.0106	0.0129	0.0195

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.0402	0.0044	0.0102	0.0033	0.0076	0.0102	0.0147
2%	0.0379	0.0042	0.0075	0.0038	0.0079	0.0104	0.0149
3%	0.0812	0.0037	0.0061	0.0042	0.0081	0.0103	0.0147
4%	0.1062	0.0034	0.0051	0.0044	0.0081	0.0102	0.0146
5%	0.1764	0.0030	0.0046	0.0046	0.0082	0.0101	0.0149
6%	0.1322	0.0030	0.0040	0.0046	0.0081	0.0101	0.0145
7%	0.1290	0.0030	0.0036	0.0046	0.0081	0.0101	0.0145
8%	0.1815	0.0027	0.0032	0.0046	0.0082	0.0101	0.0149
9%	0.1482	0.0027	0.0029	0.0046	0.0081	0.0100	0.0144
10%	0.1513	0.0027	0.0027	0.0046	0.0081	0.0100	0.0144

Mexico (New Peso)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.3016	0.0230	0.0299	0.0004	0.0207	0.0297	0.0626
2%	0.3313	0.0181	0.0149	0.0005	0.0204	0.0288	0.0628
3%	0.6560	0.0091	0.0115	0.0076	0.0266	0.0263	0.0809
4%	0.6746	0.0073	0.0093	0.0078	0.0272	0.0261	0.0836
5%	0.7538	0.0055	0.0082	0.0082	0.0307	0.0256	0.1014
6%	0.8536	0.0041	0.0075	0.0083	0.0414	0.0250	0.1557
7%	0.8022	0.0039	0.0068	0.0083	0.0343	0.0252	0.1200
8%	0.6683	0.0044	0.0059	0.0083	0.0265	0.0258	0.0790
9%	0.6236	0.0044	0.0053	0.0084	0.0253	0.0260	0.0720
10%	0.6459	0.0040	0.0049	0.0084	0.0259	0.0259	0.0754

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.2564	0.0168	0.0194	-0.0029	0.0121	0.0193	0.0418
2%	0.4498	0.0094	0.0130	0.0059	0.0172	0.0206	0.0438
3%	0.5355	0.0066	0.0102	0.0073	0.0182	0.0202	0.0459
4%	0.4714	0.0064	0.0081	0.0068	0.0176	0.0206	0.0438
5%	0.5003	0.0055	0.0069	0.0069	0.0178	0.0204	0.0449
6%	0.5702	0.0044	0.0062	0.0071	0.0185	0.0200	0.0487
7%	0.6416	0.0036	0.0058	0.0071	0.0196	0.0198	0.0549
8%	0.6085	0.0035	0.0052	0.0071	0.0190	0.0199	0.0516
9%	0.5535	0.0036	0.0047	0.0072	0.0182	0.0200	0.0470
10%	0.5714	0.0033	0.0044	0.0071	0.0185	0.0200	0.0484

Peru (New Sol)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.6771	0.0031	0.0071	0.0041	0.0073	0.0071	0.0168
2%	0.8039	0.0015	0.0058	0.0048	0.0085	0.0072	0.0208
3%	0.5744	0.0017	0.0048	0.0040	0.0071	0.0075	0.0151
4%	0.6083	0.0014	0.0044	0.0041	0.0072	0.0074	0.0157
5%	0.4925	0.0015	0.0039	0.0039	0.0069	0.0076	0.0142
6%	0.3714	0.0017	0.0034	0.0038	0.0067	0.0079	0.0132
7%	0.3776	0.0016	0.0032	0.0038	0.0068	0.0079	0.0134
8%	0.3624	0.0017	0.0028	0.0037	0.0069	0.0082	0.0139
9%	0.3597	0.0017	0.0026	0.0037	0.0070	0.0083	0.0140
10%	0.2650	0.0019	0.0023	0.0037	0.0068	0.0083	0.0130

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.4988	0.0036	0.0071	0.0031	0.0063	0.0070	0.0141
2%	0.4935	0.0026	0.0049	0.0030	0.0062	0.0070	0.0141
3%	0.4013	0.0024	0.0039	0.0027	0.0060	0.0072	0.0135
4%	0.4080	0.0021	0.0032	0.0028	0.0061	0.0072	0.0135
5%	0.4251	0.0019	0.0028	0.0028	0.0061	0.0072	0.0136
6%	0.4658	0.0016	0.0026	0.0029	0.0062	0.0071	0.0140
7%	0.4128	0.0017	0.0022	0.0028	0.0061	0.0072	0.0135
8%	0.4700	0.0014	0.0021	0.0029	0.0062	0.0071	0.0142
9%	0.4722	0.0013	0.0019	0.0029	0.0062	0.0071	0.0142
10%	0.4566	0.0013	0.0018	0.0029	0.0061	0.0071	0.0140

Venezuela (Bolivar)

Right Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.8298	0.0156	0.0139	0.0000	0.0240	0.0138	0.1049
2%	1.0274	0.0061	0.0080	0.0044	-	0.0140	-
3%	0.9737	0.0042	0.0060	0.0043	0.1022	0.0143	0.4837
4%	0.7617	0.0047	0.0042	0.0033	0.0198	0.0157	0.0719
5%	0.7413	0.0041	0.0032	0.0032	0.0190	0.0158	0.0678
6%	0.8267	0.0031	0.0027	0.0033	0.0240	0.0154	0.0933
7%	0.8490	0.0026	0.0023	0.0034	0.0263	0.0153	0.1055
8%	0.8781	0.0022	0.0020	0.0033	0.0310	0.0153	0.1288
9%	0.9145	0.0019	0.0018	0.0033	0.0414	0.0153	0.1811
10%	0.8621	0.0019	0.0016	0.0034	0.0280	0.0152	0.1139

Left Tail

%. of Excess	ξ	σ	Threshold	VaR (95%)	Expected Shortfall (95%)	VaR (99%)	Expected Shortfall (99%)
1%	0.5490	0.0087	0.0119	0.0025	0.0105	0.0118	0.0311
2%	0.4895	0.0066	0.0066	0.0017	0.0099	0.0120	0.0301
3%	0.4854	0.0053	0.0044	0.0020	0.0101	0.0121	0.0297
4%	0.5257	0.0043	0.0031	0.0022	0.0103	0.0119	0.0307
5%	0.5957	0.0034	0.0024	0.0024	0.0108	0.0116	0.0335
6%	0.6275	0.0029	0.0019	0.0025	0.0112	0.0115	0.0354
7%	0.6329	0.0026	0.0015	0.0025	0.0113	0.0115	0.0359
8%	0.5854	0.0026	0.0011	0.0025	0.0107	0.0116	0.0327
9%	0.6640	0.0021	0.0009	0.0024	0.0118	0.0115	0.0388
10%	0.6883	0.0019	0.0007	0.0024	0.0123	0.0115	0.0414

Note: The foreign exchange rate data is sourced from Bloomberg. The estimation period is from November 1, 1993 through October 29, 2001.

The values of ξ and σ under the generalized Pareto distribution are estimated with the maximum likelihood estimation on the exceedances of daily logarithm changes in the foreign exchange rates. VaR and expected shortfall are calculated using each of the estimated parameters.

Table 11 Estimation of the Bivariate Extreme Value Distribution of Daily Log Changes of the Southeast Asian Exchange Rates

Currencies		α	ξ_1	σ_1	θ_1	ξ_2	σ_2	θ_2
Indonesia (Rupiah)	Malaysia (Ringgit)	1.2658	0.4088	0.0128	0.0084	0.7371	0.0030	0.0016
Indonesia (Rupiah)	Philippines (Peso)	1.3056	0.4088	0.0128	0.0084	0.4156	0.0046	0.0035
Indonesia (Rupiah)	Singapore (Dollar)	1.3316	0.4088	0.0128	0.0084	0.3256	0.0020	0.0028
Indonesia (Rupiah)	Thailand (Baht)	1.3855	0.4088	0.0128	0.0084	0.4298	0.0051	0.0035
Malaysia (Ringgit)	Philippines (Peso)	1.2578	0.7371	0.0030	0.0016	0.4156	0.0046	0.0035
Malaysia (Ringgit)	Singapore (Dollar)	1.5288	0.7371	0.0030	0.0016	0.3256	0.0020	0.0028
Malaysia (Ringgit)	Thailand (Baht)	1.3186	0.7371	0.0030	0.0016	0.4298	0.0051	0.0035
Philippines (Peso)	Singapore (Dollar)	1.3120	0.4156	0.0046	0.0035	0.3256	0.0020	0.0028
Philippines (Peso)	Thailand (Baht)	1.4267	0.4156	0.0046	0.0035	0.4298	0.0051	0.0035
Singapore (Dollar)	Thailand (Baht)	1.4364	0.3256	0.0020	0.0028	0.4298	0.0051	0.0035

Note: The foreign exchange rate data is sourced from Bloomberg. The estimation period is from November 1, 1993 through October 29, 2001.
The estimation is for the right tails of the logarithm changes. The tail probabilities are set at $p_1 = p_2 = 0.1$.

Table 12 VaR and Expected Shortfall of the Simulated Sums of the Foreign Exchange Rates

Currencies: Indonesia (Rupiah) and Malaysia (Ringgit)
α =1.266 (Spearman's rho=0.340)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.02337	0.06852	0.06079	0.15357
Gaussian	0.02331	0.06958	0.06186	0.15783
Gumbel	0.02257	0.07041	0.06412	0.17071

Currencies: Indonesia (Rupiah) and Philippines (Peso)
α =1.306 (Spearman's rho=0.195)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.02464	0.06573	0.05481	0.12279
Gaussian	0.02464	0.06746	0.05611	0.12702
Gumbel	0.02408	0.07002	0.05855	0.13811

Currencies: Indonesia (Rupiah) and Singapore (Dollar)
α =1.332 (Spearman's rho=0.360)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.02118	0.05993	0.04980	0.11490
Gaussian	0.02133	0.06094	0.05061	0.11699
Gumbel	0.02132	0.06270	0.05203	0.12180

Currencies: Indonesia (Rupiah) and Thailand (Baht)
α =1.386 (Spearman's rho=0.203)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.02562	0.06788	0.05664	0.12629
Gaussian	0.02551	0.07015	0.05830	0.13219
Gumbel	0.02482	0.07298	0.06106	0.14513

Currencies: Malaysia (Ringgit) and Philippines (Peso)
α =1.258 (Spearman's rho=0.151)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.01161	0.03427	0.03382	0.09490
Gaussian	0.01154	0.03504	0.03550	0.10266
Gumbel	0.01111	0.03570	0.03648	0.10855

Currencies: Malaysia (Ringgit) and Singapore (Dollar)
α =1.529 (Spearman's rho=0.154)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.00844	0.02442	0.02660	0.08047
Gaussian	0.00834	0.02558	0.02844	0.08865
Gumbel	0.00811	0.02677	0.02919	0.09196

Currencies: Malaysia (Ringgit) and Thailand (Baht)
α =1.319 (Spearman's rho=0.448)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.01232	0.03692	0.03583	0.09971
Gaussian	0.01220	0.03778	0.03766	0.10826
Gumbel	0.01166	0.03850	0.03884	0.11547

Currencies: Philippines (Peso) and Singapore (Dollar)
α =1.312 (Spearman's rho=0.473)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.01043	0.02497	0.02116	0.04533
Gaussian	0.01047	0.02588	0.02179	0.04721
Gumbel	0.01035	0.02720	0.02288	0.05150

Currencies: Philippines (Peso) and Thailand (Baht)
α =1.427 (Spearman's rho=0.252)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.01455	0.03650	0.03066	0.06663
Gaussian	0.01440	0.03802	0.03185	0.07121
Gumbel	0.01395	0.03992	0.03366	0.07996

Currencies: Singapore (Dollar) and Thailand (Baht)
α =1.436 (Spearman's rho=0.411)

	VaR(95%)	VaR(99%)	ES(95%)	ES(99%)
Frank	0.01114	0.02754	0.02344	0.05152
Gaussian	0.01114	0.02882	0.02427	0.05418
Gumbel	0.01102	0.03037	0.02549	0.05885

Appendix A: Copula of Multivariate Exceedances

This appendix explains the finding of Ledford and Tawn [1996] that the copula of multivariate exceedances converges to the extreme value copula.

We consider copula of multivariate maxima before considering copula of multivariate exceedances. The following theorem gives the foundations for describing the asymptotic joint distribution of multivariate maxima (See Resnick [1987], Proposition 5.11 for the proof).

Theorem

Suppose that $\{(Z_{1j}, Z_{2j}); j = 1, \ldots, n\}$ are independent and identically distributed two-dimensional random vectors with the joint distribution function F. Also suppose that the marginal distribution of these two-dimensional random vectors is a Fréchet distribution. In other words, for each i, j, $\Pr[Z_{ij} \le z_{ij}] = \exp(-1/z_{ij})$. Define the vector of componentwise maxima as $M_{Z_i,n} = \max(Z_{i1}, Z_{i2}, \ldots, Z_{in})$. Then, the following holds:

$$\Pr[\frac{M_{Z_{1,n}}}{n} \le z_1, \frac{M_{Z_{2,n}}}{n} \le z_2] = F^n(nz_1, nz_2) \to G(z_1, z_2), \text{ as } n \to \infty,$$

where $G(z_1, z_2) = \exp\{-V(z_1, z_2)\}$,

$$V(z_1, z_2) = \int_0^1 \max\{sz_1^{-1}, (1-s)z_2^{-1}\} dH(s).$$

H is a non-negative measure on [0,1] that satisfies the following condition.

$$\int_0^1 s dH(s) = \int_0^1 (1-s) dH(s) = 1.$$

Using this theorem, Ledford and Tawn [1996] show that the copula of multivariate exceedances converges to the bivariate extreme value copula as follows.

Suppose that $\{(Z_{1j}, Z_{2j}); j = 1, \ldots, n\}$ are independent and identically distributed two-dimensional random vectors with the joint distribution function F_*. Also assume that the marginal distribution of (Z_1, Z_2) is a Fréchet distribution. In other words, for each i, $\Pr[Z_i \le z_i] = \exp(-1/z_i)$. Based on Proposition 5.15 in Resnick [1987], F_* is within the domain of attraction of G_* if and only if the following holds.

$$\lim_{t\to\infty} \frac{-\log F_*(tz_1, tz_2)}{-\log F_*(t,t)} = \frac{-\log G_*(z_1, z_2)}{-\log G_*(1,1)}. \qquad \text{(A-1)}$$

As this is an asymptotic result, Ledford and Tawn [1996] assume that this also holds with a sufficiently large value of $t = t_c$. That is, the following holds for a large value of $t = t_c$.

$$\frac{-\log F_*(t_c z_1, t_c z_2)}{-\log F_*(t_c, t_c)} = \frac{-\log G_*(z_1, z_2)}{-\log G_*(1,1)}. \qquad \text{(A-2)}$$

Define z'_j as $z'_j = t_c z_j$. With (A-2), the following holds when z'_j is above some high threshold θ_j.

$$\log F_*(z'_1, z'_2) = \log F_*(t_c, t_c) \frac{\log G_*(z'_1/t_c, z'_2/t_c)}{\log G_*(1,1)}. \qquad \text{(A-3)}$$

G_* satisfies the following condition since G_* is the extreme value distribution,

$$G(z'_1, z'_2) = \exp\{-V(z'_1, z'_2)\}, \qquad \text{(A-4)}$$

where $V(z_1, z_2) = \int_0^1 \max\{sz_1^{-1}, (1-s)z_2^{-1}\} dH(s)$.

Here, H is a non-negative measure on [0,1] that satisfies $\int_0^1 s dH(s) = \int_0^1 (1-s) dH(s) = 1$.

As V is a homogeneous function of order –1, this leads to the following relation (where z' is now expressed by z).

$$F_*(z_1, z_2) = \exp\left\{V(z_1, z_2) \frac{t_c \log F_*(t_c, t_c)}{V(1,1)}\right\} = \exp\{V(z_1, z_2)K\}, \qquad \text{(A-5)}$$

where K is a constant.

To determine the value of K we consider the value of F_* at the threshold θ_j. If we suppose that this threshold value is the $1-\lambda_j$ quantile, θ_j is derived as $\theta_j = -1/\log(1-\lambda_j)$. Setting $z_1 = \theta_1 = -1/\log(1-\lambda_1)$ and $z_2 = \infty$ in (A-5), we obtain the following.

$$F_*(-1/\log(1-\lambda_1),\infty) = \exp\{V(-1/\log(1-\lambda_1),\infty)K\}\,. \tag{A-6}$$

The left-hand side of Equation (A-6) is equal to $1-\lambda_1$ because it is the distribution function at the $1-\lambda_1$ quantile. On the other hand, the right-hand side of Equation (A-6) is equal to $\exp\{-K\log(1-\lambda_1)\}$, as shown below.

$$\begin{aligned} V(-1/\log(1-\lambda_1),\infty) &= \int_0^1 \max\{s(-\log(1-\lambda_1)),(1-s)/\infty\}dH(s) \\ &= -\log(1-\lambda_1)\int_0^1 sdH(s) = -\log(1-\lambda_1) \end{aligned}\,, \tag{A-7}$$

As $1-\lambda_1 = \exp\{-K\log(1-\lambda_1)\}$, we find that $K=-1$. Setting this into (A-5), F_* is obtained as follows.

$$F_*(z_1,z_2) = \exp\{-V(z_1,z_2)\}\,, \tag{A-8}$$

where $V(z_1,z_2)$ is the same as in (A-4).

This shows that the asymptotic joint distribution of the multivariate exceedances whose marginal distribution is a Fréchet distribution is given by (A-8).

We use this result to obtain the copula of multivariate exceedances whose marginals are not Fréchet distributions. Define u_i as $u_i \equiv \Pr[Z_i \le z_i] = \exp(-1/z_i)$. Set $z_i = -1/\log u_i$ into $G(z_1,z_2) = \exp\{-V(z_1,z_2)\}$ to obtain the following copula.

$$C(u_1,u_2) = \exp\{-V(-\frac{1}{\log u_1},-\frac{1}{\log u_2})\}\,, \tag{A-9}$$

where $V(z_1,z_2)$ is the same as (A-4).

With copula invariance, this is the copula of exceedances for all marginals since the copula is invariant under increasing continuous transformations.[34]

[34] Proposition 5.10 in Resnick [1987] shows that this approach is appropriate.

Appendix B: Tail Risk of VaR under the Generalized Pareto Distribution

This appendix analyzes the tail risk of VaR under the generalized Pareto distribution employing Feller's convolution theorem.[46] We assume that the marginal distributions of asset losses are the generalized Pareto and have the same tail index.

This assumption is different from the assumption in sections III and IV in two aspects. First, in sections III and IV, we assume that only the exceedances follow the generalized Pareto distribution. In this appendix, we assume that the both exceedances and non-exceedances follow the same generalized Pareto distribution. Second, in sections III and IV, we assume that the tail index is different among assets. In this appendix, we assume that the tail index is equal across assets. Thus, under the assumption in sections III and IV, we are unable to employ the convolution theorem used in this appendix.

Feller [1966] (p.278) and Embrechts, Klüppelberg, and Mikosch [1997] (Lemma 1.3.1) utilize the convolution theorem for regularly varying distribution functions to examine the properties of the sum of the independent random variables with the same tail index. We explain their conclusions, incorporating our concept of tail risk.

Suppose that two independent random variables Z_1 and Z_2 have the same distribution functions as follows.

$$G_{\xi,\sigma}(x) = 1 - (1 + \xi \cdot \frac{x}{\sigma})_+^{-1/\xi}. \qquad \text{(B-1)}$$

The distribution function of the sum of the two random variables Z_1 and Z_2 is derived from the convolution of Equation (B-1), as follows.

$$H(x) \equiv \Pr\{Z_1 + Z_2 \le x\} \equiv \int_0^x G_{\xi,\sigma}(x-y)\, dG_{\xi,\sigma}(y). \qquad \text{(B-2)}$$

The function $\overline{G}_{\xi,\sigma}(x) \equiv 1 - G_{\xi,\sigma}(x)$ is transformed as follows.

[46] Geluk, J., L. Peng, and C. de Vries [2000] adopts Feller's convolution theorem for analyzing the portfolio diversification effect under fat-tailed distributions.

$$\overline{G}_{\xi,\sigma}(x) = \left(1+\xi\cdot\frac{x}{\sigma}\right)_+^{-1/\xi} = x^{-1/\xi}\left(\frac{1}{x}+\frac{\xi}{\sigma}\right)_+^{-1/\xi}. \qquad \text{(B-3)}$$

Since the term $(1/x+\xi(x-\theta)/\sigma)_+^{-1/\xi}$ on the right-hand side of Equation (B-3) is *slowly-varying*,[35] using Feller's convolution theorem (see Feller [1969] <p.278>, or Embrechts, Klüppelberg, and Mikosch [1997] <Lemma 1.3.1>), the following relation holds when the value of x is sufficiently large. .

$$\overline{H}(x) \approx x^{-1/\xi}\left\{\left(\frac{1}{x}+\frac{\xi}{\sigma}\right)_+^{-1/\xi}+\left(\frac{1}{x}+\frac{\xi}{\sigma}\right)_+^{-1/\xi}\right\} = 2\left(1+\xi\cdot\frac{x}{\sigma}\right)_+^{-1/\xi}, \qquad \text{(B-4)}$$

where $\overline{H}(x) \equiv 1-H(x)$. Therefore, the distribution function of the sum of two independent random variables Z_1 and Z_2 is as follows, when the value of x is sufficiently large.

$$H(x) \approx 1-2\left(1+\xi\cdot\frac{x}{\sigma}\right)_+^{-1/\xi}. \qquad \text{(B-5)}$$

Meanwhile, the distribution function of the sum of two fully dependent random variables whose distribution function is given by (B-1) follows the same distribution as $2Z_1$. Thus, the distribution function $I(x)$ of the sum of two fully dependent variables is given below.

$$I(x) \equiv \Pr\{2Z_1 \le x\} = \Pr\{Z_1 \le x/2\} = G_{\xi,\sigma}(x/2) = 1-\left(1+\xi\cdot\frac{x}{2\sigma}\right)_+^{-1/\xi}. \qquad \text{(B-6)}$$

VaR has tail risk when the two distribution functions $H(x)$ and $I(x)$ intersect (that is, when there is a solution to $H(x)=I(x)$), and when the VaR confidence level is lower than the cumulative probability of this intersection. In the case of $\xi<1$, there is a solution to $H(x)=I(x)$, and the cumulative probability

[35] Slowly-varying functions are those functions $L(x)$ that satisfy the following condition. See Feller [1966] for the details.

$$\lim_{x\to\infty}\frac{L(tx)}{L(x)} = 1$$

$p(\xi)$ at the intersection is as follows.[36]

$$p(\xi) = 1 - 2\left(1 + \frac{2^{\xi} - 1}{1 - 2^{\xi - 1}}\right)^{-1/\xi} \quad (\xi < 1). \tag{B-7}$$

With some calculations (B-7), we find that the tail index must be 0.9 or higher for VaR to have tail risk at the confidence levels of 95% and 99%.

The tail index is 0.9 or higher only when the distribution is so fat that the 1.2-th moment is infinite. Such a fat-tailed distribution is rarely found in financial data. Thus, under the assumptions of this appendix, we find that VaR does not have tail risk as long as the confidence level is sufficiently high.

[36] When $\xi \geq 1$, VaR is not sub-additive and has no tail risk. As $H(x) < I(x)$ for all x, the VaR for independence is larger than the VaR for full dependence. This shows that VaR is not sub-additive. On the other hand, full dependence dominates independence in the first order stochastic dominance. Thus, the ordering by VaR is consistent with the ordering by the first order stochastic dominance.

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