Risk contributions in an asymptotic multi-factor framework

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Abstract

So far, regulatory capital requirements for credit risk portfolios are calculated in a bottom-up approach by determining the requirements at asset level and then adding up them. In contrast, economic capital for a credit risk portfolio is calculated for the portfolio as a whole and then decomposed into risk contributions of assets or sub-portfolios for, e.g., diagnostic purposes like identifying risk concentrations. In the "Asymptotic Single Risk Factor" model that underlies the most important part of the "Basel II Accord", bottom-up and top-down approach yield identical results. However, the model fails in detecting exposure concentrations and recognizing diversification effects. We investigate multi-factor extensions of the ASRF model and derive exact formulae for the risk contributions to Value-at-Risk and Expected Shortfall. As an application of the risk contribution formulae we introduce a new concept for a diversification index. The use of this new index is illustrated with an example calculated with a two-factor model. The results with this model indicate that there can be a substantial reduction of risk contributions by diversification effects.

1 Introduction

Credit risk, i.e. the risk that borrowers do not fully meet their obligations, is considered the most important risk banks are facing. During the past decade, therefore, banks as well as banking supervisory authorities put considerable efforts in developing models for quantitative assessments of credit risk. These efforts were accompanied by a growing interest of the academic community in credit risk models. Compared to models of market risk or actuarial models, credit risk models have some interesting special features. In particular, they produce non-normal loss distributions and reflect the empirically observed dependence of credit events.

While the more advanced large banks rely more and more on very sophisticated credit risk models in order to take into account these features, banking supervisors worldwide intend to

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implement a minimum standard of credit risk modelling which can also be met by smaller banks. The process of designing this standard ("Basel II") is coordinated by the Basel Committee on Banking Supervision (BCBS). Although some details are still under discussion, the BCBS published at the end of June 2004 a framework with the status quo of its recommendations (BCBS, 2004). Most important in this so-called "Basel II Accord" is the part on the "Internal Ratings Based Approach" (IRBA) for the determination of regulatory capital requirements for credit risk. It is regarded as a first step towards a supervisory recognition of advanced credit risk models and economic capital calculations.

According to the IRBA, the regulatory minimum capital for a credit risk portfolio is calculated in a bottom-up approach by determining capital requirements at asset level and adding up them. The capital requirements of the assets are expressed as 8 percent of the so-called "risk weights". The risk weight functions were developed by considering a special credit portfolio model, the so-called "Asymptotic Single Risk Factor Model" (ASRF model, Gordy, 2003). This model is characterized by its computational simplicity and the property that the risk weights of single credit assets depend only upon the characteristics of these assets, but not upon the composition of the portfolio ("portfolio invariance"). As a consequence, the model can reflect neither exposure concentrations nor segmentation effects (say by industry branches).

The model's inability to detect exposure concentrations entails a potential underestimation of the risk inherent in the portfolio, whereas its fault in recognizing the diversification effects following from segmentation could result in a potential overestimation of portfolio risk. The Basel Committee decided to deal in Pillar 2 of the Basel II Accord (BCBS, 2004) with the potential underestimation of risk concentration. As a consequence there is no automatism of extended regulatory capital requirements for risk concentrations, but banks will have to demonstrate to the supervisors that they have established appropriate procedures to keep concentrations under control. Nevertheless, there are methods to measure quantitatively risk concentrations. A quantitative way of tackling the exposure concentration issue was suggested in Emmer and Tasche (2005), for instance.

In the present paper, we suggest a minimal – in the spirit of Emmer and Tasche (2005) – extension of the Basel II model that allows to study the effects of segmentation on portfolio risk. Admitting several risk factors instead of a single factor only and applying the same transition to the limit as described in Gordy (2003), we arrive at versions of the model that remove the no segmentation restriction. Alternatively, our class of models can be regarded as special cases of the asymptotic models introduced by Lucas et al. (2001, Theorem 1).

As determining risk contributions or, economically speaking, capital requirements for assets or sub-portfolios, is a main purpose when using credit risk models, deriving exact formulae for risk contributions to "Value-at-risk" (VaR) and "Expected Shortfall" (ES) in the asymptotic multi risk factors setting represents the main contribution of our paper to the subject. Our results complement results on the differentiation of VaR and ES presented in Gouriéroux et al. (2000), Lemus (1999), and Tasche (1999). From a computational point of view, the resulting formulae are more demanding than in the one factor case, and – necessarily, as otherwise diversification effects could not be recognized – they are not portfolio invariant any longer. As an application of the risk contribution formulae we introduce then a new concept for a diversification index. This index can be computed at portfolio as well as at sub-portfolio or asset level, thus allowing for identifying the causes of bad diversification. The use of these new indices is illustrated with an example calculated with a two-factor model. The results with this model indicate that there can be a substantial reduction of risk contributions by diversification effects.

The material presented here is closely related to work by Pykhtin (2004) and Garcia et al. (2004). Pykhtin describes an approximation of multi-factor models by single-factor models, thus transferring the computational simplicity of single-factor models to multi-factor models. Garcia et al. propose "factor adjustments" to the risk contributions from a single-factor model in order to reflect diversification effects. As our results on the risk contribution formulae are not approximate but exact they could be used for benchmarking the results by Pykhtin and Garcia et al.

This paper is organized as follows: In Section 2 we introduce the class of models we are going to analyze and derive some basic properties. In Section 3 we shortly recall the Euler allocation principle that justifies the use of partial derivatives as risk contributions and derive then the announced formulae for risk contributions to VaR and ES in the asymptotic multi-factor setting. A potential application of the risk contribution formulae for the purpose of identifying sources of concentration risk is suggested in Section 4 where a new concept for a diversification index is introduced. Section 5 gives a numerical illustration of a potential application of the formulae and the diversification index. We conclude with some summarizing comments in Section 6.

2 Asymptotic multi-factor models: basic properties

The starting point for the factor models[1] we are going to consider is a random variable $\tilde{L}(u) = \tilde{L}(u_1, \ldots, u_n)$ that reflects the loss suffered from a portfolio of n credit assets, with respective exposures u_i. The tilde indicates that we regard the original loss variable, without any approximation procedure. The variable $\tilde{L}(u)$ can be interpreted as the absolute loss, measured in units of some currency. Then the u_i are absolute exposures[2] and amounts of money. Alternatively, $\tilde{L}(u)$ can also be understood as relative loss, indicating the percentage of the sum of all exposures that is lost. In this case the u_i are non-negative numbers without units that add up to 1.

Formally, the original loss variable $\tilde{L}(u)$ is given as

$$\tilde{L} \;=\; \sum_{i=1}^{n} u_i \, \mathbf{1}_{D_i}. \tag{2.1}$$

[1] See Bluhm et al. (2002) and the references therein for more information on credit risk models.

[2] u_i may be thought as a face value multiplied with some factor that expresses the average loss rate in case of default.

The term $\mathbf{1}_{D_i}$ is the *default indicator variable* for asset i, i.e. it takes the value of 1 if i defaults and 0 if not. As a consequence, the sum in (2.1) will be built up with only those u_i's that relate to defaulted assets i. For factor models, it is quite common to specify the default events D_i by

$$D_i \;=\; \Big\{\sum\nolimits_{j=1}^{k} \rho_{i,j}\, S_j + \omega_i\, \xi_i \le t_i\Big\}, \qquad i = 1, \ldots, n, \tag{2.2}$$

with the following for the involved constants and random variables:

- The random variables $S_1, \ldots, S_k$ are the *systematic risk factors*. They are assumed to capture the dependence of the default events. In general, we have $k \ll n$. Within this paper, we assume that the factor variables are standardized, i.e.

$$\mathrm{E}[S_j] = 0 \quad \text{and} \quad \mathrm{var}[S_j] = 1, \quad j = 1, \ldots, k. \tag{2.3}$$

  The $S_1, \ldots, S_k$ may be stochastically dependent, but they do not have do be.

- The random variables $\xi_1, \ldots, \xi_n$ are the *idiosyncratic risk drivers*. They are also standardized, i.e.

$$\mathrm{E}[\xi_i] = 0 \quad \text{and} \quad \mathrm{var}[\xi_i] = 1, \quad i = 1, \ldots, n. \tag{2.4}$$

  $\xi_1, \ldots, \xi_n, (S_1, \ldots, S_k)$ are stochastically independent. As a consequence, conditional on $(S_1, \ldots, S_k)$, the default events $D_i, i = 1, \ldots, n$ are independent.

- The constants $\rho_{i,j}, i = 1, \ldots, n,\ j = 1, \ldots, k$ are the *factor loadings* of the systematic factors. We assume that

$$\sum_{j=1,\ell=1}^{k} \rho_{i,j}\, \rho_{i,\ell}\, \mathrm{corr}[S_j, S_\ell] \;\le\; 1, \quad i = 1, \ldots, n. \tag{2.5}$$

  By (2.5) and the standardization assumption on the S_j and ξ_i the *idiosyncratic loadings* $\omega_i, i = \ldots, n$ are well defined by

$$\omega_i = \sqrt{1 - \sum_{j=1,\ell=1}^{k} \rho_{i,j}\, \rho_{i,\ell}\, \mathrm{corr}[S_j, S_\ell]}. \tag{2.6}$$

  As a further consequence of (2.5) and (2.6) and of the standardization assumptions also the *asset values changes* $\sum_{j=1}^{k} \rho_{i,j}\, S_j + \omega_i\, \xi_i$ are standardized.

- The constant $t_i, i = 1, \ldots, n$ is called *default threshold*. It can be thought as a critical loss in value of borrower i's assets that causes the borrower to default on asset i. It is common to derive t_i from borrower i's (assumed to be known) probability of default p_i. Hence, we determine t_i such that

$$\mathrm{P}[D_i] \;=\; \mathrm{P}\Big[\sum\nolimits_{j=1}^{k} \rho_{i,j}\, S_j + \omega_i\, \xi_i \le t_i\Big] \;=\; p_i, \quad i = 1, \ldots n. \tag{2.7}$$

  When the idiosyncratic risk drivers ξ_i and the factor variables are all standard normally distributed, also the asset value changes $\sum_{j=1}^{k} \rho_{i,j}\, S_j + \omega_i\, \xi_i$ are standard normal. Let Φ denote the standard normal distribution function. By (2.7) we then have $p_i = \Phi(t_i)$.

For the sake of a more concise notation we define

$$\widetilde{S} \;=\; (S_2,\dots,S_k), \qquad \widetilde{s} \;=\; (s_2,\dots,s_k) \tag{2.11a}$$

and for fixed u

$$G(v,\widetilde{s}) \;=\; \sum_{j=1}^{n} u_j\, g_j(v,\widetilde{s}). \tag{2.11b}$$

By Assumption 2.2, then, for fixed $\widetilde{s}$, $v \mapsto G(v,\widetilde{s})$ is invertible. Write $G(\cdot,\widetilde{s})^{-1}$ for the inverse function of $v \mapsto G(v,\widetilde{s})$. Write additionally $G(\cdot,\widetilde{s})^{-1}(0) = \infty$ and $G(\cdot,\widetilde{s})^{-1}(z) = -\infty$ for $z \geq \sum_{j=1}^{n} u_j$. Having fixed the assumptions and notations, we can prove a result on the calculation of moments that in particular implies that the distribution of the generalized loss variable $L(u)$ has a density.

Proposition 2.3 *Let $F : [0,1] \to \mathbb{R}$ be arbitrary and $L(u)$ be the loss variable defined by (2.10). Then, under Assumption[4] 2.2, for any $0 \leq z \leq \sum_{j=1}^{n} u_j$ and $i \in \{1,\dots,n\}$ we have:*

$$\mathrm{E}\big[F(g_i(S))\,\mathbf{1}_{\{L(u)\leq z\}}\big] = -\int_0^z \mathrm{E}\left[\frac{F\big(g_i(G(\cdot,\widetilde{S})^{-1}(t),\widetilde{S})\big)\,h\big(G(\cdot,\widetilde{S})^{-1}(t)\,|\,\widetilde{S}\big)}{\frac{\partial}{\partial v}G(v,\widetilde{S})\big|_{v=G(\cdot,\widetilde{S})^{-1}(t)}}\right] dt.$$

Proof.

$$\mathrm{E}\big[F(g_i(S))\,\mathbf{1}_{\{L(u)\leq z\}}\big] = \int \mathrm{E}\big[F(g_i(S_1,\widetilde{S}))\,\mathbf{1}_{\{G(S_1,\widetilde{S})\leq z\}}\,|\,\widetilde{S}=\widetilde{s}\big]\mathrm{P}\widetilde{S}^{-1}(d\widetilde{s}) \tag{2.12a}$$

(taking into account $G(\cdot,\widetilde{s})^{-1}(0) = \infty$)

$$= \int \int_{G(\cdot,\widetilde{s})^{-1}(z)}^{\infty} F\big(g_i(y,\widetilde{s})\big)\,h(y\,|\,\widetilde{s})\,dy\,\mathrm{P}\widetilde{S}^{-1}(d\widetilde{s}) \tag{2.12b}$$

(substituting $y = G(\cdot,\widetilde{s})^{-1}(t)$)

$$= -\int \int_0^z \frac{F\big(g_i(G(\cdot,\widetilde{s})^{-1}(t),\widetilde{s})\big)\,h\big(G(\cdot,\widetilde{s})^{-1}(t)\,|\,\widetilde{s}\big)}{\frac{\partial}{\partial v}G(v,\widetilde{S})\big|_{v=G(\cdot,\widetilde{S})^{-1}(t)}}\,dt\,\mathrm{P}\widetilde{S}^{-1}(d\widetilde{s}).$$

The assertion follows by applying Fubini's theorem. □

The choice $F = 1$ in Proposition 2.3 implies the existence of a density for the distribution of the generalized loss variable:

Corollary 2.4 *Under Assumption 2.2, the loss variable $L(u)$ from (2.10) has the density $f_{L(u)} : \big(0,\sum_{j=1}^{n} u_j\big) \to [0,\infty[$, defined by*

$$f_{L(u)}(t) \;=\; -\mathrm{E}\left[\frac{h\big(G(\cdot,\widetilde{S})^{-1}(t)\,|\,\widetilde{S}\big)}{\frac{\partial}{\partial v}G(v,\widetilde{S})\big|_{v=G(\cdot,\widetilde{S})^{-1}(t)}}\right], \quad t \in \big(0,\textstyle\sum_{j=1}^{n} u_j\big). \tag{2.13}$$

[4] In order to keep the representation of the results as intuitive and clear as possible here and in the following the proofs will not be rigorous but rather consist of calculations without consideration of continuity, differentiability etc. Moreover, for some of the results, additional assumptions on existence of moments etc. must be made.

Note that Gordy's (2003) ASRF (Asymptotic Single Risk Factor) model is a special case of (2.8) with $k = 1$. Then the expectation in (2.13) disappears, and the density h and the inverse of G do not depend on $\widetilde{s}$. Nevertheless, in the case of non-constant asset correlations or probabilities of default the calculation of the density of $L(u)$ will involve numerical inversion of G even in the simple ASRF case.

In case that an explicit representation of the conditional distribution of S_1 given $\widetilde{S}$ is known (e.g. if S is jointly normally distributed), Equation (2.12a) (with $F = 1$) yields a more efficient way to calculate the distribution function of $L(u)$ than Corollary 2.4 does. The reason is that application of Corollary 2.4 would require evaluation of a k-dimensional integral if S had a density, whereas the application of the following Proposition 2.5 would only require evaluation of a $(k-1)$-dimensional integral.

Proposition 2.5 *Under Assumption 2.2, the distribution function of the loss variable $L(u)$ as given in (2.10) can be calculated by means of*

$$\mathrm{P}[L(u) \leq z] \;=\; \int \mathrm{P}\big[S_1 \geq G(\cdot, \widetilde{S})^{-1}(z) \,|\, \widetilde{S} = \widetilde{s}\big]\, \mathrm{P}\widetilde{S}^{-1}(d\widetilde{s}). \tag{2.14}$$

Define, for $\alpha \in (0,1)$ and any real random variable X, the α-*quantile* of X by

$$q_\alpha(X) \;=\; \min\{x : \mathrm{P}[X \leq x] \geq \alpha\}. \tag{2.15a}$$

Quantiles at high levels (e.g. 99.9%) are popular metrics for determining the economic capital of portfolios. Within the financial community, the α-quantile of a loss distribution is commonly called *Value-at-Risk* (VaR) at level α. In case of the generalized loss variable $L(u)$ we write

$$q_\alpha(L(u)) \;=\; q_\alpha(u). \tag{2.15b}$$

The quantiles of $L(u)$ can be computed by numerical inversion of (2.14).

We conclude this section by providing two alternative formulae for the calculation of another popular risk measure, the Expected Shortfall[5], in the case of the asymptotic multi-factor model under consideration.

Remark 2.6 *The Expected Shortfall $\mathrm{ES}_\alpha(L(u)) = \mathrm{E}[L(u) \,|\, L(u) \geq q_\alpha(L(u))]$ at level α of the loss variable $L(u)$ from (2.10) can alternatively be calculated with recourse to Corollary 2.4 or to Proposition 2.5. Note that the existence of a density of the distribution of $L(u)$ (Corollary 2.4) implies $\mathrm{P}[L(u) \geq q_\alpha(L(u))] = 1 - \alpha$. From Corollary 2.4 we can therefore derive*

$$\mathrm{E}[L(u) \,|\, L(u) \geq q_\alpha(L(u))] \;=\; -(1-\alpha)^{-1} \int_{q_\alpha(L(u))}^{\sum_{j=1}^n u_j} t\, \mathrm{E}\left[\frac{h\big(G(\cdot, \widetilde{S})^{-1}(t) \,|\, \widetilde{S}\big)}{\frac{\partial}{\partial v} G(v, \widetilde{S})\big|_{v = G(\cdot, \widetilde{S})^{-1}(t)}}\right] dt. \tag{2.16a}$$

[5] See Acerbi and Tasche (2002) and the references given there for more details on Expected Shortfall vs. Value-at-Risk. In particular, in case of discontinuous loss distributions the definition of ES has to be slightly modified in order to make it a risk measure superior to VaR.

From Proposition 2.5 we obtain (by making use of the formula $\mathrm{E}[X] = \int_0^\infty \mathrm{P}[X \ge x]\,dx$ *for* $X \ge 0$*)*

$$\begin{aligned} &\mathrm{E}[L(u)\,|\,L(u) \ge q_\alpha(L(u))] \qquad (2.16\text{b}) \\ &= q_\alpha(L(u)) + (1-\alpha)^{-1} \int_{q_\alpha(L(u))}^{\sum_{j=1}^n u_j} \int \mathrm{P}\big[S_1 \le G(\cdot, \widetilde{S})^{-1}(z)\,|\,\widetilde{S} = \widetilde{s}\big]\, \mathrm{P}\widetilde{S}^{-1}(d\widetilde{s})\,dz. \end{aligned}$$

3 Computing the risk contributions

When economic capital for a portfolio is determined by means of a homogeneous risk measure, by the Euler allocation principle the risk contributions of assets should be calculated as partial derivatives of the portfolio-wide economic capital with respect to the exposures. In this section we first recall the Euler principle and derive then formulae for the derivatives of Value-at-Risk[6] as defined by (2.15a) and Expected Shortfall as defined in Remark 2.6 in the context of the asymptotic multi-factor model of Section 2.

3.1 Euler allocation

Suppose that real-valued random variables $X_1, \ldots, X_n$ are given that stand for the profits and losses with the assets in a portfolio. Let Y denote the portfolio-wide profit and loss, i.e. let

$$Y = \sum_{i=1}^n X_i. \qquad (3.1)$$

The economic capital EC required by the portfolio is determined with a risk measure ϱ, i.e.

$$\mathrm{EC} = \varrho(Y). \qquad (3.2)$$

Definition 3.1 *If* ϱ *is a risk measure and* V, W *are random variables such that the derivative* $\frac{d}{dh}\varrho(h\,V + W)\big|_{h=0}$ *exists, then*

$$\varrho(V\,|\,W) = \frac{d}{dh}\varrho(h\,V + W)\Big|_{h=0}$$

is called contribution of V to the risk of W in respect of ϱ*.*

Assumption 3.2 *The risk measure* ϱ *is positively homogeneous, i.e.*

$$\varrho(h\,Z) = h\,\varrho(Z)$$

for any random variable Z *in the definition set of* ϱ *and* $h > 0$*.*

[6]See Mausser and Rosen (2004) and the references therein for the practical issues when estimating VaR contributions from statistical samples.

If for every i the contribution of X_i to the risk of Y exists, then we have by Euler's theorem on the representation of positively homogenous functions

$$\varrho(Y) \;=\; \sum_{i=1}^{n} \varrho(X_i \,|\, Y). \tag{3.3}$$

The decomposition of the portfolio risk ϱ as given by (3.3) is called *Euler allocation*. The use of the Euler allocation principle was justified by several authors with different reasonings:

- Patrik et al. (1999) argued from a practitioner's view emphasizing mainly the fact that the risk contributions according to the Euler principle by (3.3) naturally add up to the portfolio-wide economic capital.
- Litterman (1996) and Tasche (1999) pointed out that the Euler principle is fully compatible with economically sensible portfolio diagnostics and optimization.
- Denault (2001) derived the Euler principle by game-theoretic considerations.
- More recently Kalkbrener (2005) presented an axiomatic approach to capital allocation and risk contributions. One of his axioms requires that risk contributions do not exceed the corresponding stand-alone risks. From this axiom in connection with more technical conditions, in the context of sub-additive and positively homogeneous risk measures, Kalkbrener concluded that the Euler principle is the only allocation principle to be compatible with the "no excession"-axiom (see also Kalkbrener et al., 2004; Tasche, 2002).

3.2 Partial derivatives of VaR and ES

Before coming to the main result on the partial derivatives of VaR with respect to the exposures of the assets in the portfolio, we will shortly discuss the case of Gordy's (2003) ASRF model.

Example 3.3 *In Gordy's (2003) ASRF model Equation (2.10) reads*

$$L(u) \;=\; \sum_{j=1}^{n} u_j \, g_j(X), \tag{3.4}$$

where X stands for a standard normally distributed random variable, the single systematic factor, and the g_j are strictly decreasing and continuous functions. As a consequence, the terms $g_j(X)$ in (3.4) are comonotonic random variables. From the comonotonic additivity of VaR and ES (see, e.g., Tasche, 2002) follows then for any $\alpha \in (0, 1)$ that

$$q_\alpha(u) = \sum_{j=1}^{n} u_j \, q_\alpha(g_j(X)) \tag{3.5a}$$

and

$$\mathrm{E}[L(u)\,|\,L(u)\geq q_\alpha(u)] = \sum_{j=1}^n u_j\,\mathrm{E}[g_j(X)\,|\,g_j(X)\geq q_\alpha(g_j(X))]. \tag{3.5b}$$

As the right-hand sides of (3.5a) and (3.5b) are linear in the exposure vector u, applying the Euler allocation principle with partial derivatives with respect to the components of u yields that the risk contributions to VaR or ES in the ASRF model model equal the corresponding stand-alone risks.

In the following we compute the derivatives of VaR and ES in the context of an asymptotic multi-factor model as given by (2.10). The validity of the results is subject to technical conditions similar to those of Tasche (1999, Section 5). For reasons of readability of the text we do not discuss these conditions here in detail.

Write again $q_\alpha(u)$ for $q_\alpha(L(u))$ and (slightly modifying the notation from (2.11b) but keeping (2.11a))

$$G(v,\widetilde{s},w) = \sum_{j=1}^n w_j\, g_j(v,\widetilde{s}) \tag{3.6a}$$

as well as

$$G^{-1}_{(\widetilde{s},w)}(z) = G(\cdot,\widetilde{s},w)^{-1}(z). \tag{3.6b}$$

Note that existence of $G^{-1}_{(\widetilde{s},w)}$ is guaranteed by Assumption 2.2. Thus prepared, we can state the main result of this paper (Theorem 3.4 and Remark 3.5), namely that in the asymptotic multi-factor model the risk contributions to VaR, calculated as partial derivatives, coincide with certain expectations conditional on the portfolio loss equalling VaR.

Theorem 3.4 *Under Assumption 2.2*[7]*, the quantiles (VaRs) at level $\alpha\in(0,1)$ of the generalized loss variable $L(u)$ as defined in (2.10) are partially differentiable with respect to the portfolio weights u_i of the single loss variables. The partial derivatives $\frac{\partial q_\alpha(u)}{\partial u_i}$ are given by*

$$\frac{\partial q_\alpha(u)}{\partial u_i} \;=\; \mathrm{E}\left[\frac{h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))\,|\,\widetilde{S}\big)}{\frac{\partial}{\partial v}G\big(v,\widetilde{S},u\big)\big|_{v=G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}}\right]^{-1}\mathrm{E}\left[\frac{g_i\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)),\widetilde{S},u\big)\,h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))\,|\,\widetilde{S}\big)}{\frac{\partial}{\partial v}G\big(v,\widetilde{S},u\big)\big|_{v=G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}}\right]. \tag{3.7}$$

Proof. Fix $z\in\big(0,\sum_{i=1}^n u_i\big)$ and observe that

$$G\big(G^{-1}_{(\widetilde{s},u)}(z),\widetilde{s},u\big) \;=\; z \quad \text{for all } \widetilde{s},u$$

[7]Some further technical conditions on uniform integrability of the random variables under consideration have to be required, cf. Tasche (1999, Lemma 5.3) for a similar result.

implies by (3.6a)

$$\begin{aligned}0 &= \frac{\partial}{\partial u_i} G\big(G^{-1}_{(\widetilde{s},u)}(z), \widetilde{s}, u\big) \\ &= \frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{s},u)}(z) \frac{\partial}{\partial v} G\big(v, \widetilde{s}, u\big)\big|_{v=G^{-1}_{(\widetilde{s},u)}(z)} + \frac{\partial}{\partial w_i} G\big(G^{-1}_{(\widetilde{s},u)}(z), \widetilde{s}, w\big)\big|_{w=u} \\ &= \frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{s},u)}(z) \frac{\partial}{\partial v} G\big(v, \widetilde{s}, u\big)\big|_{v=G^{-1}_{(\widetilde{s},u)}(z)} + g_i\big(G^{-1}_{(\widetilde{s},u)}(z), \widetilde{s}\big)\end{aligned}$$

and as a further consequence

$$\frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{s},u)}(z) \;=\; -\frac{g_i\big(G^{-1}_{(\widetilde{s},u)}(z), \widetilde{s}\big)}{\frac{\partial}{\partial v} G\big(v, \widetilde{s}, u\big)\big|_{v=G^{-1}_{(\widetilde{s},u)}(z)}}. \tag{3.8a}$$

Additionally, we have

$$\frac{\partial}{\partial z} G^{-1}_{(\widetilde{s},u)}(z) \;=\; \left(\frac{\partial}{\partial v} G\big(v, \widetilde{s}, u\big)\big|_{v=G^{-1}_{(\widetilde{s},u)}(z)}\right)^{-1}. \tag{3.8b}$$

Assuming existence[8] of $\frac{\partial q_\alpha(u)}{\partial u_i}$, it can be implicitly determined as follows:

$$\begin{aligned}\alpha &= \mathrm{P}[L(u) \le q_\alpha(u)] \\ &= \mathrm{E}\big[\mathrm{P}[S_1 \ge G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,|\, \widetilde{S}]\big] \\ &= \mathrm{E}\left[\int_{G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}^{\infty} h(y \,|\, \widetilde{S})\, dy\right]\end{aligned}$$

implies

$$0 = -\mathrm{E}\left[\frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))\, h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,|\, \widetilde{S}\big)\right]. \tag{3.9}$$

By (3.8a) and (3.8b) we obtain

$$\begin{aligned}\frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) &= \frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{S},u)}(z)\big|_{z=q_\alpha(u)} + \frac{\partial}{\partial z} G^{-1}_{(\widetilde{S},u)}(z)\big|_{z=q_\alpha(u)} \frac{\partial q_\alpha(u)}{\partial u_i} \\ &= \left(\frac{\partial}{\partial v} G\big(v, \widetilde{S}, u\big)\big|_{v=G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}\right)^{-1} \left(\frac{\partial q_\alpha(u)}{\partial u_i} - g_i\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)), \widetilde{S}\big)\right).\end{aligned} \tag{3.10}$$

Replacing $\frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))$ in (3.9) by the right-hand side of (3.10) and solving for $\frac{\partial q_\alpha(u)}{\partial u_i}$ yields the assertion. □

Remark 3.5 *Equation* (3.7) *may equivalently be written as*

$$\frac{\partial q_\alpha(u)}{\partial u_i} \;=\; \mathrm{E}[g_i(S) \,|\, L(u) = q_\alpha(u)]. \tag{3.11}$$

[8]Under appropriate smoothness and moment conditions, existence can be proven by means of the implicit function theorem.

This follows from Proposition 2.3 and Corollary 2.4. For by Corollary 2.4, we have for any $z \in \big(0, \sum_{i=1}^n u_i\big)$

$$\mathrm{E}\big[g_i(S)\,\mathbf{1}_{\{L(u)\le z\}}\big] \;=\; -\int_0^z \mathrm{E}[g_i(S)\,|\,L(u)=t]\,\mathrm{E}\left[\frac{h\big(G(\cdot,\widetilde{S})^{-1}(t)\,|\,\widetilde{S}\big)}{\frac{\partial}{\partial v}G(v,\widetilde{S})\big|_{v=G(\cdot,\widetilde{S})^{-1}(t)}}\right]dt. \tag{3.12a}$$

On the other hand, Proposition 2.3 implies that

$$\mathrm{E}\big[g_i(S)\,\mathbf{1}_{\{L(u)\le z\}}\big] \;=\; -\int_0^z \mathrm{E}\left[\frac{g_i(G(\cdot,\widetilde{S})^{-1}(t),\widetilde{S})\,h\big(G(\cdot,\widetilde{S})^{-1}(t)\,|\,\widetilde{S}\big)}{\frac{\partial}{\partial v}G(v,\widetilde{S})\big|_{v=G(\cdot,\widetilde{S})^{-1}(t)}}\right]dt. \tag{3.12b}$$

Equating the right-hand sides of (3.12a) and (3.12b) respectively implies by taking the derivative with respect to z and then letting $z = q_\alpha(u)$ that $\mathrm{E}[g_i(S)\,|\,L(u) = q_\alpha(u)]$ equals the right-hand side of (3.7). □

A result analogous to Theorem 3.4 for VaR holds for ES as the following corollary shows.

Corollary 3.6 *Under Assumption 2.2, the Expected Shortfall risk measure $\mathrm{E}[L(u)\,|\,L(u) \ge q_\alpha(u)]$ of the generalized loss variable as defined in (2.10) is partially differentiable with respect to the weights u_i. The partial derivatives can be computed as*

$$\frac{\partial}{\partial u_i}\mathrm{E}[L(u)\,|\,L(u)\ge q_\alpha(u)] \;=\; \mathrm{E}[g_i(S)\,|\,L(u)\ge q_\alpha(u)], \quad i=1,\dots,n. \tag{3.13}$$

Proof. A straight-forward calculation as in the proof of Proposition 2.3 (see (2.12b)) yields

$$\begin{aligned}
&(1-\alpha)\,\frac{\partial}{\partial u_i}\mathrm{E}[L(u)\,|\,L(u)\ge q_\alpha(u)] \\
&= \frac{\partial}{\partial u_i}\left(\sum_{j=1}^n u_j\,\mathrm{E}\left[\int_{-\infty}^{G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))} g_j(y,\widetilde{S})\,h(y\,|\,\widetilde{S})\,dy\right]\right) \\
&= \sum_{j=1}^n u_j\,\mathrm{E}\left[\frac{\partial}{\partial u_i}G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))\,g_j\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)),\widetilde{S}\big)\,h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))\,|\,\widetilde{S}\big)\right] \\
&\quad + \mathrm{E}\big[g_i(S)\,\mathbf{1}_{\{L(u)\ge q_\alpha(u)\}}\big].
\end{aligned} \tag{3.14}$$

Making use of identity (3.10) and of the definition of $G^{-1}_{(\widetilde{S},u)}$ (see (3.6a) and (3.6b)) we obtain

$$
\begin{aligned}
&\sum_{j=1}^n u_j \,\mathrm{E}\left[\frac{\partial}{\partial u_i} G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))\, g_j\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)), \widetilde{S}\big)\, h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,\big|\, \widetilde{S}\big)\right] \\
&\quad= \frac{\partial q_\alpha(u)}{\partial u_i}\,\mathrm{E}\left[\frac{\Big(\sum_{j=1}^n u_j\, g_j\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)), \widetilde{S}\big)\Big)\, h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,\big|\, \widetilde{S}\big)}{\frac{\partial}{\partial v} G\big(v, \widetilde{S}, u\big)\big|_{v = G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}}\right] \\
&\qquad - \mathrm{E}\left[\frac{\Big(\sum_{j=1}^n u_j\, g_j\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)), \widetilde{S}\big)\Big)\, g_i\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)), \widetilde{S}\big) h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,\big|\, \widetilde{S}\big)}{\frac{\partial}{\partial v} G\big(v, \widetilde{S}, u\big)\big|_{v = G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}}\right] \\
&\quad= q_\alpha(u)\Bigg\{\frac{\partial q_\alpha(u)}{\partial u_i}\,\mathrm{E}\left[\frac{h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,\big|\, \widetilde{S}\big)}{\frac{\partial}{\partial v} G\big(v, \widetilde{S}, u\big)\big|_{v = G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}}\right] \\
&\qquad - \mathrm{E}\left[\frac{g_i\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)), \widetilde{S}\big) h\big(G^{-1}_{(\widetilde{S},u)}(q_\alpha(u)) \,\big|\, \widetilde{S}\big)}{\frac{\partial}{\partial v} G\big(v, \widetilde{S}, u\big)\big|_{v = G^{-1}_{(\widetilde{S},u)}(q_\alpha(u))}}\right]\Bigg\} \\
&\quad= 0,
\end{aligned}
$$

by Theorem 3.4. By means of (3.14) this implies the assertion. □

Remark 3.7 *At first sight, formulae (3.11) and (3.13) look very much like corresponding formulae for the derivatives of VaR and ES in Gouriéroux et al. (2000), Lemus (1999), and Tasche (1999). Note, however, that those formulae were not derived in an asymptotic multi-factor setting like the ones here. On the other hand, the validity of the results by Gouriéroux et al. (2000), Lemus (1999), and Tasche (1999) is not restricted to the case of bounded loss variables of the assets. Therefore, the results from this paper and the earlier results complement each other.*

4 Defining a diversification index

During the last few years three properties of risk measures ϱ turned out to be potentially most important:

- **Positive homogeneity.** See Assumption 3.2 for the formal definition. This assumption seems very natural as long as the asset or sub-portfolio under consideration does not dominate the portfolio and is not subject to liquidity risk. Moreover, by (3.3), positive homogeneity implies that, within a portfolio, the risk contributions add up to the total risk. This additivity property is of high practical importance.
- **Sub-additivity.** Sub-additivity of a positively homogeneous risk measures is equivalent to the property that risk contributions are not larger than the corresponding stand-alone

risks. Speaking in terms of Sub-section 3.1,

$$\varrho(V+W) \le \varrho(V) + \varrho(W) \text{ for all } V, W \;\Leftrightarrow\; \varrho(V\,|\,W) \le \varrho(V) \text{ for all } V, W, \tag{4.1}$$

if ϱ is positively homogeneous (Tasche, 2002, Proposition 2.5).

- **Comonotonic additivity.** Random variables V and W are called comonotonic if they can be represented as non-decreasing functions of a third random variable Z, i.e.

$$V = h_V(Z) \quad \text{and} \quad W = h_W(Z) \tag{4.2a}$$

for some non-decreasing functions h_V, h_W. As comonotonicity is implied if V and W are correlated with correlation coefficient 1, it generalizes the concept of linear dependence. A risk measure ϱ is called comonotonic additive if for any comonotonic random variables V and W

$$\varrho(V+W) \;=\; \varrho(V) + \varrho(W). \tag{4.2b}$$

Thus comonotonic additivity can be interpreted as a specification of the worst case scenarios for the sub-additivity (4.1): nothing worse can occur than comonotonic random variables – which seems quite natural.

Note that VaR is positively homogeneous and comonotonic additive but not sub-additive and that ES is positively homogeneous, comonotonic additive and sub-additive (see, e.g. Tasche, 2002). As a consequence, finding worst case scenarios for given marginal distributions of V, W in (4.1) is easy in case of ES (take the comonotonic scenario) and non-trivial in case of VaR (see Embrechts et al., 2003; Luciano and Marena, 2003).

As for positively homogeneous, comonotonic additive and sub-additive risk measures nothing worse than the comonotonic case can happen, it seems natural[9] to measure diversification by comparison with the comonotonic scenario. This suggests the following definition[10].

Definition 4.1 *Let $X_1, \ldots, X_n$ be real-valued random variables and let $Y = \sum_{i=1}^n X_i$. If ϱ is a risk measure such that $\varrho(Y), \varrho(X_1), \ldots, \varrho(X_n)$ are defined, then*

$$\mathrm{DI}_\varrho(Y) \;=\; \frac{\varrho(Y)}{\sum_{i=1}^n \varrho(X_i)}$$

denotes the diversification index *of portfolio Y with respect to the risk measure ϱ.*
The fraction

$$\mathrm{DI}_\varrho(X_i\,|\,Y) \;=\; \frac{\varrho(X_i\,|\,Y)}{\varrho(X_i)}$$

with $\varrho(X_i\,|\,Y)$ being the risk contribution of X_i as in Definition 3.1 denotes the diversification index *of sub-portfolio X_i with respect to the risk measure ϱ.*

[9] Martin and Tasche (2005) suggest another approach to measuring diversification as they calculate the proportions of systematic and idiosyncratic risk within the total risk of the portfolio.

[10] Without calling the concept "diversification index", Memmel and Wehn (2005) calculate a diversification index for the German supervisor's market price risk portfolio.

If ϱ is sub-additive and positively homogeneous, then by (4.1) both $\mathrm{DI}_\varrho(Y)$ and $\mathrm{DI}_\varrho(X_i \,|\, Y)$ will be bounded by 1. If ϱ is additionally comonotonic additive, then the bound 1 can be reached by portfolios with comonotonic risks. Thus, with a reasonable risk measure, $\mathrm{DI}_\varrho(Y)$ being close to 1 will indicate that there is no significant diversification in the portfolio. Similarly, a value of $\mathrm{DI}_\varrho(X_i \,|\, Y)$ close to 1 will indicate that there is almost no diversification effect with asset i.

Note that VaR by practical experience can be considered an almost sub-additive risk measure (but see Kalkbrener et al., 2004, for an example of a sub-additivity violation from practice). In the following section we will illustrate the use of the diversification indices from Definition 4.1 by a numerical example.

5 Numerical example

In this section, we illustrate the application of the formulae for the loss distribution function (Proposition 2.5) and the risk contributions to VaR (Theorem 3.4) with a simple example. We consider a special case of model (2.10) with two independent normally distributed systematic factors and normally distributed idiosyncratic risk drivers as in Example 2.1.

We assume that all but one of the assets are exposed to the first systematic factor only. Only one asset in the portfolio has additionally got an exposure to the second systematic factor. By varying the extent of this exposure to the second systematic factor we will obtain a dynamic picture of the effect of diversification by dependence on more than one systematic factors. Additionally, we will fix the exposure to the second systematic factor but vary the weights of the assets within the portfolio in order to get an impression of the dynamics of the diversification indices defined in Section 4.

Note that our example model can be considered a special case of segmentation, as there is one segment of assets with exposure to the first systematic factor only and another (degenerated to a single asset) segment with exposure to both systematic factors.

Example 5.1 *Consider the loss variable $\tilde{\tilde{L}}(u)$ from Example 2.1 with independent standard normally distributed systematic factors S_1 and S_2 and independent (also of (S_1, S_2)) standard normally distributed idiosyncratic risk drivers $\xi_1, \ldots, \xi_n$. We consider the case of relative (to the total exposure) loss, i.e. the case $\sum_{i=1}^n u_i = 1$. Additionally we assume*

$$p_i \;=\; p, \qquad i = 1, \ldots, n, \tag{5.1}$$

so that all assets have the same probability of default. With respect to the correlations with the systematic factors, we fix some $\varrho \in (0,1)$ and let

$$\varrho_{1,1} = \sqrt{\varrho\, w}, \quad \varrho_{1,2} = \sqrt{\varrho\,(1-w)}, \tag{5.2a}$$

$$\varrho_{i,1} = \sqrt{\varrho}, \qquad \varrho_{i,2} = 0, \quad i = 2, \ldots n, \tag{5.2b}$$

where $w \in [0,1]$ is a weight parameter controlling the exposure of the first asset to the second factor.

The square-root representation in (5.2a) and (5.2b) was chosen in order to make the correlations comparable in size with those from BCBS (2004, §272). $w = 1$ means that we are in a single factor model, whereas $w = 0$ implies that the default indicator of the first asset is independent from the rest of the portfolio. Note that due to the almost homogeneous structure of the portfolio under consideration the approximate loss variable $\tilde{\tilde{L}}(u)$ obtains a relatively simple form, not depending any longer on n, namely

$$\tilde{\tilde{L}}(u) \;=\; u\,\Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\varrho\, w}\, S_1 - \sqrt{\varrho\,(1-w)}\, S_2}{\sqrt{1-\varrho}}\right) + (1-u)\,\Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\varrho}\, S_1}{\sqrt{1-\varrho}}\right). \tag{5.3}$$

Since by independence of S_1 and S_2 the random variables S_1 and $\sqrt{w}\, S_1 + \sqrt{1-w}\, S_2$ are both standard normally distributed, both random parts of $\tilde{\tilde{L}}(u)$ are identically distributed. This way, we ensure that in the example all observed differences in VaR or in the risk contributions are due to the factor structure only.

For the first calculations we choose

$$p \;=\; 0.1, \quad \varrho \;=\; 0.1, \quad u \;=\; 0.1. \tag{5.4}$$

This choice is mainly driven by the desire to come up with illustrative results. Nevertheless, the values from (5.4) are not too far away from reality, as 10% probability of default may be observed in some retail credit portfolios. The value 10% for ϱ is somewhere in the center of the span provided by BCBS (2004). A 10% weight for a single asset might appear quite high. However, this single asset could be seen as having been created from a non-degenerated portfolio segment by transition to the limit in the sense of (2.8).

In order to assess the impact of factor diversification at portfolio level, first we calculate[11] VaR-figures at different levels both for the single factor model as in (5.3) with $w = 1$ as well as for the two factor model with $w = 0$. Table 1 shows that the impact even in the case of an independent second factor and for high levels of VaR remains limited.

This picture changes dramatically if we consider UL contributions with respect to VaR instead of total UL. "UL" means "unexpected loss" and is defined by choosing

$$\varrho(V) \;=\; \mathrm{VaR}_\alpha(V) - \mathrm{E}[V] \;=\; q_\alpha(V) - \mathrm{E}[V] \;=\; \mathrm{UL}(V) \tag{5.5}$$

in Definition 3.1. In Figure 1 we plot the relative contribution to UL with respect to 99.9%-VaR of the first asset in the model in Example 5.1 against the extent of the asset's exposure to the first factor (low values of w correspond to low exposure, values of w close to 1 correspond to high exposure). It turns out that the size of the risk contribution of the first asset can be reduced

[11] The more intricate calculations for this paper were conducted by means of the statistics software package R (cf. R Development Core Team, 2003).

Assuming conditional independence of the default events, given the realizations of the systematic factors, entails by the law of large numbers that the loss variable $\tilde{L}(u)$ can be reasonably approximated by a modified loss variable $\tilde{\tilde{L}}(u)$. The approximate loss variable $\tilde{\tilde{L}}(u)$ then depends on the systematic factor only (cf. Gordy, 2003; Lucas et al., 2001). This is interpreted as elimination of the idiosyncratic risk by diversification. In general, the quality of the approximation depends on conditions like the number of credit assets in the portfolio, the granularity of the portfolio, or the correlations of the asset value changes with the systematic factors. $\tilde{\tilde{L}}(u)$ is obtained from $\tilde{L}(u)$ by replacing the default indicators $\mathbf{1}_{D_i}$ with their best predictors given the systematic factors, i.e. with the conditional probabilities $\mathrm{P}[D_i \,|\, (S_1, \ldots, S_k)]$. Hence $\tilde{\tilde{L}}(u)$ is given by

$$\tilde{\tilde{L}}(u) \;=\; \sum_{i=1}^{n} u_i \, \mathrm{P}[D_i \,|\, (S_1, \ldots, S_k)]. \tag{2.8}$$

Example 2.1 *If the default events are given by (2.2) and the idiosyncratic risk drivers ξ_i are standard normally distributed, then the approximate loss variable $\tilde{\tilde{L}}(u)$ can be written as*

$$\tilde{\tilde{L}}(u) \;=\; \sum_{i=1}^{n} u_i \, \Phi\Big(\frac{t_i - \sum_{j=1}^{k} \rho_{i,j} \, S_j}{\omega_i}\Big). \tag{2.9}$$

Example 2.1 suggests to consider a – compared to (2.8) – slightly *generalized loss variable* $L(u)$

$$L(u) \;=\; \sum_{i=1}^{n} u_i \, g_i(S) \;=\; \sum_{i=1}^{n} u_i \, g_i(S_1, \ldots, S_k), \tag{2.10}$$

with $g_i : \mathbb{R}^k \to [0, 1]$, $i = 1, \ldots, n$ decreasing[3] at least in one (always the same) component of the vector argument. When investigating the generalized loss variable $L(u)$ we will need some technical conditions as specified in the following assumption.

Assumption 2.2

1. *The exposures $u_i, i = 1, \ldots, n$ in definition (2.10) are non-negative.*
2. *For any fixed $(k-1)$-tuple $(s_2, \ldots, s_k)$ the mapping*

$$s_1 \;\mapsto\; \sum_{i=1}^{n} u_i \, g_i(s_1, \ldots, s_k), \quad \mathbb{R} \to [0, \infty[$$

   is strictly decreasing, continuous, and onto $\big]0, \sum_{i=1}^{n} u_i\big[$.
3. *There is a conditional density $h(s_1 \,|\, s_2, \ldots, s_k)$ of S_1 given $(S_2, \ldots, S_k)$.*

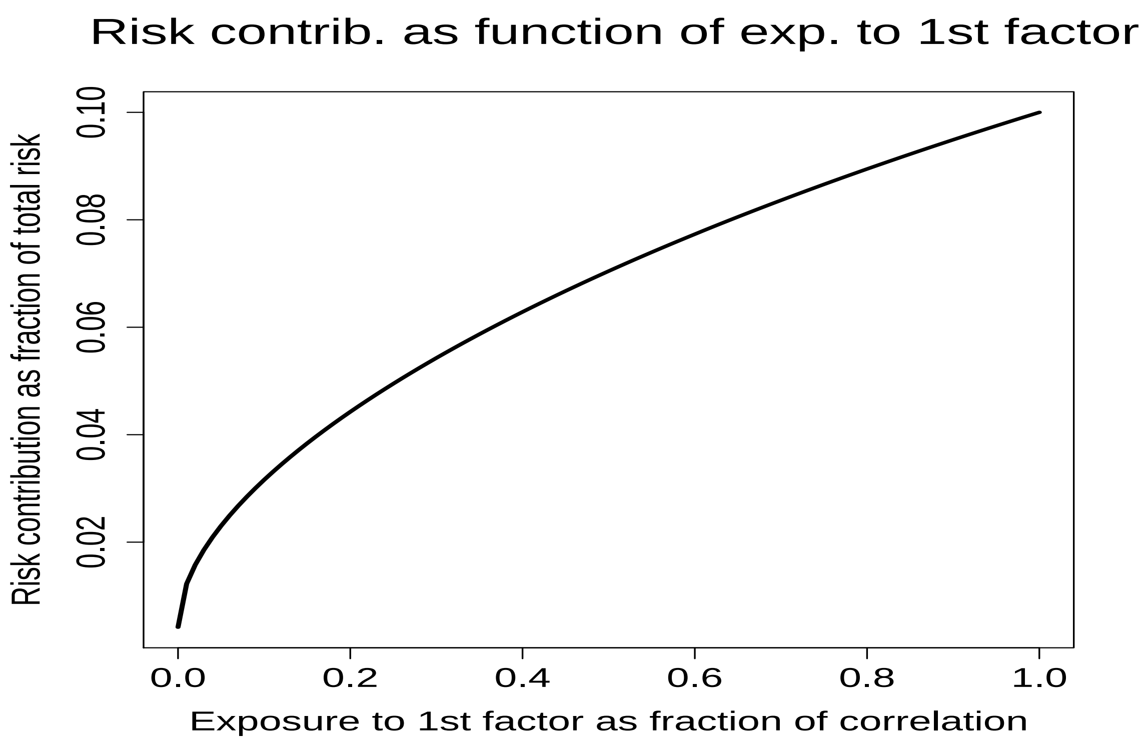
[3]The results of this paper hold also when "decreasing" is replaced by "increasing". Some of the formulae then must be appropriately adapted.

Table 1: *VaRs at different levels α for the asymptotic single ($w = 1$) and two-factor ($w = 0$) models as in Example 5.1. Parameter values as in* (5.4).

α	75%	90%	95%	97.5%	99%	99.9%	99.95%
VaR (single factor)	13.0%	17.8%	21.1%	24.3%	28.3%	37.4%	40.0%
VaR (two factors)	12.7%	17.0%	20.0%	22.9%	26.5%	34.7%	37.0%
"two/single"	97.9%	95.8%	94.9%	94.3%	93.7%	92.8%	92.6%

to almost 0 when it is exposed to the independent second systematic factor only. The rate of the reduction becomes the smaller the stronger the exposure to the first systematic factor is but remains significant.

Figure 1: *Relative contribution to 99.9%-VaR of the first asset in the model in Example 5.1 as function of the extent of the asset's exposure to the first factor (measured by $w \in [0, 1]$). Parameter values as in* (5.4).



In order to illustrate the functioning of the diversification indices defined in Section 4 we fix the exposure of the first asset to the second factor by setting $w = 0.5$. We then make the weight u (see (5.3)) of the first asset in the portfolio move from 0% to 100%. We do so with two different values for the probability of default p of the first asset. First, we choose the same parameter setting as in (5.4). Then we change the value of p to 0.2. Figure 2 shows that the portfolio-wide diversification index identifies a most diversified portfolio. In case of the fully symmetric setting of (5.4) the most diversified portfolio is the one where the weights of the two assets are equal. When the probability of default of the first asset is changed to a higher value – 20% – the weight of the first asset in the most diversified allocation is reduced to a value significantly less than 50% – as should be expected.

Figure 2: *Diversification index with respect to UL in sense of Definition 4.1 of loss variable* $\tilde{\tilde{L}}(u)$ *from Example 5.1. Represented for 2 values of default probability of first asset as function of weight* $u \in [0, 1]$ *of first asset.*

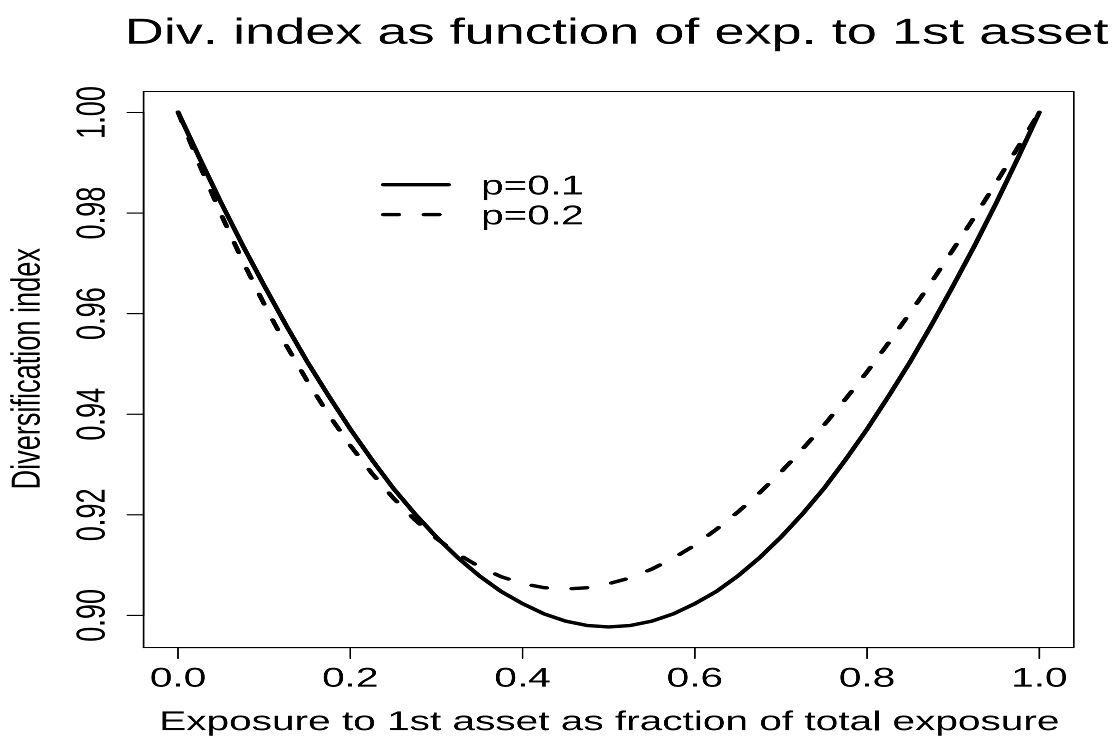
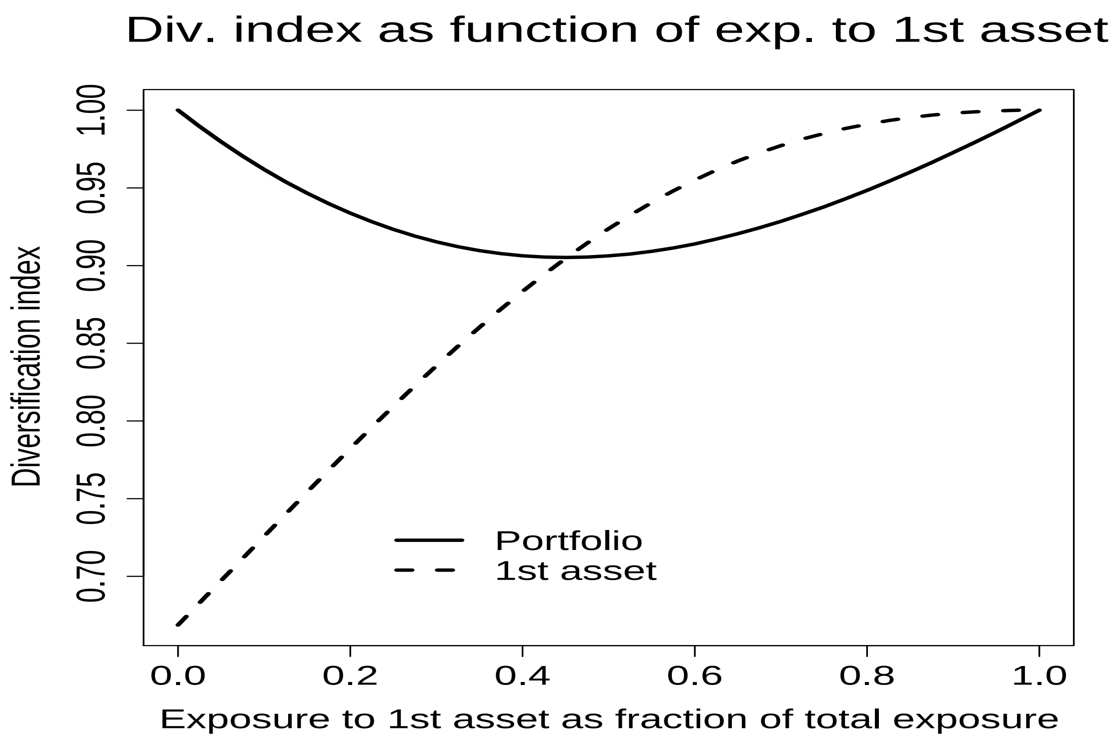


Figure 3 illustrates the connection between the two diversification indices from Definition 4.1. The solid line is – up to the scaling – identical to the dashed line in Figure 2, showing again the portfolio-wide diversification index of the loss variable from (5.3). The dashed line in Figure 3 reflects the corresponding risk contribution-based diversification index of the first asset. From the definition of the risk contribution by means of a derivative (see Definition 3.1) follows that the two lines intersect at just the weight of the first asset that yields the most diversified portfolio. According to Figure 3, well-diversified portfolios are those portfolios where portfolio-wide and risk contribution-based diversification indices are close together. A wide range of the diversification indices of a portfolio indicates that the portfolio is not very well diversified.

Figure 3: *Diversification indices with respect to UL in sense of Definition 4.1 of loss variable* $\tilde{\tilde{L}}(u)$ *and first asset from Example 5.1. Represented as function of weight* $u \in [0,1]$ *of first asset.*



6 Conclusions

In this paper we have derived explicit formulae for risk contributions to VaR and ES in the context of asymptotic multi-factor models, thus generalizing the capital requirements as provided by the Basel II Accord in the context of the ASRF (Asymptotic Single Risk Factor) model. The effort needed for the numerical calculations is higher than in the ASRF case but, as a numerical example shows, remains feasible at least in the case of two-factor models. The example also indicates that the effect of factor diversification on portfolio-wide economic capital is moderate but can be significant for risk contributions of single assets or sub-portfolios.

The risk contributions we analyze in the first sections of the paper can be used for calculating diversification indices for sub-portfolios or assets in a portfolio. If these indices, considered for all the assets in the portfolio, take a wide range, then there is a high potential for diversification in the portfolio. If, in contrast, the range of the indices is narrow, there is no potential left for diversification by changing the weights of the assets in the portfolio. In this case, more diversification can be only reached by adding new assets or by removing assets from the portfolio.

This observation suggests the use of the newly developed diversification indices for reflecting factor diversification: assets found well-diversified by an index close to the portfolio-wide index could receive a reduction of capital requirements. The sizes of such reductions could be estimated by means of an asymptotic two-factor model. Of course, the concrete choice of the model and its underlying parameters might have a strong impact on the estimates. Further research in this direction seems necessary.

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