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Basel Committee on Banking Supervision
Bank for International Settlements
Centralbahnplatz 2
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Dear Members of the Basel Committee on Banking Supervision:

Algorithmics Inc. appreciates this opportunity to comment on the "Guidelines for Computing Capital for Incremental Risk in the Trading Book" consultative document ("the Guidelines") issued in July 2008 by the Basel Committee on Banking Supervision ("the Committee"). As a provider of risk-related software and services to more than 300 international financial institutions, we have undertaken discussions of the Guidelines with a wide range of banks that will be directly and immediately impacted by the changes. For your reference, we provide a corporate overview of Algorithmics in Annex D.

We believe that the general direction of the revised proposal provides for a more comprehensive and risk-sensitive capitalization standard for the trading book. We further commend the Committee for its rapid response to events in the credit markets and its attempt to articulate attainable standards of practice for international financial institutions.

In particular, the extension of the incremental risk charge from default risk to default, migration, spread and equity risks – including correlations within and across those risks – provides simultaneously for realism and appropriate conservatism in capitalization standards for the trading book. The broader definition better captures the full risk profile of products most impacted by the recent market turmoil. Incorporating correlation across the modeled risk factors is, we agree, essential in capturing all material components of price risk found in many structured products today including cross-asset trading and hedging strategies employed by many trading desks. Given the enlarged scope of IRC, however, we do question whether a fully integrated model capturing general, specific and incremental risks might be more tractable.

Our main concern with the Guidelines is the option of adopting a constant level of risk approach to the calculation of incremental risks. We feel that this option is counterproductive: it adds complexity, reduces action-ability, clouds transparency and provides capital relief only through a subtle manipulation of input correlations. We presume the Committee initially adopted this

approach to redress industry concerns over the one-year horizon and 99.9% quantile. However, our studies show that capital relief provided by the approach, if any, is due to an artificial downplaying of correlations. A detailed discussion, including examples of the issues focusing on default and migration risks, is provided in Annex A. Annex B speaks to this subject from the equity risk perspective.

The lack of transparency in the constant level of risk approach and unclear action-ability of its measures are of pressing concern. Specifically, fictitious or virtual issuers, created as part of the roll-over at each liquidity horizon, lead to a lack of transparency by preventing risk managers from attributing the final capital requirement to specific, known names. Using standard risk attribution methods would give an attribution to (for example) Ford Motor Company as part of the current portfolio when in fact this capital must support Ford Motor Company and (under some circumstances) an unnamed, unknown substitute created within the model. Such opaqueness complicates business decisions, particularly hedging.

The issue of correlation dissolution highlighted in Annex A compounds the virtual issuer problem since correlations also dissipate insidiously and counter-intuitively within the model, reducing the portfolio manager's ability to, amongst other potential activities, assess macro hedges. Having a single measurement horizon and adopting the constant position approach would address the lack of transparency and forced inaction created by the constant level of risk approach with its underlying assumptions and implications.

In light of the lack of actionable information, the discrepancies between the methodologies and the complexities of implementation and oversight, **we strongly urge the Committee to consider removing the option of adopting a constant level of risk approach**. A compromise single, shorter liquidity horizon, specified directly by the Committee would both provide a more even playing field across institutions and relieve the implicit regulatory burden, without compromising realistic capitalization standards. As supporting evidence, we enclose Annex C. It illustrates the integrated measurement of market and credit risks in the trading book of one of our clients.

Beyond this serious concern, we also offer the following comments on the specific issues for which the Committee sought feedback, as outlined in section IX of the Guidelines. We limit our responses to the first seven (7) questions.

Scope and coverage

1. Under the proposal, the IRC would reflect all price risks except those directly attributable to movements in commodity prices, foreign exchange rates, or the term structure of default-free interest rates ("non-IRB market factors").

(a) Would it be preferable for supervisors to list specific types of events that must be captured (eg defaults, migrations, and only certain types of movements in credit spreads and equity prices)? What should be the basis for determining which types of events would be included, and how could the Committee ensure that the framework was not largely backward looking?

In general, definitions are preferable to lists because they adapt to individual firms' circumstances and market changes more effectively. However, in this case, as seen in the answer to question #2 below, the distinction amongst the risk types is unclear in the Guidelines. If the Guidelines are meant to read that all spread and equity price movements are included, then they

are sufficiently clear. If this is not the case, as some of the proposed changes to the Accord[1], s.718(xciii) seem to indicate, then an inclusionary or exclusionary list would prove helpful.

(b) Would it be worthwhile to expand the scope and coverage of the IRC to capture price risks associated with commodity prices, foreign exchange rates and the term structure of default-free interest rates?

Assuming that such an extension would effectively combine the measurement processes and standards for general, specific and incremental market risks into a single framework, we support the idea of broadening the scope. A fully integrated measure of risk would be beneficial to both the financial institution and the regulator for several reasons:

- More clarity: a fully integrated measure of risk would remove the necessity of artificially defining and categorizing events and risks and so more easily enable consistent, comprehensive modeling and measurement programs.
- More comprehensive: when all risks are measured consistently, there is not the same likelihood that risks will go uncounted. Likewise, there is not the chance that risks will be double-counted. The current rules are subject to the risk that models, processes, roles and responsibilities, incentive schemes, limits or measures will fail to acknowledge certain risks due to misclassification.
- More actionable: incorporating all risks into a single framework allows for more effective decision-making; this increases compliance with the use test and effectively aligns capital requirements with business practice.
- More realistic: an integrated model for risk is better suited to the realities of trading book positions in general, and complex derivatives in particular. It should also capture concentration and correlation risks more effectively.

Annex C presents a case study based on the trading book of one of our clients. It illustrates part of their current ability in integrating risk, and is representative of a significant proportion of internationally active financial institutions.

General versus specific risks

2. For covered IRC positions, Pillar 1 charges would depend in various ways on three types of risks: general market risks and specific risks, as defined under the current MRA, and IRC covered risks. Are the differences among these types of risks clear and measurable?

Unfortunately, the differences amongst the risk types are not clear. Unlike the previous definition of incremental default risk, the current definition of incremental risk is ambiguous. For example, s.11 and s.12 of the Guidelines indicate equity price risks are included in their entirety under IRC. They also imply that LIBOR rates and (potentially) some sovereign interest rate curves (e.g., emerging markets or non-G7) would be included in IRC as spread risks. At the same time s.718(xciii) of the proposed revisions to the Accord[2] indicates that only risks incremental to those captured by the VaR-based calculations need be included. This is a material discrepancy in the definition of the charge.

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[1] See "Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version", June 2006.

[2] See "Proposed revisions to the Basel II market risk framework", July 2008.

In the case of an outright holding of an exchange-traded equity, for example, it is demonstratively punitive in comparison to capitalization requirements for similar holdings in the banking book or using alternative methods in the trading book. We have two specific observations on this point:

1. The fall-back feature of the regulations, requiring the standardized method to be used, results in the lowest capital requirements across all trading and banking book models.

2. The total capital requirements under the trading book rules produce capital requirements far above the IRB credit risk framework while making assumptions on the equity holding period that is at most one year, compared to the five year effective maturity assumed in the IRB standard.

This discrepancy raises concerns about motivation in moving to more advanced approaches and creates a potentially unfair playing field. Annex B discusses equity risks in more detail and provides background examples in support of these observations.

Double-counting adjustments

3. While the capital horizons and confidence levels underlying the IRC and the 10-day VaR charge would differ, the risk factors captured by these risk measures would overlap to a significant degree. However, any adjustments to offset double-counting would complicate the framework and diminish the Pillar 1 importance of the 10-day VaR calculations including incentives to estimate the 10-day VaR as accurately as possible. Is it possible to provide double-counting adjustments that do not raise such concerns? How?

Based on our studies already presented in the Annexes, the double-counting adjustment is unlikely to be the sole cause of diminishing the importance of the 10-day VaR calculations. The magnitude of the incremental risk charge (default, migration, equity and spread related) on the stated one-year time horizon is likely to be so large, with or without a constant level of risk approach, so as to overwhelm the 10-day VaR. Thus, the capital charge based on 10-day VaR will be immaterial to the overall capital level, regardless of any position taken on double-counting. Should the IRC rules be revised to address some of the scope or measurement horizon concerns, the double-counting issue might re-emerge as a concern.

Capital horizon and confidence level

4. The proposal stipulates that an IRC model incorporate a one-year capital horizon, a 99.9 percent confidence level, and a liquidity horizon appropriate for each trading position. The Committee recognises that such an approach could present considerable practical challenges, including the need for data to calibrate key parameters.

(a) What alternative guidelines would achieve the Committee's objectives, but in a manner that would be less costly or difficult to implement?

The use of an **integrated model across a single, common horizon**, without the complexity of roll-overs to create a constant level of risk, would be a more feasible alternative. Under this approach, data collection would be less costly to implement because the practical challenges around the assignment of a liquidity horizon, giving due consideration to hedging and position-level issues, would be avoided.

Such an alternative would also simplify the model and the estimation of model parameters; in particular the mechanics of correlation assumptions. In an environment where liquidity horizons vary (primarily) by product type, it is difficult to combine positions for a single name and assess incremental risks for the entire name. It is also difficult to correlate across names. While enticing, the concept of roll-overs under the constant level of risk approach adds greatly to the complexity of modeling correlations appropriately. A reduction in complexity is of critical concern because the correlations are usually a key driver of capital requirements at the 99.9% level.

The analysis presented in Annex A was facilitated by the simplifying assumption that the liquidity horizon for all positions was one month. The constant level of risk approach with a consistent roll-over assumption meant we could calculate the one month distribution and then just convolute it twelve times to calculate the one-year distribution. The mechanics of allowing for different liquidity horizons would effectively eliminate any remaining vestiges of the estimated correlation structure.

For instance, if we estimated capital requirements for each segment (i.e., product type) of the book, as characterized by its liquidity horizon separately then aggregated after the fact, the one-month segment's distribution would be convoluted twelve times, the three-month segment's distribution four times and the one-year segment's distribution used outright. The resultant distributions could be convoluted together to get the final distribution. However, not only does this significantly reduce correlation between issuers but it also ignores correlation between multiple instruments referencing a single issuer.

Another implication of roll-overs is the creation of a short (one-month) horizon for many asset classes. One month is problematic in the accurate estimation of probability of default values as they typically become exceedingly small on this horizon. A longer horizon would be better suited to effective estimation of probability of default. A single, common measurement horizon is likely, we presume, to exceed one month for more fundamental reasons thereby mitigating this concern.

A final thought on the constant level of risk approach relates specifically to equities: It produces higher capital requirements than the constant position approach for equity portfolios, counter to the stated objective of the Guidelines. We again refer to the details provided in Annex B.

(b) Given the current state of risk modelling, is it feasible to estimate the portfolio loss distribution (excluding non-IRC market factors) over a one-year capital horizon at a 99.9 percent confidence level?

This question can be answered in three parts.

First, the issues around estimating incremental risk, excluding non-IRC market factors, are similar in scope to those in the estimation of credit risk capital in the banking and trading books. In our experience, many financial institutions perform these calculations quite readily today for default and migration risks. The inclusion of equity and spread risk in an integrated model with default and migration risks adds to overall model complexity. However, many financial

institutions today are already able to do this type of calculation using sophisticated, publicly available[3] approaches.

While we see no new quantitative modeling issues in measuring these risks over a one-year capital horizon at a 99.9% confidence level, long-standing issues remain. For example, value-at-risk is an incoherent measure (mathematically speaking) and as a result other industries such as insurance have moved to conditional tail expectations. Further, while modeling is possible at the one-year, 99.9% level, other processes are less developed. For example, back-testing and model parameter estimation remain particularly challenging in this extreme part of the tail.

The inclusion of varying liquidity horizons and the assumption of a constant level of risk create the need for complex, multi-level roll-over schemes. This in turn leads to many issues in both the modeling and interpretation of results. Many financial institutions are struggling with these concepts. There is a risk that inappropriate or ill-considered models and practices may develop in the pursuit of perceived capital relief. The Committee can encourage firms to focus on real issues such as calibration, coherence, back-testing and implementation without the distraction of complications from the proposed roll-over options by mandating a single measurement horizon.

(c) Would it be worthwhile to allow banks to use a single horizon for all covered positions (eg three months) and a lower confidence level (eg 99 percent), together with a supervisory scaling factor that was calibrated to achieve broad comparability with the IRB Framework for the banking book? Would such an approach be as useful for internal risk management purposes as the proposed IRC?

We feel that such a measure, provided the supervisory scaling factor is well considered, will prove more effective than the currently-proposed IRC. Having a scaling factor to adjust for the measurement horizon would be beneficial in assuring appropriate calibration across banking and trading activities. However, given the nature of the risks to be measured, we suggest that it would prove more informative to estimate the 99.9% quantile itself, rather than rescaling from a lower quantile.

As discussed in our opening remarks, the constant level of risk approach presents difficulties in interpretation of results. For IRC to be successfully adopted as a risk measure for internal use, it must be both transparent and actionable. The constant level of risk approach neither transparently identifies sources of risk, nor promotes positive risk management action.

Validation

5. Given the IRC soundness standard of a one year time horizon and 99.9th percentile loss, the Committee seeks comment on how the resulting risk measure might be validated quantitatively. For example, would it be reasonable to validate the underlying model at shorter horizons and/or at lower percentiles? If so, how might one ensure that the validation exercise is relevant for the one year 99.9th percentile standard? Also, would different aspects of the model likely require different validation approaches?

Validation of the model at a one year time horizon and 99.9th percentile will necessarily differ from the back-testing techniques employed today for general market risks. Validating at lower

[3] For example, see the conference proceedings for the "Conference on the Interaction of Market and Credit Risk", December 6-7, 2007, (hosted by the Research Task Force of the Basel Committee on Banking Supervision, et. al.). For an alternative approach, see "Compound scenarios: An efficient framework for integrated market-credit risk", Algorithmics Research Paper Series, May 2007 available at: http://www.algorithmics.com.

quantiles is unlikely to be informative concerning the soundness of the capital requirement because it is exactly the atypical, stressful conditions that the model must address. Mathematically speaking, fitting too closely at lower quantiles might imply a lack of fit at the higher quantiles rather than the desired outcome of assuring a good fit. This issue is not new: it is shared by the general market risk models and validation techniques to some extent. In that case, scaling has become a de facto standard and could be used judiciously under the constant position approach. Other approaches such as survival probability analysis against historical events and plausible scenarios could be used to both enhance comparability across firms and foster confidence in the models.

As discussed earlier, the assumption of rolling-over positions at the liquidity horizon makes practical interpretation of results created under the constant level of risk difficult. In turn, this difficulty makes it harder to envision back-testing or other validation processes. We see no consistent way to validate the completely synthetic positions in assets of unknown issuers created under the constant level of risk approach. Further, as a key part of the model, the roll-over assumption would presumably also require validation. Validation of a hypothetical process is potentially problematic.

6. The flexibility built into the proposed IRC potentially could make Pillar 1 charges for trading positions less comparable across banks. How might the framework ensure greater comparability without unduly limiting firms modelling choices? In particular, would it be productive to require banks to calculate risk measures for standardised test decks of trading portfolios, which could be used to compare model results across banks.

The idea of all firms calculating incremental risk measures for a standard portfolio (or set of portfolios) holds great merit. It would provide a quantitative measure of the variability across banks. Extremely high and low values could be used to pinpoint the aspects of the model to be reviewed most closely. At the same time, we do foresee some difficulties in operationalizing the process. For example, incremental risk is related directly to specific issuers and issues. As such, a bank that did not hold issues used in the standard portfolio might model them less effectively than those institutions holding the issues. This could create an imbalance in the model review process. Careful definition of the standard portfolio and the comparison process would mitigate this concern.

Speaking to the flexibility of the Guidelines, we found them to be difficult to interpret, leaving several open questions. In the face of uncertain interpretation of the regulations, banks are likely to adopt those interpretations producing lower capital levels – perhaps contrary to the spirit of the Guidelines (e.g. see response to question #2 above where the Standardized Method, while being less risk-sensitive, results in far lower capital requirements than the proposed Internal Models Methods). Refining the language in key areas would be helpful.

Implementation timeline

7. Is the proposed implementation schedule feasible? If not, which IRC guidelines, and what specific types of positions or risk factors, are most problematic?

The timeline of January 2010 will be challenging for many banks, including some of our clients. The issue is not necessarily the details of the measurement method or list of risk types considered, but rather the policy, data and processes around the calculations and their validation. In this regard, interim rules or a reduction in complexity are likely to prove beneficial.

One significant issue blocking immediate adoption of the Guidelines is the combination of many implementation options and the lack of clear benefits and drawbacks of each option. This is likely to result in the need for significant testing before a regulatory submission or application is put forth. Many firms will be unable to make a selection within the stated timeframe, particularly as concerns equity risk.

The inclusion of mandated, varying liquidity horizons is likely to create data mapping issues and significant theoretical debate. It is unlikely that this can be completely resolved in the given timeframe for large trading operations under current market conditions.

Further, reinventing, re-implementing or reconfiguring models used for other default risk measures to adequately assess liquidity horizon considerations will present a technical and methodological challenge.

To summarize, it is our view that integrated measures of a comprehensive set of risk types provide for the best representation of reality without introducing artificial boundaries that create concerns (e.g., double counting of risks, precise categorization of risks for quantification). Data and validation issues are very real, but can be addressed with time and creativity. For management purposes, using scaling factors to extend and shrink time horizons may be an acceptable approximation, but this is unlikely to be the case across quantiles. Stress testing, trend analysis, relative value comparisons and reconciliation to a base model are most likely to provide the assurances sought in model validation exercises.

The main issues we encountered with the Guidelines lie (1) in the language around the risks to be measured and (2) in the, perhaps unexpected, implications of measuring those risks as specified under the constant level of risk approach. Specifically, a reconsideration of the latter approach is encouraged.

We thank the Committee for its diligent review of our concerns and comments. We would welcome queries or requests for further detail on any of the topics raised, or related issues. We can be reached by telephone at +1 416 217 1500. Alternatively, we are available via e-mail to Michael.Zerbs@Algorithmics.com or Ben.DePrisco@Algorithmics.com.

Sincerely,

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Supporting Documents

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Algorithmics

Losing Default Correlations, Step by Step

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October 2005

Abstract

The behaviour of default correlation is assessed analytically for a discrete, multistep approach known as the Bucketing-Incremental (BI) credit-risk model. The impact of three key factors is evaluated: (1) asset correlation, (2) time horizon, and (3) number and density of time steps. The results reveal that the default correlations are significantly undervalued in comparison with the more realistic approach known as the Continuous-Cumulative (CC) model.

1 Introduction

The BI (Bucketing-Incremental) multistep model is a rough approximation to the CC (Continuous-Cumulative) model[1] for correlated defaults among counterparties. Both are structural (or Merton-type) models; the essential difference being that the BI model is, loosely speaking, a concatenation of the single-step model(s) over the entire time horizon: at the end of each time period, the creditworthiness index[2] essentially restarts at the zero value. In the two-state Default/non-Default case, this means that counterparties are restored to perfect health at the beginning of each time period, provided that they haven't defaulted in the previous time period. In contrast, in the CC multistep model, if a counterparty has almost defaulted, i.e., their creditworthiness index has landed very close to but has not crossed the default boundary, then they are more poised for default in the next time period, as they are closer to the next boundary point.

Thus, conditional on a "bad" credit-driver scenario, if two counterparties have positive (but not perfect) default correlation, there is the possibility that during a time step, one of them may just default while the other one may just survive, depending on the sizes and signs of their creditworthiness indices' idiosyncratic components. In the BI model, the lucky counterparty will be restored to perfect health and will be less likely to default in the next time step(s). The overall effect will be a reduction in clustering of defaults and, therefore, an effective reduction in default correlation. The logical conclusion is that there will be an understatement of concentration risk in the tails.

The purpose of this paper is to supply some rigour to the previous heuristic analysis. For (notational) simplicity, we work with a single credit driver and two counterparties of identical characteristics. Assuming that they are not perfectly correlated, for a fixed time horizon we show that, asymptotically, in the BI model, as the time-step size tends to zero, the default correlation tends to zero, regardless of the value of the credit-driver weight. We also provide some numerical results to illustrate the degeneracy when there are many macroscopic time steps.

In the next section we set out the notation, restate the BI and CC model definitions mathematically, and state the main results. The technical parts of the proofs are given in the Appendix. In the third section, some illustrative numerical experiments are presented.

[1] The CC model is the accepted standard at Algorithmics, Inc.
[2] The creditworthiness index relates changes in an obligor's credit state to changes in a set of set of credit-risk drivers such as market indexes or macroeconomic factors. For details, see [1].

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2 Models and main results

2.1 Notation and problem definition

We denote the fixed time horizon by the interval $[0, T]$. There are n intermediate time steps, offset from *today* (time 0), and denoted by $t_0 = 0, t_1, \ldots, t_n = T$; and $\Delta t := \max_i(t_i - t_{i-1})$. We track only the Default/non-Default status of each of two counterparties, CP1 and CP2, and denote their common Probability of Default Curve (PD) by $p(t),\ t \in [0, T]$:

$$p(t) = \mathbb{P}(\text{Default by time } t)$$

For brevity, we denote $p(t_i)$ by $p_i,\ 0 \le i \le n$.

The conditional probabilities of default for each period, given survival up to that period, are given by

$$\tilde{p}_i = \frac{p_i - p_{i-1}}{1 - p_{i-1}}, \qquad 1 \le i \le n.$$

The creditworthiness index for each counterparty, is

$$Y_i^k = \beta X_i + \sigma \epsilon_i^k, \qquad k = 1, 2;\ 0 \le i \le n; \tag{1}$$

where

$$\beta^2 + \sigma^2 = 1$$

and the credit driver, X, and idiosyncratic components, ϵ^k, are independent, Normal random walks;[3] so that $X_i, \epsilon_i^k, Y_i^k \sim N(0, t_i)$, and their increments, over $[t_{i-1}, t_i]$, are independent $N(0, t_i - t_{i-1})$-distributed random variables.

Set

$$\delta_T^k = \begin{cases} 1, & \text{if counterparty } k \text{ defaults by time } T; \\ 0, & \text{otherwise.} \end{cases} \qquad k = 1, 2.$$

Our interest is in the behaviour of $\text{Corr}(\delta_T^1, \delta_T^2)$, as $n \to \infty$. Now,

$$\text{Corr}(\delta_T^1, \delta_T^2) = \frac{\text{Cov}(\delta_T^1, \delta_T^2)}{\sqrt{\text{Var}(\delta_T^1)}\sqrt{\text{Var}(\delta_T^2)}};$$

$$\text{Var}(\delta_T^k) = p(T)(1 - p(T)), \qquad k = 1, 2;$$

and

$$\begin{aligned} \text{Cov}(\delta_T^1, \delta_T^2) &= \mathbb{E}\left[\delta_T^1 \delta_T^2\right] - \mathbb{E}\left[\delta_T^1\right]\mathbb{E}\left[\delta_T^2\right] \\ &= \mathbb{P}(\text{Both CP1 \& CP2 default by time } T) - p(T)^2. \end{aligned}$$

Thus

$$\text{Corr}(\delta_T^1, \delta_T^2) = \frac{\mathbb{P}(\text{Both CP1 \& CP2 default by time } T) - p(T)^2}{p(T)(1 - p(T))} \tag{2}$$

and $\mathbb{P}$(Both CP1 & CP2 default by time T) is the only term that depends on the number of time steps, n.

[3] i.e., the discrete-time (but still continuous state-space) analogue of Brownian motion

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We denote by Δ, the difference operator on sequences $(a_i;\ 0 \le i \le n) : \Delta a_i := a_i - a_{i-1}$. Also, the standard univariate normal cdf is denoted by Φ, and $\bar{\Phi} := 1 - \Phi$. The standard bivariate normal cdf, with correlation ρ, is denoted by $\Phi_\rho^{(2)}$. For brevity, we write $\Phi_\rho^{(2)}(x) \equiv \Phi_\rho^{(2)}(x,x)$. Clearly, $\Phi_0^{(2)}(x) = \Phi^2(x)$ and $\Phi_1^{(2)}(x) = \Phi(x)$.

We omit the superscript from the symbols CP, δ_T, etc., in expressions which have the same value for both counterparties.

There is one theoretical result (see Proposition 1) which holds very generally; namely that the default correlation is nonnegative in any model in which defaults are independent among counterparties, conditional on some auxiliary random variable, possibly vector-valued (the credit-driver path in our model). We also conjecture that the default correlation is eventually a decreasing function of the time horizon, T. The behaviour for small to moderate values of T, is model-dependent, as we shall see.

Proposition 1 *The default correlation is nonnegative in any model in which defaults are independent among counterparties, conditional on some auxiliary random variable.*

Proof. Denote the conditioning random variable by X and the default times of the two counterparties by τ_1 and τ_2. Also denote the conditional probability, $\mathbb{P}(\cdot \,|\, X)$ by $\mathbb{P}_X$. Using the symmetry of the counterparties, and conditioning on X to access their independence, we calculate

$$\begin{aligned}
\mathbb{P}(\text{Both CP1 \& CP2 default by time } T) &= \mathbb{P}(\tau_1 \le T,\ \tau_2 \le T) \\
&= \mathbb{E}\left[\mathbb{P}_X(\tau_1 \le T,\ \tau_2 \le T)\right] \\
&= \mathbb{E}\left[\mathbb{P}_X(\tau_1 \le T)\mathbb{P}_X(\tau_2 \le T)\right] \\
&= \mathbb{E}\left[\mathbb{P}_X(\tau \le T)^2\right] \\
&\ge \left(\mathbb{E}\left[\mathbb{P}_X(\tau \le T)\cdot 1\right]\right)^2 \\
&= \mathbb{P}(\tau \le T)^2 \equiv p(T)^2,
\end{aligned}$$

where, in the third-to-last step, we have used the Cauchy-Schwarz inequality. Thus the numerator (covariance) in (2) is nonnegative. □

This proof also shows we have zero correlation if and only if we have equality in our application of the Cauchy-Schwarz inequality. The latter occurs if and only if $\mathbb{P}_X(\tau \le T)$ is identically constant (with respect to X). This can only happen if $\beta = 0$, in the definition (1) of the creditworthiness index.

Conjecture 1 *In any default model, the default correlation is a decreasing function of the time horizon, T, for $T \ge$ median of p, i.e., for all T such that $p(T) \ge 1/2$.*

We will see some evidence in support of this conjecture in Section 3.2.

2.2 BI model; exact results

In the BI model, counterparty defaults occur independently, conditional on a given credit driver scenario, $X = x$. (Here x denotes the entire sample path, $(x_i; 0 \le i \le n)$, with $x_0 = 0$.) Moreover, default occurs at time, t_i, if $\Delta Y_i^k < b_i$ and $\Delta Y_j^k \ge b_j$ for all $j < i$. By the independence of Y^k's increments, this characterization determines the default boundary points, b_i, through the equation,

$$\tilde{p}_i = \mathbb{P}(\Delta Y_i^k < b_i \,|\, \Delta Y_j^k \ge b_j, \forall j < i) = \mathbb{P}(\Delta Y_i^k < b_i);$$

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and since X_i^k is $N(0,t_i)$-distributed,

$$b_i = \sqrt{\Delta t_i}\Phi^{-1}(\tilde{p}_i).$$

The conditional independence and independent increments properties allow us to make the following reductions in the calculation of the basic joint default probability. The first one was already used in the proof of Proposition 1.

$$\begin{aligned}\mathbb{P}(\text{Both CP1 \& CP2 default by time } T) &= \mathbb{E}\left[\mathbb{P}(\text{Both CP1 \& CP2 default by time } T \,|\, X)\right] \\ &= \mathbb{E}\left[\mathbb{P}(\text{CP defaults by time } T \,|\, X)^2\right] \qquad (3)\end{aligned}$$

$$\begin{aligned}\mathbb{P}(\text{CP defaults by time } T \,|\, X = x) &= \sum_{i=1}^{n} \mathbb{P}(\text{CP defaults in } (t_{i-1}, t_i] \,|\, X = x) \\ &= \sum_{i=1}^{n} \mathbb{P}(\Delta Y_i < b_i;\ \Delta Y_j \geq b_j\ \forall j < i \,|\, X = x) \\ &= \sum_{i=1}^{n} \mathbb{P}\left(\frac{\Delta\epsilon_i}{\sqrt{\Delta t_i}} < \frac{b_i - \beta\Delta x_i}{\sigma\sqrt{\Delta t_i}};\ \frac{\Delta\epsilon_j}{\sqrt{\Delta t_j}} \geq \frac{b_j - \beta\Delta x_j}{\sigma\sqrt{\Delta t_j}}\ \forall j < i\right) \\ &= \sum_{i=1}^{n} \Phi\left(\frac{b_i - \beta\Delta x_i}{\sigma\sqrt{\Delta t_i}}\right) \prod_{j<i} \bar{\Phi}\left(\frac{b_j - \beta\Delta x_j}{\sigma\sqrt{\Delta t_j}}\right). \qquad (4)\end{aligned}$$

The evaluation of the expectation of the square of the last result, is carried out in the Appendix. We state the end result here.

Proposition 2

$$\begin{aligned}&\mathbb{P}(\textit{Both CP1 \& CP2 default by time } T) \\ &= \sum_{i=1}^{n} \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_i)) \prod_{k<i}[1 - 2\tilde{p}_k + \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k))] \\ &\quad + 2 \sum_{1\leq i<j\leq n} \{\tilde{p}_i - \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_i))\}\{\prod_{k<i}[1 - 2\tilde{p}_k + \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k))]\}\{\tilde{p}_j\}\{\prod_{i<\ell<j}[1 - \tilde{p}_\ell]\} \qquad (5) \\ &= \sum_{i=1}^{n} \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_i)) \prod_{k<i}[1 - 2\tilde{p}_k + \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k))] \\ &\quad + 2 \sum_{1\leq i\leq n-1} \{\tilde{p}_i - \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_i))\}\{\prod_{k<i}[1 - 2\tilde{p}_k + \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k))]\}\{\frac{p_n - p_i}{1 - p_i}\}. \qquad (6)\end{aligned}$$

The expression (6) results from a simplification of the last pair of factors in the second term in (5):

$$\begin{aligned}\tilde{p}_j\{\prod_{i<\ell<j}[1 - \tilde{p}_\ell]\} &= \frac{\Delta p_j}{1 - p_i}; \\ \sum_{j=i+1}^{n} \tilde{p}_j\{\prod_{i<\ell<j}[1 - \tilde{p}_\ell]\} &= \frac{p_n - p_i}{1 - p_i}.\end{aligned}$$

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The expression (6) is more convenient for numerical calculations.

We describe three situations in which the formula in Proposition 2 simplifies. The first two are the extreme cases, $\beta = 0$ or $\beta = 1$, representing independent or completely correlated counterparties. The third situation is when all $\tilde{p}_i$ coincide, with common value $\tilde{p}$. This arises, in particular when the PD, $(p_i\,;\, 1 \le i \le n)$, are derived from a 2×2 transition matrix and all time steps are of the same length, denoted Δt.

Case $\boldsymbol{\beta = 0}$: Substituting Φ^2 everywhere for $\Phi_0^{(2)}$ in (5), we obtain

$$\sum_{i=1}^{n} \tilde{p}_i^2 \prod_{k<i} [1-\tilde{p}_k]^2 + 2 \sum_{1\le i<j\le n} \{\tilde{p}_i[1-\tilde{p}_i]\}\{\prod_{k<i}[1-\tilde{p}_k]^2\}\{\tilde{p}_j\}\{\prod_{i<\ell<j}[1-\tilde{p}_\ell\}. \tag{7}$$

This last expression is easily recognized as

$$\left[\sum_{i=1}^{n} \tilde{p}_i \prod_{k<i}[1-\tilde{p}_k]\right]^2 \equiv \left[\mathbb{E}\left[\delta_T\right]\right]^2$$

as it must, since $\mathrm{Corr}(\delta_T^1, \delta_T^2) = 0$, in this case. In the next section, we present the asymptotic version of this result, for $\beta^2 < 1$, as $\Delta t \to 0$. It turns out, under appropriate hypotheses, to be the same as for $\beta = 0$.

Case $\boldsymbol{\beta = 1}$: Substituting Φ everywhere for $\Phi_1^{(2)}$ in (5), we obtain (with $\delta_T^1 = \delta_T^2 \equiv \delta_T$.)

$$\begin{aligned}
\sum_{i=1}^{n} \tilde{p}_i \prod_{k<i}[1-\tilde{p}_k] + \; 2 \sum_{1\le i<j\le n} 0 \;&=\; \sum_{i=1}^{n} \Delta p_i, \text{ by (6)} \\
&=\; p(T) = \mathbb{E}\left[\delta_T\right] = \mathbb{E}\left[(\delta_T)^2\right] \\
&=\; \mathbb{E}\left[\delta_T^1 \delta_T^2\right].
\end{aligned}$$

This is consistent with the known result, $\mathrm{Corr}(\delta_T^1, \delta_T^2) = 1$, in this case.

Case "all $\boldsymbol{\tilde{p}_i}$ coincide": When all $\tilde{p}_i$ coincide, the sums in the general result (5) are geometric and can be summed explicitly:

Corollary 3 *In the case where $\tilde{p}_i \equiv \tilde{p}$, for all $1 \le i \le n$,*

$$\begin{aligned}
&\mathbb{P}(\textit{Both CP1 \& CP2 default by time } T) \\
&= \; 1 - \frac{1}{2\tilde{p} - \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}))} \left\{ (\Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}))) \left(1 - 2\tilde{p} + \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}))\right)^n \right. \\
&\qquad \left. + \, 2(\tilde{p} - \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}))) \left(1 - 2\tilde{p} + \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}))\right)^{n-1} \right\} \qquad (8) \\
&\qquad - \, 2(1-\tilde{p})^n \left[1 - \left(1 - \frac{\tilde{p} - \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}))}{1-\tilde{p}} \right)^{n-1} \right] \qquad (9)
\end{aligned}$$

We will use this result in the next section, when we study the behaviour of the default correlation for a fixed step size and long time horizon.

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2.3 BI model: asymptotic results

2.3.1 T fixed, $\Delta t \to 0$

In this section we study the limiting correlation, as the number of intermediate dates, $\{t_i;\ 1 \le i \le n\}$, increases in such a way that $\lim_{n\to\infty} \Delta t = 0$. Recall that Δt is the largest step size among all the steps, $t_i - t_{i-1}$. In order to obtain a theoretical result, we must impose some mild regularity condition on the PD curve. We assume that $\lim_{n\to\infty} \max_{1\le i\le n} \tilde{p}_i = 0$, at a rate comparable to Δt. A sufficient condition for this to hold, would be that the PD curve exists as a smooth curve with bounded derivative, on the time interval $[0, T]$; and that the PD is bounded away from 1 on that interval. Indeed, in that case, by the Mean Value Theorem of Calculus, for each i $(1 \le i \le n)$ there is some $t^* \in (t_{i-1}, t_i)$ such that

$$\tilde{p}_i = \frac{p(t_i) - p(t_{i-1})}{1 - p(t_{i-1})} = \frac{p'(t^*)\Delta t_i}{1 - p(t_{i-1})} \le C \times \Delta t_i$$

where the constant $C = \max_{t\in[0,T]} p'(t)/[1 - p(T)]$.

Proposition 4 *Under the assumptions*

(i) $0 \le \beta < 1$;
(ii) $\lim_{n\to\infty} \Delta t = 0$, $(\Delta t = \max_{1\le i\le n} \Delta t_i)$;
(iii) there exists a constant, C, independent of n and $1 \le i \le n$ such that $\tilde{p}_i \le C\Delta t_i$, for all $1 \le i \le n$.

the $\lim_{n\to\infty} \mathrm{Corr}(\delta_T^1, \delta_T^2) = 0$.

The proof of this result is given in the Appendix. The method is to show that, $\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_\cdot))$ tends to 0 sufficiently rapidly to allow us, *asymptotically*, to replace the terms $\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_\cdot))$ in (5), by $\tilde{p}^2_\cdot$, whereupon the sum of the terms in (5) becomes the expression (7), corresponding to the case $\beta = 0$.

It follows from the proof, that an estimate of the rate of convergence can be given as $const_\beta \times (\Delta t)^\gamma$, where $\gamma = (1 - \beta^2)/(1 + \beta^2)$ and $const_\beta$ is a bounded multiple of $(1 - \beta)^{-1/2}$. Thus the convergence would appear to be fastest for β near 0 and slowest near $\beta = 1$. In particular, we do not obtain uniform convergence with respect to β in the interval $[0, 1)$. The latter is not surprising, as the correlation is always 1 when $\beta = 1$. Rather than proving the absence of uniform convergence rigorously, we provide a numerical illustration in the Numerical Examples section.

2.3.2 Δt fixed, $T \to \infty$

In this section, we study the behaviour of the default correlation for large T when the step size is held constant. The BI model is Markovian – a finite (two) state Markov chain, for each counterparty. For the BI model, the correlation will be shown to tend to 0. This is to be expected because the survival of one counterparty should be independent of the other once the latter defaults, and that happens with certainty due to the Markovian nature of the BI model: we have a sequence of independent Bernoulli trials (over the time steps) for default. Moreover one expects that the correlation should converge geometrically fast, again by the Markovian nature of the BI model. We make this heuristic discussion precise in the following proposition, under the assumption that the PD is of the form, $p_i = 1 - (1 - p_1)^i$, $1 \le i \le n$, in which case all $\tilde{p}_i$ have the common value, $\tilde{p} := p_1$.

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Proposition 5 *Under the assumptions*

(i) $0 \le \beta < 1$;
(ii) $\Delta t_i \equiv \Delta t$, *a constant, for all* $1 \le i \le n$;
(iii) $p_i = 1 - (1 - p_1)^i$, $1 \le i \le n$.

the $\lim_{T\to\infty} \mathrm{Corr}(\delta_T^1, \delta_T^2) = 0$ *and the rate of convergence is geometric.*

The proof of Proposition 5 is based on the representation in Corollary 3 and is given in the Appendix. Note that the denominator in the correlation, (2), tends to 0, for large T; so it must be shown that the numerator tends to zero at a faster rate. It is also shown in the proof (cf. (A.23)), that the rate of convergence to is essentially $[1 - (\tilde{p} - \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p})))/(1-\tilde{p})]^{T/\Delta t}$, which, as $\beta \to 1$, degenerates to $1^T = 1 \not\to 0$, as $T \to \infty$. Thus, as with Proposition 4, the convergence is not uniform in $\beta \in [0, 1)$.

2.4 CC model

The characterization of default in this model is as follows: default occurs at time t_i if $Y_i^k < b_i$ and $Y_j^k \ge b_j$ for all $j < i$. This introduces a non-Markovian nature to the default process and is the key feature which distinguishes it from the BI model. The default boundary, (b_i $1 \le i \le n$), for the CC model differs from that in the BI model. There are no analytical expressions for the default boundary and conditional default probabilities beyond the first time step,[4] in this model; instead one must determine the boundary via Monte Carlo (MC) simulation (typically with several million paths), as empirical quantiles. (For further details on the CC model, see [1].) Having thus determined the default boundary, one can then calculate the probability of joint default in either one of two ways: (i) numerically, using a recursive (along time steps) numerical integral equation conditionally, followed by a numerical integration on the credit-driver's distribution; (ii) direct MC simulation. We will follow the latter, simpler approach, as we are more interested in stability than extreme accuracy in this study.

We have conjectured, in general, that the default correlation is eventually decreasing in T; but based on some numerical examples in the next section, we also conjecture for the CC model, that it is initially increasing (provided that $p(t_2) < 1/2$, which is the case in practice). Moreover, the numerical examples indicate that the waiting period until the decreasing behaviour takes effect, is longer than most horizons seen in practice. Thus the CC and BI default correlations have quite the opposite behaviour in practice: they decrease immediately over time in the BI model, while they increase over time in the CC model (for realistic time horizons). It is not known if the limiting default correlation, as $T \to \infty$, is zero, as one would expect.

Conjecture 2 *In the CC model, if* $p(t_2) < 1/2$, *then the default correlation is an increasing function of* T, *for* $0 \le T \le$ *median of* p, *i.e., for* T *such that* $p(T) \le 1/2$.

3 Numerical examples

3.1 Example 1: T fixed, $\Delta t \to 0$

To illustrate the convergence of the default correlation to zero, we graph the default correlation as a function of asset correlation, β^2, for several values of n, the number of time steps.

[4] At the first time step, the default boundary is the same as that in the BI model.

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In the first example, the total horizon, $T = 5$ years, is partitioned into 1, 5, 10, 20, and 130 equal time steps, creating the five curves shown in Figure 1. The reason for including the last case, 130 time steps, is to dispel the possible belief, based on only the first four graphs, that the default correlation converges to a nonzero value. The last case clearly illustrates the convergence to zero. The data supporting Figure 1 is given in Table 1. The common 5-year default probability is $p(5) = 0.036257228293999$ (approximately 3.626%) and the probabilities for the intermediate times are set in a manner consistent with a 2×2 transition matrix for the first time step, in each of the five cases; namely, $1 - p_i = (1 - p_n)^{i/n}$, with $p_n = p(5)$. The values of β were: 0.0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999, 1.0.

We compare the empirical rate of convergence against the theoretical (over)estimate of the rate of convergence, in the BI model. The latter was of the order $n^{-\gamma}$, where $\gamma = (1 - \beta^2)/(1 + \beta^2)$. Thus if n is doubled, we expect that the default correlation will drop by a factor of about $2^{-\gamma}$. Empirically, this is observed to hold quite closely in the present example, by comparing the ratios of the default correlations when $n = 10$ vs. $n = 5$; and when $n = 20$ vs. $n = 10$. The comparison is shown in Table 2.

In sharp contrast, in the CC model, as a function of the number of time steps, the default correlation is stable, even convergent. For each value of β, we determined the approximate joint default correlation by averaging over 10 MC simulations. We report the relevant statistics, to assess the reliability of the method. Figure 2 shows the CC model's results which stand in sharp contrast to those of Figure 1 for the BI model. The data supporting Figure 2 is given in Table 3.

There being no difference between the BI and CC models in the single-step case, we expect the correlations to match. In essence, the BI results obtained, serve as an analytic benchmark for the CC model in this case. The results of the MC estimation of the correlation are indeed quite close to the analytic values, with the largest discrepancies (1% to 6.5%) occurring for low values of β. This is expected: it is difficult to accurately estimate a number that is very close to 0, using the Monte Carlo method.

This last issue with MC estimation also appears in the very small negative result for the 10-time-step case when $\beta^2 = 0.01$. This may be attributed to an insufficient number of simulations used in the averaging. In Table 4, we report the standard deviations of the 10 MC simulations, as an indication of the accuracy of the values in Table 3. The standard deviations are generally quite small, but the one corresponding to the negative correlation, is sufficiently large that positive values (including the 5-time-step result, 0.000607) are within one standard deviation of the negative value, -0.0000368. For graphical purposes we have reset the latter value to 0.

Finally, we give graphical illustrations of the comparisons between the BI and CC results for the 5- and 10-step cases, in Figure 3.

3.2 Example 2: Δt fixed, $T \to \infty$

In this subsection, we examine the behaviour when there are many time steps but their size is fixed; i.e., we illustrate the behaviour of the default correlation when the time horizon, T, is large. Here we shall see that the convergence of the default correlation to zero, is more pronounced for higher levels of the incremental default probability.

We fix the step size to be one year and generate the PD from the initial value, $p_1 \equiv p(1)$, in the same way as for the previous example ($p_i \equiv p(i) = 1 - (1 - p_1)^i$), by regarding p_1 as a parameter for the correlation calculation. We compare the behaviour for two values of p_1 : 0.00735896013719

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(approximately 73.59bp), which is the value that was used in the previous example, and 0.05 or 5%. The results for the BI model are shown graphically in Figures 4 and 5; and numerically in Tables 5 and 6.

This behaviour is not exhibited by the CC model. In fact, we have the opposite result: the correlations initially increase with the number of steps, before decreasing. This phenomenon, as predicted in Conjectures 1 and 2, is illustrated in Figures 6 and 7 for the case $p_1 = 0.05$.

The horizon after which the correlations are decreasing is, as predicted by Conjecture 1, the minimum solution to: $1 - (1 - p_1)^T \geq 1/2$. Solving for T, yields $T \geq -\log 2/\log(0.95) \doteq 13.5$. The plot in Figure 7 shows that this effect is pronounced: the default correlation reaches its maximum at $T = 14$.

We have chosen the rather large value of $p_1 = 0.05$ mainly for graphical purposes. For the value $p_1 = 0.00735896013719$, the correlations are predicted, according to Conjecture 1, to decrease as soon as $1 - (1 - p_1)^T \geq 1/2$. Solving for T, yields $T \geq -\log 2/\log(1 - p_1) \doteq 94$. It is worthwhile seeing if almost a century is actually *required* in this case, for a typical value of β, e.g., 0.6. We found,[5] as with the previous examples, that the initial increasing monotonicity of the default correlation, persists until the 94th time step. (Beyond that, in the last few of the 100 time steps in the simulation, the values vary ever so slightly. The monotonicity is masked by small simulation noise – in spite of the fact that 10 million scenarios were used.) Thus, for practical purposes, the default correlation is an increasing function of the time horizon in the CC model.

From another point of view, the predicted horizon for decreasing default correlations, would be 10 years provided $1 - (1 - p_1)^{10} \geq 1/2$; i.e., if $p_1 \geq 1 - 0.5^{0.1} \doteq 6.7\%$. For this value of p_1 and $\beta = 0.6$, we found[5] that the correlation climbs from about 0.140, at $T = 1$, to its maximum value of about 0.217, at $T = 10$ and then descends. Thus, once again, the conservative prediction turns out to be sharp and we have initially increasing correlations over time in the CC model.

Acknowledgement

During this investigation, I benefited greatly from discussions with Yijun Jiang and Alex Kreinin. I am also grateful to Diane Reynolds for her suggestions which led to improvements of the exposition.

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[1] Iscoe, I. and Kreinin, A. (2001) Default boundary problem. Algorithmics technical paper. Available at: www.finance-research.net/enaa13520t4_kreinin.html

[5] not depicted here

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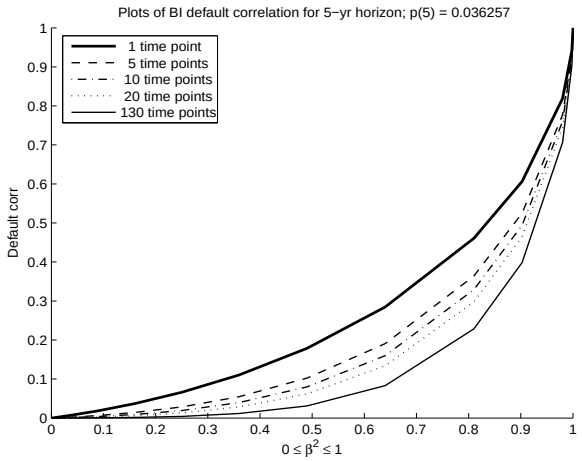

Figure 1: Convergence of default correlations for BI model.

β^2	Number of time steps				
	1	5	10	20	130
0.0	0	0	0	0	0
0.01	0.00045445	0.0001409	8.1893e-05	4.6795e-05	9.6946e-06
0.025	0.0018398	0.00057632	0.00033651	0.0001932	4.0552e-05
0.04	0.0077186	0.0025191	0.0014988	0.0008772	0.00019435
0.09	0.018768	0.0065582	0.0040281	0.0024368	0.00059341
0.16	0.037061	0.014231	0.009141	0.0057957	0.0016177
0.25	0.06591	0.028457	0.019339	0.013012	0.0043202
0.36	0.11042	0.054587	0.039598	0.028537	0.011565
0.49	0.17865	0.1024	0.079732	0.061878	0.030997
0.64	0.28468	0.19079	0.15988	0.13389	0.08308
0.81	0.4607	0.36413	0.32908	0.29768	0.22843
0.9025	0.60639	0.52483	0.49383	0.46521	0.39827
0.9801	0.81951	0.77739	0.76082	0.74517	0.70685
0.998001	0.9426	0.92876	0.92329	0.91809	0.90523
1.0	1	1	1	1	1

Table 1: Default correlation data for BI model.

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	β^2						
Ratio	0.01	0.025	0.04	0.09	0.16	0.25	0.36
Theoretical	0.5017	0.5069	0.5274	0.5606	0.6054	0.6598	0.7217
Empirical 5 : 10	0.5812	0.5839	0.5950	0.6142	0.6423	0.6796	0.7254
Empirical 10 : 20	0.5714	0.5741	0.5853	0.6049	0.6340	0.6728	0.7207

	β^2					
Ratio	0.49	0.64	0.81	0.9025	0.9801	0.998001
Theoretical	0.7888	0.8589	0.9298	0.9651	0.9931	0.9993
Empirical 5 : 10	0.7786	0.8380	0.9038	0.9409	0.9787	0.9941
Empirical 10 : 20	0.7761	0.8374	0.9046	0.9420	0.9794	0.9944

Table 2: Default correlation ratios for BI model.

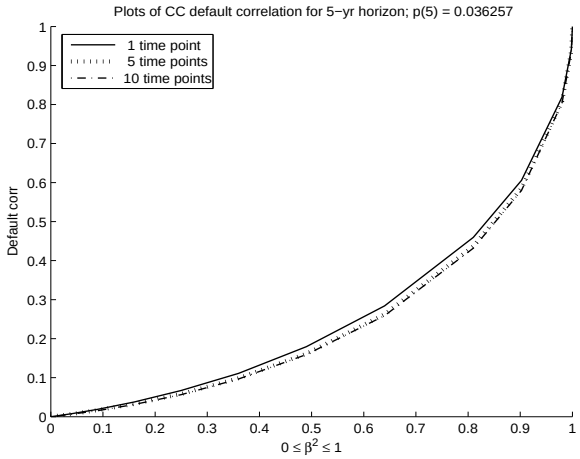

Figure 2: Convergence of default correlations for CC model.

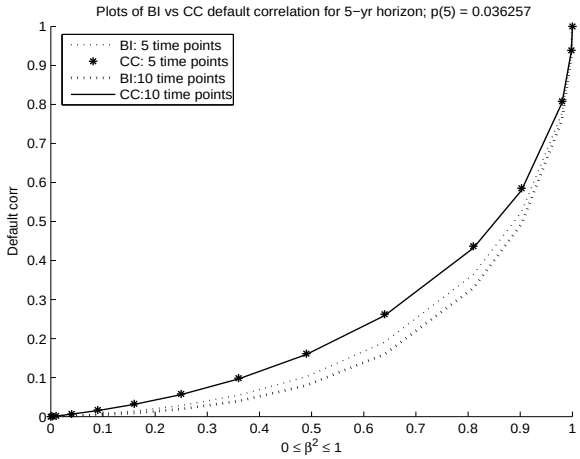

Figure 3: Comparison of default correlations: BI vs CC.

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β^2	Number of time steps		
	1	5	10
0.0	0	0	0
0.01	0.00042107	0.000607	-0.0000368
0.025	0.00177758	0.001852	0.001294
0.04	0.007664	0.0068	0.0064
0.09	0.019000	0.0168	0.0155
0.16	0.037756	0.0328	0.0313
0.25	0.06714	0.0583	0.0566
0.36	0.11096	0.0986	0.0967
0.49	0.17955	0.1614	0.1592
0.64	0.28413	0.2625	0.2588
0.81	0.4596	0.4367	0.4317
0.9025	0.60581	0.5851	0.5799
0.9801	0.81895	0.8082	0.8045
0.998001	0.9419	0.9387	0.9369
1.0	1	1	1

Table 3: Default correlation data for CC model.

β^2	Number of time steps		
	1	5	10
0.0	0	0	0
0.01	0.001	0.001	0.001
0.025	0.001	0.001	0.001
0.04	0.001	0.001	0.001
0.09	0.002	0.001	0.001
0.16	0.002	0.001	0.001
0.25	0.001	0.002	0.002
0.36	0.002	0.002	0.002
0.49	0.003	0.002	0.002
0.64	0.002	0.003	0.002
0.81	0.003	0.004	0.003
0.9025	0.003	0.004	0.003
0.9801	0.005	0.004	0.004
0.998001	0.005	0.005	0.004
1.0	0	0	0

Table 4: Std devs of MC runs for CC model.

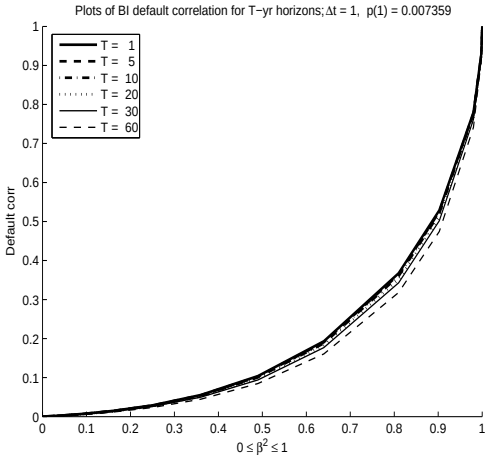

Figure 4: BI default correlations for $\Delta t = 1$, $p_1 \doteq 0.007$.

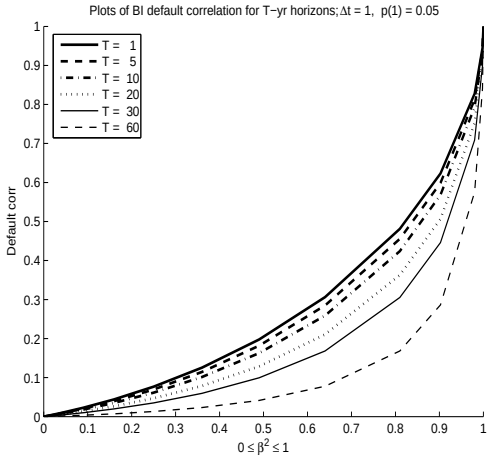

Figure 5: BI default correlations for $\Delta t = 1$, $p_1 = 0.05$.

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β^2	T = Number of time steps					
	1	5	10	20	30	60
0.0	0	0	0	0	0	0
0.025	0.00014301	0.0001409	0.0001383	0.0001332	0.00012822	0.00011408
0.01	0.00058492	0.00057632	0.00056568	0.00054481	0.00052447	0.00046663
0.04	0.0025566	0.0025191	0.0024727	0.0023816	0.0022929	0.0020405
0.09	0.0066555	0.0065582	0.0064379	0.0062018	0.0059716	0.0053166
0.16	0.01444	0.014231	0.013972	0.013463	0.012967	0.011555
0.25	0.02887	0.028457	0.027947	0.026944	0.025965	0.023175
0.36	0.055357	0.054587	0.053635	0.051761	0.04993	0.044696
0.49	0.10377	0.1024	0.10071	0.097363	0.094089	0.084684
0.64	0.19309	0.19079	0.18794	0.18231	0.17676	0.16071
0.81	0.36756	0.36413	0.35985	0.35135	0.34291	0.31805
0.9025	0.52851	0.52483	0.52022	0.51101	0.50179	0.47421
0.9801	0.77993	0.77739	0.77419	0.76774	0.7612	0.74115
0.998001	0.92973	0.92876	0.92754	0.92506	0.92253	0.91464
1.0	1	1	1	1	1	1

Table 5: BI default correlation data for $\Delta t = 1$, $p_1 \doteq 0.007$.

β^2	T = Number of time steps					
	1	5	10	20	30	60
0.0	0	0	0	0	0	0
0.025	0.00056174	0.00050566	0.00044121	0.00033052	0.00024251	8.5745e-05
0.01	0.0022698	0.0020436	0.0017835	0.0013366	0.00098116	0.00034738
0.04	0.0094491	0.0085139	0.0074373	0.0055845	0.0041071	0.0014624
0.09	0.022688	0.020471	0.017913	0.013498	0.0099617	0.0035848
0.16	0.044038	0.039824	0.034948	0.026483	0.019656	0.0071968
0.25	0.076692	0.069592	0.061334	0.046884	0.035106	0.013204
0.36	0.12543	0.11441	0.10149	0.078599	0.059644	0.023379
0.49	0.19768	0.18167	0.16271	0.12852	0.099513	0.041573
0.64	0.30633	0.28477	0.2588	0.21065	0.16829	0.077791
0.81	0.48118	0.45563	0.424	0.3627	0.30545	0.16828
0.9025	0.62293	0.59871	0.56807	0.50648	0.44606	0.28629
0.9801	0.82768	0.81282	0.79342	0.75228	0.70888	0.57544
0.998001	0.94525	0.93982	0.93261	0.91683	0.89947	0.841
1.0	1	1	1	1	1	1

Table 6: BI default correlation data for $\Delta t = 1$, $p_1 = 0.05$.

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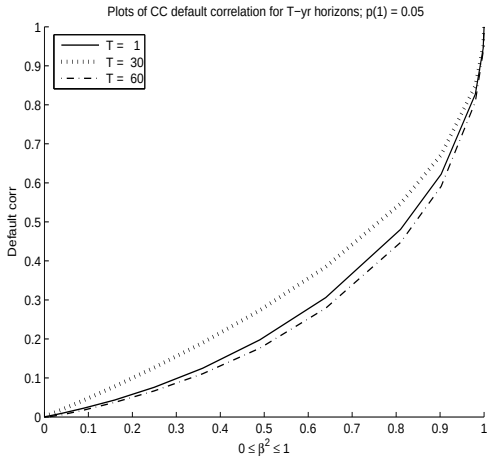

Figure 6: CC default correlations for $\Delta t = 1$, $p_1 = 0.05$.

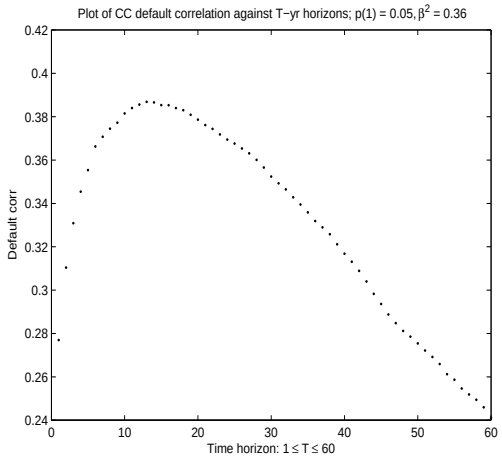

Figure 7: CC default correlations over $[0, T]$; $p_1 = 0.05$, $\beta^2 = 0.36$.

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Appendix

In this appendix, we provide the proofs of the propositions stated in Section 2. We begin with an evaluation of some Gaussian expectations, followed by the proof of Proposition 2; then some asymptotic results for Gaussian cdfs, followed by the proof of Proposition 4.

Lemma A.1 *Let $Z \sim N(0,1)$ and a, c two constants.*

$$\mathbb{E}\left[\Phi(a - cZ)\right] = \int_{-\infty}^{+\infty} \Phi(a - cz)\, d\Phi(z) = \Phi\left(\frac{a}{\sqrt{1+c^2}}\right);$$

$$\mathbb{E}\left[(\Phi(a - cZ))^2\right] = \int_{-\infty}^{+\infty} (\Phi(a - cz))^2\, d\Phi(z) = \Phi_\rho^{(2)}\left(\frac{a}{\sqrt{1+c^2}}\right), \quad \rho = \frac{c^2}{1+c^2}.$$

Proof. For the first identity, let $Z' \sim N(0,1)$ and independent of Z. Then $Z' + cZ \sim N(0, 1+c^2)$ and

$$\int_{-\infty}^{+\infty} \Phi(a - cz)\, d\Phi(z) = \mathbb{E}\left[\mathbb{P}(Z' \le a - cZ \mid Z)\right] = \mathbb{P}(Z' + cZ \le a) = \Phi(a/\sqrt{1+c^2}).$$

For the second identity, let Z_1, Z_2, Z_3 be iid $N(0,1)$ random variables. Then $Z_1 + cZ_3, Z_2 + cZ_3$ are jointly standard normal random variables, with correlation $c^2/(1+c^2)$; so

$$\mathbb{P}(Z_1 + cZ_3 \le a,\, Z_2 + cZ_3 \le a) = \Phi_\rho^{(2)}\left(\frac{a}{\sqrt{1+c^2}}\right), \quad \rho = \frac{c^2}{1+c^2}.$$

On the other hand,

$$\begin{aligned}\mathbb{P}(Z_1 + cZ_3 \le a,\, Z_2 + cZ_3 \le a) &= \mathbb{E}\left[\mathbb{P}(Z_1 \le a - cZ_3,\ Z_2 \le a - cZ_3 \mid Z_3)\right] \\ &= \mathbb{E}\left[(\Phi(a - cZ_3))^2\right] = \int_{-\infty}^{+\infty} (\Phi(a - cz))^2\, d\Phi(z).\end{aligned}$$

□

Proof of Proposition 2. We return to the calculations at (3) and (4). Set

$$a_i = \frac{b_i}{\sigma\sqrt{\Delta t_i}}, \quad c = \frac{\beta}{\sigma}, \quad Z_i = \frac{\Delta X_i}{\sqrt{\Delta t_i}}$$

and note that

$$\frac{a_i}{\sqrt{1+c^2}} = \Phi^{-1}(\tilde{p}_i) \tag{A.10}$$

$$\frac{c^2}{1+c^2} = \beta^2 \tag{A.11}$$

$$Z_i \sim N(0,1) \text{ and are iid.} \tag{A.12}$$

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Continuing from (3) and (4), we have, in this notation,

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{P}(\text{CP defaults by time } T \,|\, X)^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[(\Phi(a_i - cZ_i))^2 \prod_{k<i} (\bar{\Phi}(a_k - cZ_k))^2\right] \\
&\quad + 2 \sum_{1\le i<j\le n} \mathbb{E}\left[\Phi(a_i - cZ_i) \prod_{k<i} \bar{\Phi}(a_k - cZ_k)\Phi(a_j - cZ_j) \prod_{\ell<j} \bar{\Phi}(a_\ell - cZ_\ell)\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[(\Phi(a_i - cZ_i))^2\right] \prod_{k<i} \mathbb{E}\left[(\bar{\Phi}(a_k - cZ_k))^2\right] \\
&\quad + 2 \sum_{1\le i<j\le n} \mathbb{E}\left[\Phi(a_i - cZ_i)\bar{\Phi}(a_i - cZ_i)\right] \cdot \\
&\qquad\qquad \cdot \prod_{k<i} \mathbb{E}\left[(\bar{\Phi}(a_k - cZ_k))^2\right] \mathbb{E}\left[\Phi(a_j - cZ_j)\right] \prod_{i<\ell<j} \mathbb{E}\left[\bar{\Phi}(a_\ell - cZ_\ell)\right]
\end{aligned}$$

where in the last equality, we used the independence (A.12) of the "increments" $\{Z_i;\, 1 \le i \le n\}$, rearranging the factors in the second summation (to avoid overlaps) in order to take advantage of the independence. Now we apply Lemma A.1 to each of the expectations, using the identities (A.10) and (A.11). □

Lemma A.2 *Let $0 < p < 1$ and set $b = -\Phi^{-1}(p)$. The following asymptotics are valid, as $p \to 0$.*

$$b^2 = 2\log(p^{-1}) - \log\log(p^{-1}) - \log 2 - 2\log(\sqrt{2\pi}) + o(1);$$
$$b = \sqrt{2\log(p^{-1})}(1 + o(1)).$$

Proof. The correspondence between p and $b > 0$ is bijective, strictly decreasing. The following asymptotics for $1 - \Phi$ is well known:

$$1 - \Phi(b) = \frac{e^{-b^2/2}}{\sqrt{2\pi}b}(1 + O(b^{-1})), \text{ as } b \to \infty.$$

Therefore, with $p = \Phi(-b) = 1 - \Phi(b)$,

$$2\log(p^{-1}) = 2\log(\sqrt{2\pi}) + \log b^2 + b^2 + O((b^2)^{-1/2}), \tag{A.13}$$

where we have used the elementary result, $\log(1 + \epsilon) = O(\epsilon)$, as $\epsilon \to 0$. Making the change of variables, $u = 2\log(p^{-1})$, $x = b^2$, we can express (A.13) in terms of u and x, as

$$u = x + \log x + c + O(x^{-1/2}), \text{ as } x \to \infty;\ c \equiv 2\log(\sqrt{2\pi}). \tag{A.14}$$

Clearly, $u/x \to 1$, as $x \to \infty$; so that $x/u \to 1$, as $u \to \infty$. Therefore, $b^2 = 2\log(p^{-1})(1 + o(1))$, which yields the second result of the lemma.

To obtain the finer asymptotics for b^2, we apply the method of reversion to (A.14), by rewriting it in the form,

$$\begin{aligned}
x &= u - \log x - c + O(x^{-1/2}) \\
&= u - \log(u[1 + o(1)]) - c + O(u^{-1/2}[1 + o(1)]), \text{ by using } x = u[1 + o(1)] \\
&= u - \log u - c + o(1).
\end{aligned}$$

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Translating the latter back to the original variables, p and b, yields the first result of the lemma. □

Lemma A.3 *Let $0 \le \rho < 1$ and $b > 0$. The following asymptotics is valid as $b \to \infty$.*

$$\Phi_\rho^{(2)}(-b,-b) = \frac{(1+\rho)^2}{2\pi\sqrt{1-\rho^2}} \frac{e^{-b^2/(1+\rho)}}{b^2}(1+o(1))$$

Proof. By symmetry, the asymptotics will be obtained for the equivalent probability,

$$\int_b^\infty \int_b^\infty \phi_\rho^{(2)}(x,y)\,dx\,dy, \qquad \phi_\rho^{(2)}(x,y) := \frac{e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}}{2\pi\sqrt{1-\rho^2}}.$$

Making the change of variables, $x \mapsto bx$, $y \mapsto by$, and setting $\theta = b^2$, transforms the integral to

$$\theta \int_1^\infty \int_1^\infty \frac{e^{-\frac{\theta}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}}{2\pi\sqrt{1-\rho^2}}\,dx\,dy.$$

The dominant contribution to the integral comes from any neighbourhood of the point, in the region of integration, which minimizes the quadratic form in the exponent of the integrand. It is straightforward to check that the quadratic form has no critical points in the region and, along each of the two boundary lines, it is an increasing function. Therefore the minimum is attained at the corner point, (1,1), and for the purposes of asymptotics, we change coordinates to make (1,1) our new origin.

Express the quadratic form in terms of $x-1$ and $y-1$ (an exact Taylor expansion):

$$x^2 - 2\rho xy + y^2 = 2(1-\rho) + 2(1-\rho)(x-1) + 2(1-\rho)(y-1) + (x-1)^2 - 2\rho(x-1)(y-1) + (y-1)^2.$$

Making the change of variables, $x \mapsto (x+1)/\theta$, $y \mapsto (y+1)/\theta$, and using the latter expansion, transforms the integral (and its preceding factor, θ) to

$$e^{-\theta/(1+\rho)}\theta^{-1} \int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{1+\rho}(x+y)-\frac{1}{2\theta(1-\rho^2)}(x^2-2\rho xy+y^2)}}{2\pi\sqrt{1-\rho^2}}\,dx\,dy.$$

As $\theta \to \infty$, the integral converges to the constant

$$\int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{1+\rho}(x+y)}}{2\pi\sqrt{1-\rho^2}}\,dx\,dy = \frac{1}{2\pi\sqrt{1-\rho^2}}\Big(\int_0^\infty e^{-x/(1+\rho)}\,dx\Big)^2 = \frac{(1+\rho)^2}{2\pi\sqrt{1-\rho^2}}.$$

The lemma is now immediate, as $\theta = b^2$. □

Proof of Proposition 4. Hypotheses (ii), (iii) of the current Proposition imply that $\tilde{p}_i$ tends to 0, uniformly in i, as $n \to \infty$. Therefore, combining Lemma A.2 (second order asymptotics) and Lemma A.3, we obtain, with $\rho = \beta^2 < 1$ and $\gamma := (1-\rho)/(1+\rho)$:

$$\begin{aligned}\Phi_\rho^{(2)}(\Phi(\tilde{p}_i)) &= const_\rho \times \frac{\tilde{p}_i^{2/(1+\rho)}(\log \tilde{p}_i^{-1})^{1/(1+\rho)}}{\log \tilde{p}_i^{-1}} \times [1+o(1)] \\ &= const_\rho \times \tilde{p}_i^{1+\gamma}(\log \tilde{p}_i^{-1})^{-\rho/(1+\rho)} \times [1+o(1)]\end{aligned}$$

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where $const_\rho$ is some constant which depends on ρ. (Note that if $\beta = 1$ were allowed, then $const_\rho$ would be infinite.) The "$o(1)$" term is with respect to n and is uniform in i, $1 \le i \le n$. Allowing now $const_\rho$ to be a generic constant which can vary from line to line, we can use the uniform estimate (iii) on $\tilde{p}_i$ to obtain the following estimates:

$$\begin{aligned}\sum_{i=1}^{n} \Phi_\rho^{(2)}(\Phi(\tilde{p}_i)) &\le const_\rho \times \sum_{i=1}^{n} (\Delta\tilde{p}_i)^{1+\gamma} (\log \tilde{p}_i^{-1})^{-\rho/(1+\rho)} \\ &\le const_\rho \times (\log(\Delta t)^{-1})^{-\rho/(1+\rho)} \sum_{i=1}^{n} (\Delta t_i)^{1+\gamma} \\ &\le const_\rho \times T(\log(\Delta t)^{-1})^{-\rho/(1+\rho)} (\Delta t)^{\gamma} \\ &\le O((\Delta t)^{\gamma}); \end{aligned} \tag{A.15}$$

and similarly

$$\sum_{1\le i<j\le n} \Phi_\rho^{(2)}(\Phi(\tilde{p}_i))\tilde{p}_j \le const_\rho \times T^2(\log(\Delta t)^{-1})^{-\rho/(1+\rho)}(\Delta t)^{\gamma} \le O((\Delta t)^{\gamma}). \tag{A.16}$$

Now, the products in the two terms in (5) have values between 0 and 1 because they represent products of expectations of Φ or $\bar{\Phi}$ or their squares, composed with some random variables. Therefore they can be dropped in any overestimation. As such, estimate (A.15) shows that the first term in (5) tends to 0, as $n \to \infty$. In particular, we can replace it with the first term in (7), as it also tends to zero – it is the case $\rho = 0$ in the present analysis. Estimate (A.16) shows that part of the second term in (5) tends to 0; namely

$$\sum_{1\le i<j\le n} \{-\Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}_i))\}\{\prod_{k<i}[1 - 2\tilde{p}_k + \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}_k))]\}\{\tilde{p}_j\}\{\prod_{i<\ell<j}[1-\tilde{p}_\ell]\} = O((\Delta t)^{\gamma}).$$

Therefore, asymptotically, we can replace this expression with the corresponding part of the second term in (7),

$$\sum_{1\le i<j\le n} \{-(\tilde{p}_i))^2\}\{\prod_{k<i}[1-\tilde{p}_k]^2\}\{\tilde{p}_j\}\{\prod_{i<\ell<j}[1-\tilde{p}_\ell]\}$$

as it also tends to zero (at the rate Δt – it is the case $\rho = 0$ in the present analysis).

The final replacement to be made is in the first part of the second term in (5); namely

$$S_\beta := \sum_{1\le i<j\le n} \tilde{p}_i\{\prod_{k<i}[1 - 2\tilde{p}_k + \Phi_{\beta^2}^{(2)}(\Phi^{-1}(\tilde{p}_k))]\}\{\tilde{p}_j\}\{\prod_{i<\ell<j}[1-\tilde{p}_\ell]\}. \tag{A.17}$$

S_0 is the corresponding part of the term in (7):

$$S_0 = \sum_{1\le i<j\le n} \tilde{p}_i\{\prod_{k<i}[1-\tilde{p}_k]^2\}\{\tilde{p}_j\}\{\prod_{i<\ell<j}[1-\tilde{p}_\ell]\}.$$

We will show below that

$$S_\beta - S_0 = S_0 \times O((\Delta t)^{\gamma}) \tag{A.18}$$

and that

$$S_0 \text{ is bounded by an absolute constant.} \tag{A.19}$$

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This implies that the replacement of S_β with S_0, incurs an error of at most $O((\Delta t)^\gamma)$.

Taken together, these replacements show that

$$\mathrm{Corr}_\beta(\delta_T^1, \delta_T^2) = \mathrm{Corr}_0(\delta_T^1, \delta_T^2) + O((\Delta t)^\gamma) = 0 + O((\Delta t)^\gamma) \to 0,$$

as $n \to \infty$, where we have made the dependence of the correlations on β explicit as a subscript to the correlation operator, Corr.

It remains to verify (A.18) and (A.19). For the latter, we can use the estimate in hypothesis (iii):

$$\begin{aligned} S_0 &= \sum_{1 \le i < j \le n} \frac{\tilde{p}_i}{1-\tilde{p}_i} \{\prod_{k<i}[1-\tilde{p}_k]\}\{\tilde{p}_j\}\{\prod_{\ell<j}[1-\tilde{p}_\ell]\} \\ &\le \frac{1}{1-C\Delta t}(\sum_{i=1}^{n} \tilde{p}_i \prod_{k<i}[1-\tilde{p}_i]^2)^2 \\ &= \frac{p(T)^2}{1-C\Delta t}. \end{aligned}$$

As $\Delta t \to 0$, eventually $C\Delta t \le 1/2$; then $S_0 \le 2p(T)^2$.

For (A.18), we first obtain a similar estimate for the products over $k < i$ in logarithmic form, again using the estimate in hypothesis (iii):

$$\begin{aligned} \log\left[\frac{\prod_{k<i}[1-2\tilde{p}_k + \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k))]}{\prod_{k<i}[1-\tilde{p}_k]^2}\right] &= \sum_{k<i} \log\left[1 + \frac{\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k)) - \tilde{p}_k^2}{[1-\tilde{p}_k]^2}\right] \\ &= \sum_{k<i} O((\Delta t)^{1+\gamma}) = O((\Delta t)^\gamma). \end{aligned}$$

Exponentiating this estimate yields

$$\prod_{k<i}[1-2\tilde{p}_k + \Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}_k))] = \{\prod_{k<i}[1-\tilde{p}_k]^2\}\{1 + O((\Delta t)^\gamma)\}$$

which, when substituted into (A.17), implies that (A.18) holds:

$$S_\beta - S_0 = S_0 \times (1 + O((\Delta t)^\gamma)) - S_0 = S_0 \times O((\Delta t)^\gamma).$$

□

Proof of Proposition 5. To simplify the notation, we assume that $\Delta t = 1$, so that $n = T$. As such, we have the relation

$$1 - p(T) = (1-\tilde{p})^T. \tag{A.20}$$

Now we rewrite the expression, (2), in the form

$$\mathrm{Corr}(\delta_T^1, \delta_T^2) = \frac{\mathbb{P}(\text{Both CP1 \& CP2 default by time } T) - 1}{p(T)(1-p(T))} + \frac{1+p(T)}{p(T)}. \tag{A.21}$$

Clearly $p(T) \to 1$, as $T \to \infty$, so that the second term on the right-hand side of (A.21) tends to 2, as $T \to \infty$. The rate of convergence can be seen to be essentially $(1-\tilde{p})^T$, by rewriting the second term in the form

$$\frac{1+p(T)}{p(T)} = 1 + \frac{1}{p(T)} = 1 + \frac{1}{1-(1-\tilde{p})^T} = 2 + \frac{(1-\tilde{p})^T}{1-(1-\tilde{p})^T}. \tag{A.22}$$

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The latter fraction is asymptotic to $(1-\tilde{p})^T$.

We next apply Corollary 3 to the first term on the right-hand side of (A.21). In doing so, we first rewrite the terms in (8) which involve powers of T, in a form similar to (9) by writing

$$\left(1-2\tilde{p}+\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}))\right)^T = (1-\tilde{p})^T\left(1-\frac{\tilde{p}-\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}))}{1-\tilde{p}}\right)^T \equiv (1-\tilde{p})^T r^T$$

and similarly for the term involving the power $T-1$. We will show that $0<r<1$ in due course. The first term on the right-hand side of (A.21) can then be expressed, after some lengthy algebra, as simply

$$\frac{1}{p(T)}\left[-2+r^T\right].$$

Combining this result with (A.22) and the identity,

$$\frac{1}{p(T)} = 1+\frac{(1-\tilde{p})^T}{1-(1-\tilde{p})^T},$$

yields the result,

$$\mathrm{Corr}(\delta^1_T,\delta^2_T) = r^T - \frac{(1-\tilde{p})^T}{1-(1-\tilde{p})^T}\left[1-r^T\right]. \tag{A.23}$$

Thus the correlation tends to 0 at a geometric rate, as $T\to\infty$.

It remains to show to that $0<r<1$. As $r=[1-2\tilde{p}+\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p}))]/[1-\tilde{p}]$, the upper bound on r is equivalent to the inequality,

$$\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p})) < \tilde{p}. \tag{A.24}$$

To see that the latter holds, set $z=\Phi^{-1}(\tilde{p})$ and let Z_1, Z_2 be jointly Normally distributed according to $\Phi^{(2)}_{\beta^2}$. The left-hand side of the inequality (A.24) is

$$\mathbb{P}(Z_1\le z, Z_2\le z) < \mathbb{P}(Z_1\le z) = \Phi(z) = \tilde{p}.$$

For the lower inequality on r, we note, as in the proof of Proposition 2, that we can appeal to Lemma A.1 to see that the numerator of r can be expressed as

$$1-2\tilde{p}+\Phi^{(2)}_{\beta^2}(\Phi^{-1}(\tilde{p})) = \mathbb{E}\left[(\bar{\Phi}(a-cZ))^2\right] \ge 0$$

where $a=\Phi^{-1}(\tilde{p})/\sqrt{1-\beta^2}$ and $c=\beta/\sqrt{1-\beta^2}$ (so that $\rho=\beta^2$). □

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Measuring Equity VaR with the Constant Level of Risk Approach

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Abstract

We compare the new constant level of risk approach to Value-at-Risk capital calculations with the more traditional constant position approach for a single equity and a portfolio of equities. Analytical results for an equity portfolio are derived and compared with Monte Carlo simulation. In particular, we demonstrate that the constant level or risk approach leads to much higher capital requirements irrespective of assumed correlation, quantile, liquidity horizon or volatility and portfolio diversification effects. We conclude that the constant level of risk approach may produce VaR measures exceeding the initial investment leading to unrealistic capital requirements.

Keywords: Value-at-Risk, Constant level of risk, Liquidity horizon.

Introduction

The main idea behind estimating portfolio Value at Risk (VaR) is to find the distribution of the portfolio value, V_t, at time t and the resulting q-percentile of the loss distribution, $L_t = V_0 - V_t$, for q sufficiently close to 1. The 99-percentile is traditionally used to measure general market risk. When estimating portfolio credit risk, practitioners typically use 99.9% and higher values of q. In the case the portfolio value at time t has a continuous distribution, the portfolio VaR, ℓ_q, can be found from the equation

$$\mathbb{P}\left(V_0 - V_t \leq \ell_q\right) = q.$$

In the calculation of incremental risk for the trading book an alternative method to calculating VaR, called the constant level of risk approach (CLRA) is proposed. CLRA is based on the concept of replacing risky instruments at the end of their liquidity horizon by similar-starting risky instruments thereby preserving a constant level of risk over the full time horizon. It is assumed that the liquidity horizon, Δt, is much smaller

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than time horizon t and that the risk profile of the replacement instrument is identical to that of the original instrument in the portfolio at time 0.

Thus, by time t the portfolio will accumulate losses

$$\mathcal{L}_t = \sum_{k=1}^{n_t} \hat{L}_k, \qquad n_t = \frac{t}{\Delta t},$$

where $\hat{L}_k$, $(k = 1, 2, \ldots, n_t)$, are independent, identically distributed random variables having the same distribution as the random variable $\hat{L}_1 = V_0 - V_{\Delta t}$.

CLRA leads to VaR estimates that are often very different from the results obtained using the traditional VaR methodology, also known as the constant position approach (CPA). Below we compare the CPA methodology with CLRA for equities paying special attention to the implied capital requirements under each method and the ability of CLRA to reduce capital over CPA.

Single equity VaR analysis

Constant position approach

Consider a stock whose value, S_t, follows a GBM process

$$S_t = S_0 \exp\left(\sigma W_t + \mu t\right), \qquad t \geq 0,$$

where S_0 is the initial value of the stock, governed by the Brownian Motion process, W_t, and as usual, σ is the stock volatility and μ is the stock growth rate.

The mean value of the stock at time t is

$$\mathbb{E}S_t = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}, \quad t \geq 0.$$

The loss L_t is

$$L_t = S_0 - S_0 e^{\sigma W_t + \mu t}.$$

By definition, the VaR at time t, $\ell_q(t)$, satisfies the equation

$$\mathbb{P}\left(L_t \leq \ell_q(t)\right) = q,$$

where q is the quantile probability ($q \sim 1$). It follows (see Appendix) that

$$\ell_q(t) = S_0 \cdot \left(1 - e^{\mu t - z_q \sigma \sqrt{t}}\right). \tag{1}$$

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Constant level of risk approach

The Guidelines provide for a one-year time horizon, 99.9% quantile measure and a minimum liquidity horizon of one-month for CLRA. Accordingly, we find the distribution of portfolio losses, $\hat{L}_1$, over the minimum liquidity horizon, $\Delta t = \frac{1}{12}$, and find the distribution of the random variable

$$\mathcal{L}_t = \sum_{k=1}^{n_t} \hat{L}_k. \tag{2}$$

Since losses over disjoint liquidity intervals are independent and identically distributed[1] the distribution of $\mathcal{L}_t$, $F_I(x) = \mathbb{P}(\mathcal{L}_t \leq x)$, is the 12-fold convolution of the distribution of $\hat{L}_1$. The quantile, $\hat{\ell}_q(t)$, of this distribution is the CLRA VaR. The distribution of $\hat{L}_1$ (see Appendix) is

$$\mathbb{P}\left(\hat{L}_1 \leq x\right) = 1 - \Phi\left(\frac{\ln(S_0 - x) - \ln S_0 - \mu t}{\sigma\sqrt{t}}\right), \quad x \leq S_0.$$

In this case, Monte Carlo (MC) is an efficient way to compute $\hat{\ell}_q(t)$. The number of MC scenarios used in the numerical experiment herein is $N = 10^7$.

The additive structure of Equation (2) and independence of the random variables $\hat{L}_k$ allows us to apply an approximation based on the Central Limit Theorem (CLT) for sufficiently large n_t. The approximate value of the quantile, $\hat{\ell}_q(t)$, can be written as a linear combination of the expected losses and the standard deviation of losses:

$$\hat{\ell}_q(t) = \mathbb{E}\mathcal{L}_t + z_q\sigma(\mathcal{L}_t), \tag{3}$$

where

$$\mathbb{E}\mathcal{L}_t = n_t S_0\left(1 - e^{\Delta t(\mu + \frac{1}{2}\sigma^2)}\right),\ \sigma(\mathcal{L}_t) = \sqrt{n_t} S_0 e^{\mu\Delta t + \frac{1}{2}\sigma^2\Delta t}\sqrt{e^{\sigma^2\Delta t} - 1} \ \text{ and } \ t = n_t\Delta t.$$

Comparison of CPA and CLRA

Using an analytical formula for CPA VaR and Monte Carlo simulation for CLRA VaR we obtain the results presented in Figures 1 through 5.

In Figure 1 we compare VaR computed over a one year time horizon ($t = 1$) using CPA, CLRA and the normal approximation (NA) to CLRA using CLT. The liquidity horizon is one month, $\Delta t = \frac{1}{12}$. The initial value of the stock is $S_0 = 100$. The volatility parameter, $\sigma = 0.3$. The quantile is $q = 0.999$. At the regulatory quantile of $q = 0.999$ the difference between CLRA and CPA VaR is positive at all time horizons and close to 50% at the 1 year time horizon. The CLRA will therefore produce higher capital requirements irrespective of the time horizon used, counter to the stated objective of the Guidelines.

From a practical standpoint the approximation is a sufficient substitute to the full Monte Carlo simulation given the more conservative results. Overall, the approximation tracks the simulated CLRA measure relatively well with only a small bias.

[1] We assume that the lengths of the liquidity intervals are the same, $\Delta t = t/n_t$.

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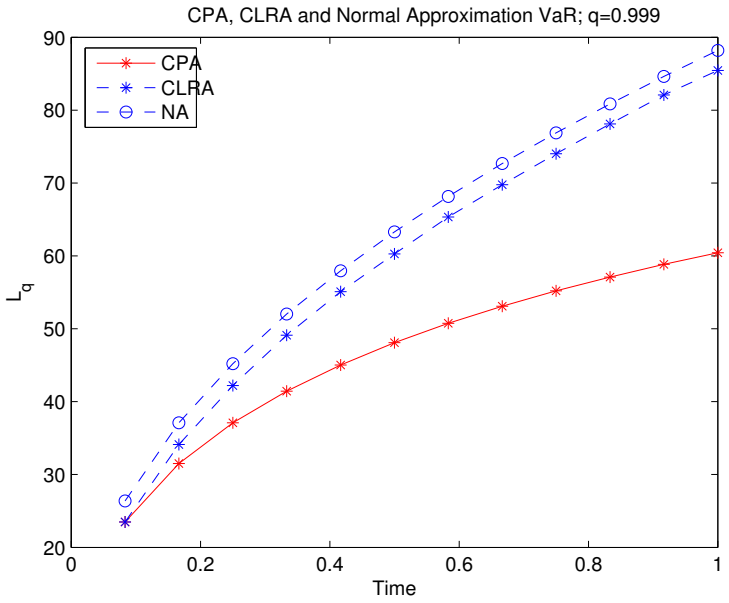

Figure 1: Equity VaR: $\sigma = 0.3, \mu = 0, S_0 = 100$.

In Figures 2 and 3 we illustrate the relative difference between the CLRA and CPA results,

$$\Delta_q(\sigma) = \frac{\hat{\ell}_q - \ell_q}{\ell_q},$$

computed as a function of the volatility parameter, σ, using $n_t = 12$ and $n_t = 4$ liquidity horizons respectively. In both cases the relationship is linear with the difference between CLRA ($\hat{\ell}_q$) and CPA (ℓ_q) increasing with asset volatility. Both approaches imply increasing capital requirements with rising volatility, but CLRA is substantially more punitive, doubling CPA at $\sigma = 0.68$. The general intuition as to why CLRA continually overstates VaR relative to CPA is borne from the observation that equity prices are no longer floored at zero. The effect of the roll-over is to allow a series of n_t losses, each of which is in the range $[0, S_0]$, creating a maximum loss of $n_t S_0$. In this case, the price distribution is more similar to a normal distribution, allowing losses greater than the amount invested and unlike a lognormal distribution with its zero floor for the equity price.

Figure 4 shows CLRA VaR as a function of the equity volatility, demonstrating that CLRA is also "almost" a linear function of the stock volatility parameter, σ, as we would expect to see in a normal distribution. By design, CLRA thus allows for VaR measures that can exceed the investment in the equity. In this case we see that VaR is greater than the initial value, $S_0 = 100$, for $\sigma > 0.4$. This implication is a serious concern with the use of CLRA as a capital measurement tool for equities given the distortion to the final loss distribution. The counterintuitive results create issues with the actionability of the derived measures.

Figure 5 examines the impact of quantile selection on the relative results of the two models. As the quantile

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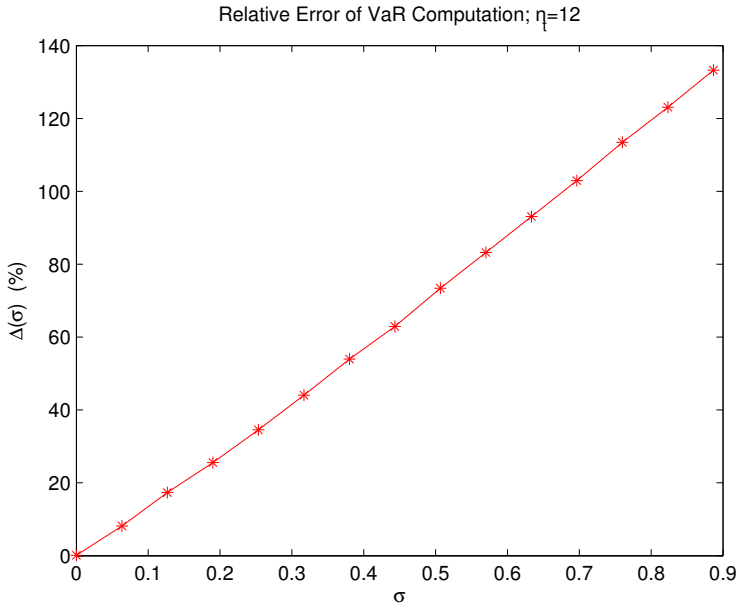

Figure 2: $\Delta_q(\sigma)$ for $q = 0.999$, $n_t = 12$.

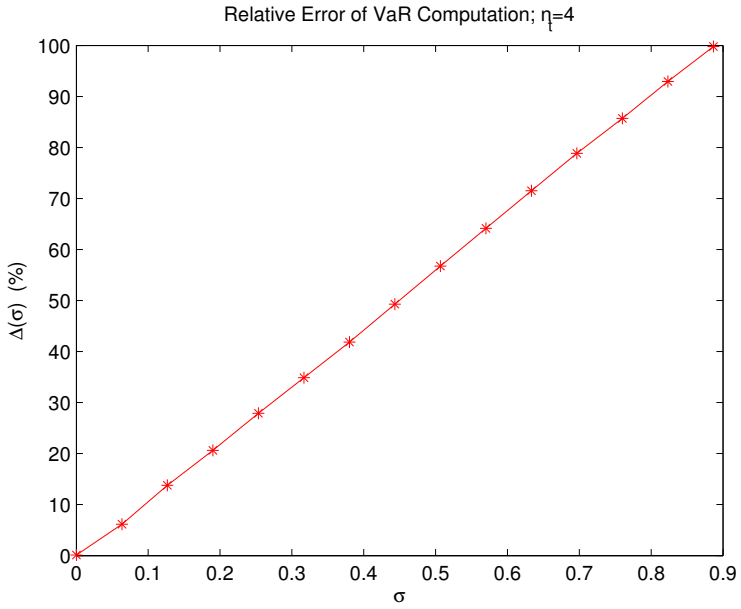

Figure 3: $\Delta_q(\sigma)$ for $q = 0.999$, $n_t = 4$.

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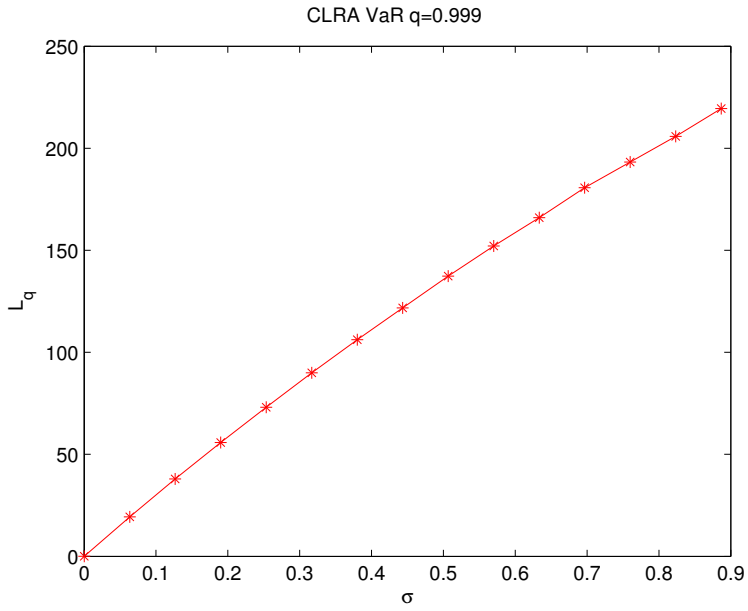

Figure 4: CLRA $\hat{\ell}_q(\sigma)$

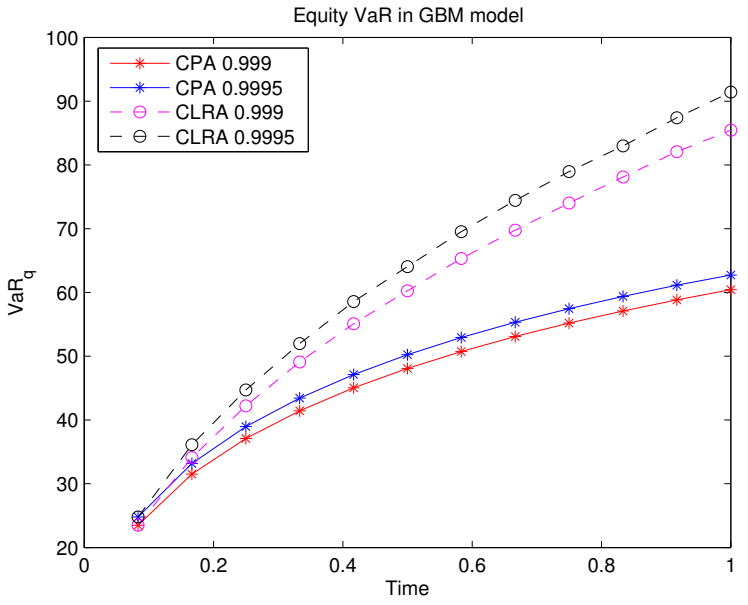

Figure 5: Equity VaR for $q = 0.999$ and $q = 0.9995$

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increases the spread between CLRA and CPA also increases implying divergence in the tail of the distributions. This means that the amount by which capital requirements from CLRA exceed those from CPA expands in an increasing manner as the quantile increases.

Overall, for a single equity, the constant level of risk VaR measure implies capital requirements higher than those under the constant position approach when using higher equity volatility, higher quantiles, and shorter liquidity horizons. Next we extend our analysis to a portfolio of equities.

Portfolio VaR analysis

Consider the case of VaR estimation for a portfolio of equities. We assume that there are n stocks in the portfolio, each with a price process described by the equation

$$S_t^{(k)} = S_0^{(k)} \exp\left(\sigma_k W_t^{(k)} + \mu_k t\right), \qquad k = 1, 2, \ldots, n,$$

where $W_t^{(k)}$ are correlated Brownian Motions (BM) such that the correlation coefficient is,

$$\rho_{kl} = \frac{\mathbb{E}[W_t^{(k)} W_t^{(l)}]}{\sigma\left(W_t^{(k)}\right)\sigma\left(W_t^{(l)}\right)} = \beta_k \beta_l, \quad k, l = 1, 2, \ldots, n.$$

The simplest implementation of this correlation structure is based on the following representation of the processes $W_t^{(k)}$:

$$W_t^{(k)} = \beta_k W_t + \sqrt{1 - \beta_k^2} B_t^{(k)}, \quad k = 1, 2, \ldots, n, \quad t > 0,$$

where W_t is a systemic equity risk represented by a BM process independent of the mutually independent BM processes $B_t^{(k)}$ representing the individual components of equity risk.

The value of the portfolio at time t is

$$V_t = \sum_{k=1}^{n} S_t^{(k)}.$$

In particular,

$$V_0 = \sum_{k=1}^{n} S_0^{(k)}$$

and the portfolio losses, L_t satisfy the equation

$$L_t = V_0 - \sum_{k=1}^{n} S_t^{(k)}.$$

As in the case of a single equity, we compare the portfolio CPA VaR with portfolio CLRA VaR, computed as follows. Suppose we would like to estimate portfolio CLRA VaR at $t = 1$ year with a one month liquidity horizon for all equity positions. The time horizon is partitioned into $n_t = 12$ liquidity intervals as in the case

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of a single equity. At the end of each liquidity horizon, the portfolio losses $\hat{L}_j$ have the same distribution as $L_{\Delta t} = V_0 - V_{\Delta t}$. We then find

$$\mathcal{L}_t = \sum_{k=1}^{n} \hat{L}_k,$$

where the random variables, $\hat{L}_k$, are independent and identically distributed.

Taking into account the additive structure of the portfolio value we propose an analytical CLT approximation of the quantiles of the portfolio losses. For that, we need to estimate the portfolio mean value and variance. We have

$$\mathbb{E}V_t = \sum_{k=1}^{n} \mathbb{E}S_t^{(k)} = \sum_{k=1}^{n} S_0^{(k)} \cdot \exp\left(\mu_k t + \frac{1}{2}\sigma^2 t\right). \tag{4}$$

Computation of the portfolio variance is more involved because of the covariance term:

$$\sigma^2(V_t) = \sum_{k=1}^{n} \sigma^2\left(S_t^{(k)}\right) + 2 \sum_{1 \le i < j \le n} \operatorname{cov}\left(S_t^{(i)}, S_t^{(j)}\right). \tag{5}$$

In the Appendix we show that

$$\operatorname{cov}\left(S_t^{(i)}, S_t^{(j)}\right) = \mathbb{E}S_t^{(i)} \cdot \mathbb{E}S_t^{(j)} \cdot \left(e^{\hat{\beta}_{ij} t} - 1\right), \tag{6}$$

where $\hat{\beta}_{ij} = \beta_i \beta_j \sigma_i \sigma_j$.

Since

$$\sigma\left(S_t^{(k)}\right) = S_0^{(k)} \cdot \exp\left(\mu_k t + \frac{1}{2}\sigma_k^2 t\right) \sqrt{(\exp(\sigma_k^2 t) - 1},$$

we obtain from (6) and (5) that

$$\sigma^2(V_t) = \sum_{k=1}^{n} \left[S_0^{(k)}\right]^2 e^{2\mu_k t + \sigma_k^2 t} \left(e^{\sigma_k^2 t} - 1\right) + 2 \sum_{1 \le i < j \le n} S_0^{(i)} S_0^{(j)} e^{\mu_{ij} t} \left(e^{\hat{\beta}_{ij} t} - 1\right), \tag{7}$$

where $\mu_{ij} = \mu_i + \mu_j + (\sigma_i^2 + \sigma_j^2)/2$.

From Equations (4) and (7) we find an approximation of the quantile, $\ell_q(t)$, of the equity portfolio distribution at time t:

$$\ell_q(t) \approx V_0 - \mathbb{E}V_t + \sigma(V_t) z_q. \tag{8}$$

Tests not presented herein showed that this approximation appears to be quite accurate for correlation coefficients $\rho_{ij} \le 0.64$.

The constant level of risk approach requires a distribution of portfolio losses accumulated over n_t liquidity horizons:

$$\mathcal{L}_t = \sum_{i=1}^{n_t} \hat{L}_i.$$

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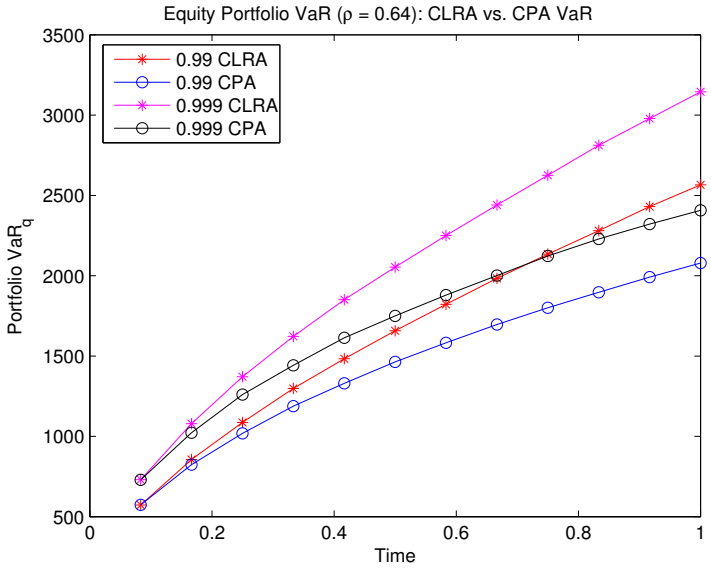

Figure 6: Comparison of CLRA and CPA: $\rho_{ij} = 0.64$

Assuming independence of the portfolio losses between liquidity horizons, we find the distribution of the random variable $\hat{L}_1 = V_0 - V_{\Delta t}$ representing the portfolio loss distribution over the first interval and compute n_t-fold convolution.

For a sufficiently large number of time steps, n_t, the distribution of $\mathcal{L}_t$ can be approximated by a Normal distribution, $\mathcal{N}\left(a(t), \sigma^2(t)\right)$, where

$$a(t) = n_t \cdot (V_0 - \mathbb{E}V_{\Delta t}), \quad \sigma^2(t) = n_t \cdot \sigma^2 (V_{\Delta t}).$$

Then we obtain the following approximation for the CLRA VaR of the portfolio:

$$\hat{\ell}_q(t) \approx a(t) + z_q \sigma(t), \quad z_q = \Phi^{-1}(q). \tag{9}$$

Figures 6 and 7 compare CLRA and CPA VaR for a portfolio of 50 stocks. Equity positions are modeled by a multivariate GBM process under two assumed levels of correlation: $\rho_{ij} = 0.64$ and $\rho_{ij} = 0.04$ for all $i \neq j$.

The number of Monte Carlo scenarios used was $N = 10^7$. The difference, $\Delta_q = \hat{\ell}_q(t) - \ell_q(t)$, between the CLRA ($\hat{\ell}_q(t)$) and the CPA ($\ell_q(t)$) VaR at the end of the one-year time horizon is 26% for $q = 0.99$ and is 32% for $q = 0.999$ using $\rho_{ij} = 0.64$. The differences are about half for $\rho_{ij} = 0.04$. The quantile impact is more pronounced with a higher correlation but in both cases even the $q = 0.99$ CLRA VaR exceeds $q = 0.999$ CPA VaR at the 1 year horizon!

Further, we analyze the relative difference between the approaches shown as a function of the correlation

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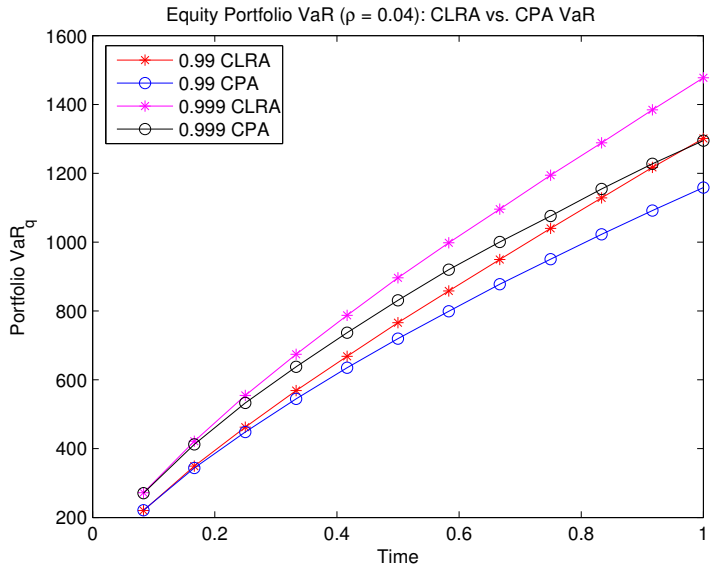

Figure 7: Comparison of CLRA and CPA approaches: $\rho_{ij} = 0.04$

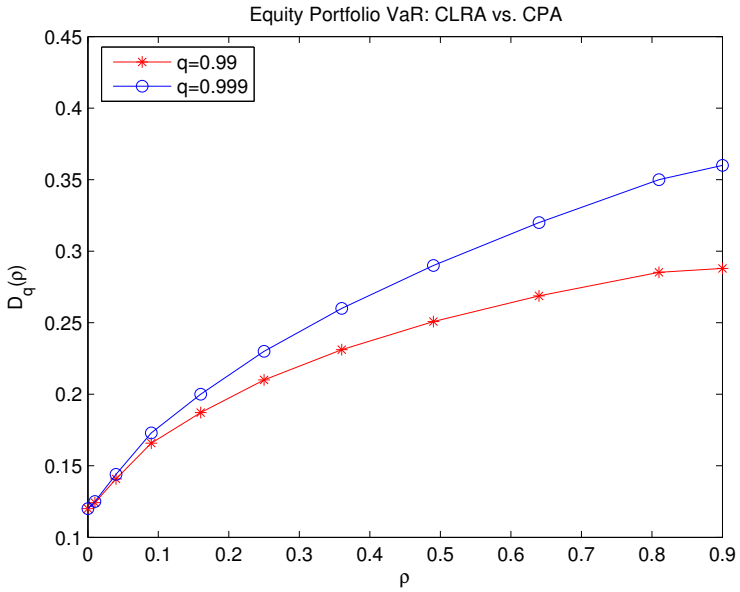

Figure 8: Relative difference $\mathcal{D}_q(\rho)$.

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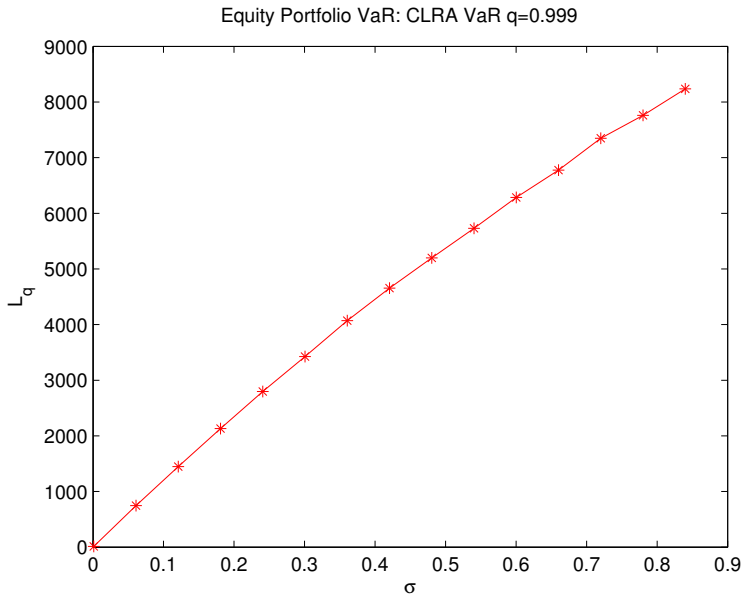

Figure 9: Portfolio IVaR as a function of σ; $q = 0.999$

parameter. Figure 8 displays the relative difference between CLRA and CPA VaR,

$$\mathcal{D}_q(\rho) = \frac{\Delta_q(\rho)}{\ell_q(\rho)}$$

at the end of the 1 one-year time horizon, for the portfolio with $n = 50$ stocks using quantile levels 0.99 and 0.999 and a correlation coefficient, ρ, in the interval $\beta \in [0, 0.9]$. Our numerical results show that $\mathcal{D}_q(\rho)$ is a monotone function of the parameter ρ. The higher the correlation between the stocks in the portfolio, the higher is the difference between approaches.

We note that the Basel II IRB mandates correlation assumptions ranging from 0.12 to 0.24. This range of correlation assumptions falls in the zone where the CLRA VaR will substantially exceed CPA VaR. In general, VaR measures under both methods will decrease as portfolio diversification increases, however the diversification effect is insufficient to reduce the CLRA capital requirements below those of the CPA. The substantial difference between CLRA VaR and CPA VaR can be explained intuitively as follows: the CLRA estimator, $\hat{\ell}_q(t)$ depends on the number of liquidity horizons, n_t, as

$$\hat{\ell}_q(t) \approx a(\Delta t)n_t + z_q\sigma(\Delta t)\sqrt{n_t}, \quad t = n_t \cdot \Delta t.$$

As n_t increases, $\hat{\ell}_q(t)$ may become very large, even greater than the initial portfolio value V_0. This can never happen with CPA VaR.

Finally, in Figure 9 we examine the impact of volatility on the portfolio CLRA VaR. As in the case of a single stock, VaR increases with the asset volatility exceeding the initial value of the portfolio, $V_0 = 5000$,

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for all $\sigma > 0.45$. Similarily to the impact of correlation, CLRA capital requirements can easily surpass the notional value of the portfolio, reaching values above 150% for values of $\sigma \approx 0.9$.

Conclusion

In this paper, we compared the computation of VaR under the constant level of risk and constant position approaches for both an individual equity and a portfolio of equities. The CLRA leads to significantly higher Value-at-Risk for equities with limited benefits deriving from portfolio diversification effects. Capital requirements are much higher under the constant level of risk approach irrespective of the correlation, quantile, liquidity horizon or volatility assumptions. However, the least desirable property of CLRA for equities is the high probability of VaR measures exceeding the initial investment amount. Overall the result of applying the CLRA, particularly in periods of stress and high volatility, may produce unintentionally punitive capital requirements.

A Appendix

Distribution of losses

In this section we derive Equation (1):

$$\ell_q(t) = S_0 \cdot \left(1 - e^{\mu t - z_q \sigma \sqrt{t}}\right).$$

Our derivation is based on the following general statement for monotone instrument value functions.

Lemma 1 (Monotone instruments) *Suppose that a financial instrument has a value function,*

$$V_t = f(X_t), \quad t \geq 0,$$

depending on a scalar risk factor X_t where $f(\cdot)$ is a monotone function. Denote the q-quantile of X_t by $x_q(t)$, $(0 \leq q \leq 1)$:

$$\mathbb{P}\left(X_t \leq x_q(t)\right) = q.$$

Then VaR of the instrument is

$$\ell_q(t) = \begin{cases} f(0) - f(x_{1-q}(t)), & \text{if } f(\cdot) \text{ is a monotonically increasing function}, \\ f(0) - f(x_q)(t), & \text{if } f(\cdot) \text{ is a monotonically decreasing function.} \end{cases} \tag{10}$$

Proof. We consider the case when $f(\cdot)$ is a monotonically increasing function. For simplicity, we assume that X_t has a continuous distribution. Then from the quantile definition

$$\mathbb{P}\left(L_t \leq \ell_q(t)\right) = q$$

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it follows that

$$\mathbb{P}(f(0) - f(X_t) \leq \ell_q(t)) = q.$$

Therefore,

$$\mathbb{P}(f(X_t) > f(0) - \ell_q(t)) = q.$$

Since f is a monotonically increasing function, the inverse function, $f^{-1}(x)$, is also monotone and the latter equation is equivalent to

$$\mathbb{P}(X_t > f^{-1}(f(0) - \ell_q(t))) = q.$$

Then we have

$$\mathbb{P}(X_t \leq f^{-1}(f(0) - \ell_q(t))) = 1 - \mathbb{P}(X_t > f^{-1}(f(0) - \ell_q(t))) = 1 - q.$$

Therefore

$$f^{-1}(f(0) - \ell_q(t)) = x_{1-q}(t),$$

and

$$f(0) - \ell_q(t) = f(x_{1-q}(t)),$$

which is eqivalent to the first case in (10). The proof in the second case is almost identical to that in the first case. ■

In the case of the GBM model, the value function can be represented in the form

$$f(x) = C \exp(x)$$

and the risk factor $X_t = \mu t + \sigma W_t$. Therefore at time t the distribution of X_t is normal, $\mathcal{N}(\mu t, \sigma^2 t)$ and

$$x_q(t) = \mu t + \sigma\sqrt{t} z_q, \tag{11}$$

where z_q is the q-quantile of the standard normal distribution, $\mathcal{N}(0,1)$. Note that

$$z_{1-q} = -z_q.$$

Choosing $C = S_0$ and substittuting (11) into Equation (10) we obtain Formula (1).

The distribution of equity losses, $L_t = S_0 - S_t$, can be obtained as follows:

$$\begin{aligned}
\mathbb{P}(L_t \leq l) &= \mathbb{P}(S_0 - S_0 \exp(\sigma W_t + \mu t) \leq l)) \\
&= \mathbb{P}(S_0 \exp(\sigma W_t + \mu t) > S_0 - l)) \\
&= \mathbb{P}\left(\sigma W_t + \mu t > \ln\left(\frac{S_0 - l}{S_0}\right)\right) \\
&= \mathbb{P}\left(\frac{W_t}{\sqrt{t}} > \frac{\ln(S_0 - l) - \ln S_0 - \mu t}{\sigma\sqrt{t}}\right) \\
&= 1 - \Phi\left(\frac{\ln(S_0 - l) - \ln S_0 - \mu t}{\sigma\sqrt{t}}\right).
\end{aligned}$$

In the last line we used equality in distribution of the random variables

$$\frac{W_t}{\sqrt{t}} \stackrel{d}{=} \xi, \qquad \xi \sim \mathcal{N}(0,1).$$

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Thus,

$$\mathbb{P}\left(L_t \le l\right) = 1 - \Phi\left(\frac{\ln(S_0 - l) - \ln S_0 - \mu t}{\sigma\sqrt{t}}\right), \qquad l \le S_0,$$

as was to be proved.

Covariance of stock values

Let us derive Equation (6). We have

$$\mathrm{cov}\left(S_t^{(i)}, S_t^{(j)}\right) = \mathbb{E}\Big[S_t^{(i)} S_t^{(j)}\Big] - \mathbb{E}S_t^{(i)}\mathbb{E}S_t^{(j)}.$$

The expected value of the product of the stock values is computed as follows.

$$\begin{aligned}
\mathbb{E}\Big[S_t^{(i)} S_t^{(j)}\Big] &= \mathbb{E}\left[S_0^{(i)} S_0^{(j)} \exp\left(\sigma_i(\beta_i W_t + \sqrt{1-\beta_i^2}B_t^{(i)})\right) \exp\left(\sigma_j(\beta_j W_t + \sqrt{1-\beta_j^2}B_t^{(j)})\right)\right] \\
&= S_0^{(i)} S_0^{(j)} e^{(\mu_i+\mu_j)t}\mathbb{E}\Big[e^{\sigma_i(\beta_i W_t + \sqrt{1-\beta_i^2}B_t^{(i)}) + \sigma_j(\beta_j W_t + \sqrt{1-\beta_j^2}B_t^{(j)})}\Big] \\
&= S_0^{(i)} S_0^{(j)} e^{(\mu_i+\mu_j)t}\mathbb{E}\Big[e^{(\sigma_i\beta_i + \sigma_j\beta_j)W_t + \sigma_i\sqrt{1-\beta_i^2}B_t^{(i)} + \sigma_j\sqrt{1-\beta_j^2}B_t^{(j)}}\Big].
\end{aligned}$$

Consider the process

$$X_t = (\sigma_i\beta_i + \sigma_j\beta_j)W_t + \sigma_i\sqrt{1-\beta_i^2}B_t^{(i)} + \sigma_j\sqrt{1-\beta_j^2}B_t^{(j)}.$$

It is not difficult to see that X_t is a Gaussian process having a Normal $\mathcal{N}(0, \sigma_*^2 t)$ distribution with the variance

$$\sigma_*^2 = (\sigma_i\beta_i + \sigma_j\beta_j)^2 + \sigma_i^2(1-\beta_i^2) + \sigma_j^2(1-\beta_j^2) = \sigma_i^2 + \sigma_j^2 + 2\sigma_i\sigma_j\beta_i\beta_j.$$

Therefore

$$\mathbb{E}\Big[\exp\left(X_t\right)\Big] = \exp\left(\frac{1}{2}\sigma_*^2 t\right).$$

Then we derive

$$\mathbb{E}\Big[S_t^{(i)} S_t^{(j)}\Big] = S_0^{(i)} S_0^{(j)} \exp\left(t\cdot(\mu_i + \mu_j + \frac{1}{2}(\sigma_i^2 + \sigma_j^2))\right)\cdot\Big[\exp\left(\beta_i\sigma_i\beta_j\sigma_j t\right) - 1\Big],$$

and, finally, we obtain (6). ■

The correlation coefficient, $\rho(S_t^{(i)}, S_t^{(j)})$, of Equity values is

$$\rho(S_t^{(i)}, S_t^{(j)}) = \frac{\exp\left(\beta_i\beta_j\sigma_i\sigma_j t\right) - 1}{\Big[\exp(\sigma_i^2 t) - 1\Big]^{1/2}\Big[\exp(\sigma_j^2 t) - 1\Big]^{1/2}}. \tag{12}$$

In Figure 10 the correlation function $\rho(S_t^{(i)}, S_t^{(j)})$ is displayed. As shown, the correlation coefficient is not sensitive to t. The limit value as $t \to 0$ is

$$\lim_{t\to 0}\rho(S_t^{(i)}, S_t^{(j)}) = \beta_i\beta_j.$$

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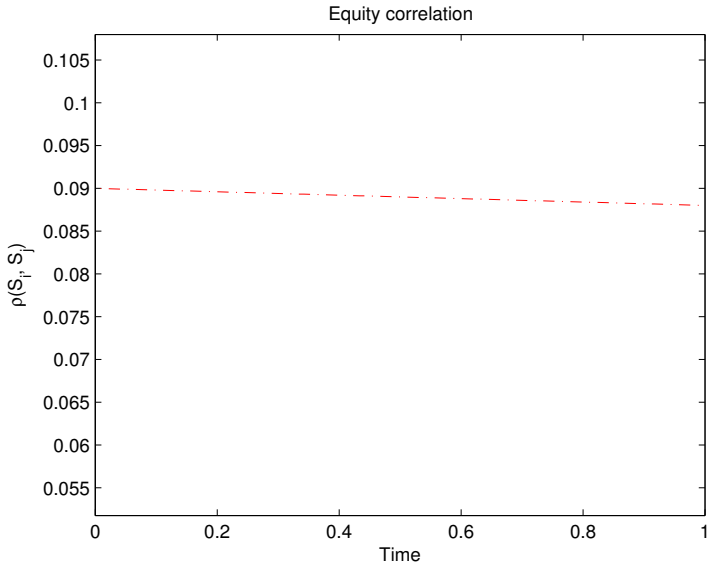

Figure 10: Correlation $\rho(S_t^{(i)}, S_t^{(j)})$: $\sigma_i = \sigma_j = 0.3, \beta_i = \beta_j = 0.15$.

Thus, the correlation coefficient of the risk factors in the joint multifactor GBM model determines correlation of values of equities.

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Annex A: Studies on Default and Migration Risks

October 2008

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Content

Summary 3

Relevant Regulation 3

Introduction 4

Example 1: Simple Default 6

Example 2: PD Effects 9

Example 3: Simple Migration 9

Example 4: Simple Correlations 11

Analysis of the Examples 13

Conclusions 15

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SUMMARY

This Annex discusses the capital implications of measuring default and migration risks as described in the Guidelines. The discussion focuses on paragraphs 20, 21 and 22, inclusive of footnotes; specifically, the statement that the "constant level of risk approach" ("CLRA") provides capital relief compared to the IRB framework and the "constant position approach" ("CPA"). Our studies show that any capital relief provided by CLRA compared to CPA is a happy accident of relative parameterization, as illustrated by experiments on a representative portfolio.

Through a series of examples we investigate, bottom up, some of the issues associated with the CLRA models, and note that under many circumstances the CLRA measured over one year with one-month liquidity horizons leads to a capital requirement at least as high as that implied by the CPA. In one special case, we do see a capital benefit from CLRA: when correlations are high under CPA but broken by the implementation of roll-overs in CLRA.

RELEVANT REGULATION

This document references three separate documents published by the Committee:

- "Guidelines for Computing Capital for Incremental Risk in the Trading Book", July 2008, referred to as simply "the Guidelines".
- "Proposed revisions to the Basel II market risk framework", July 2008, referred to as "the Proposed Revisions".
- "Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version", June 2006, referred to as simply "the Accord".

The Guidelines present information about the constant position approach ("CPA") and the constant level of risk approach ("CLRA") and liquidity horizons in several sections. For the purposes of this Annex, the most relevant paragraphs are 20, 21 and 22, including footnote 3 to paragraph 20.

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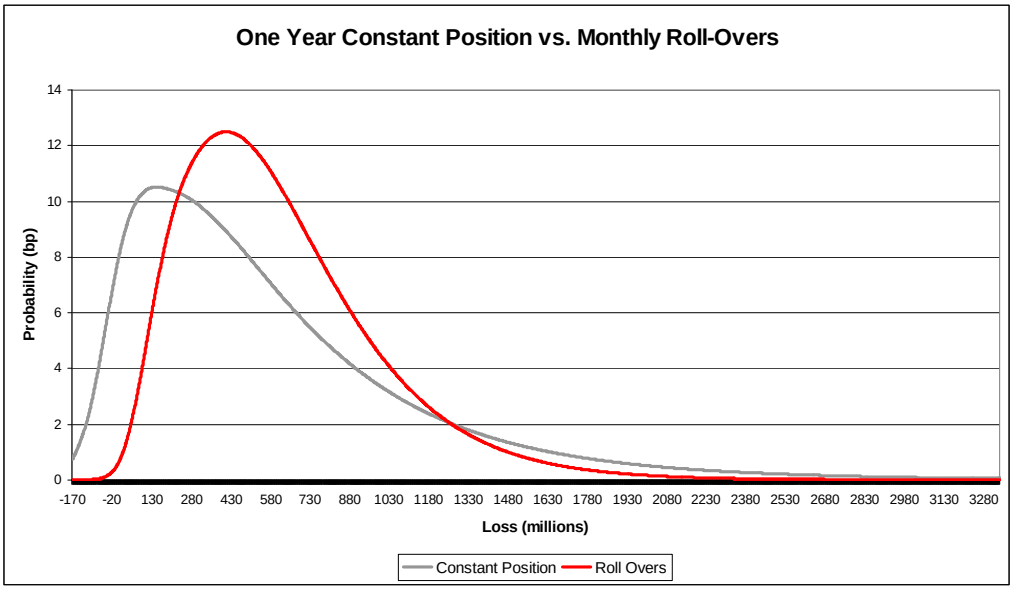
INTRODUCTION

The studies and examples compiled in this Annex began in response to a simple question: "How big a capital reduction can banks expect when using CLRA compared to CPA?" The answer to this question is, naturally, of great interest to Algorithmics' clients because the potential for capital relief is gained at the expense of model complexity and, for most organizations, an upfront investment.

Our initial testing showed some examples producing significant capital reductions from CLRA. To illustrate, we adopt the portfolio used by ISDA/IACPM in a joint study on portfolio credit risk[1]. While not specific to trading operations, this portfolio is fairly diverse and contains a total of 3000 names making it suitably representative of typical trading operations that might seek approval under the Guidelines. The loss distributions from the CLRA and CPA techniques are compared in Figure 1. Numerically, the 99.9% quantile of the loss distributions is 4.3B under CPA and 2.5B under CLRA, leading to a 40% reduction. This level of reduction is well within common-sense expectations.

Figure 1: Loss distributions for the representative portfolio



[1] Source: "Convergence of Credit Capital Models", Rutter Associates LLC, sponsored by International Association of Credit Portfolio Managers (IACPM) and International Swaps and Derivatives Association (ISDA), February 2006.

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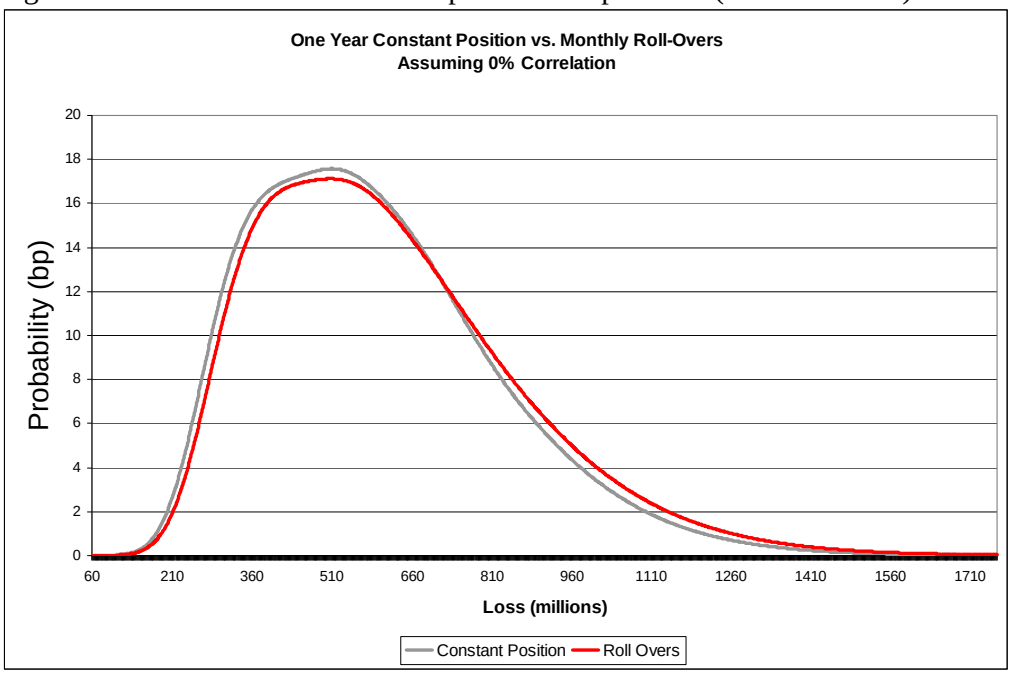
Figure 2: Loss distributions for the representative portfolio (0% correlation)



Taking our investigations further, we looked at the relative capital impact of various standard stress tests. Most tests showed expected patterns of behaviour. Stress tests around PD levels produced results slightly differently than originally expected, but nothing of great concern. However, shifting the correlations did produce unexpected results.

To illustrate these results, we recalculate the loss distributions for the ISDA/IACPM portfolio under each method, but this time assume zero (0) correlation. The loss distributions are compared in Figure 2. From the figure, it is clear that the two methods produce very similar results for the zero-correlation case. While the tails are harder to see in the graph, we note that the 99.9% loss quantiles (Loss(99.9%)) for the methods are 1.54B and 1.65B. This time, however, it is CLRA that produces the higher result! Various error analyses and simulation investigations yielded an inescapable conclusion: the CLRA really does produce a higher result. Specifically, the difference is statistically significant given the error bounds of the Monte Carlo simulation.

This Annex derives from our efforts to understand this phenomenon, starting with the simple case of a single issuer, and examining key factors (default, migration, PD, and correlation) to assess their impact on the two approaches. Our analysis and the resulting conclusions, detailed at the end of the Annex, were not comforting.

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The bulk of the Annex comprises a series of illustrative examples. In all examples, the CLRA is presented using a liquidity horizon of one month. The examples are as follows:

- Example 1: Simple Default – examines a single issuer in the case of default
- Example 2: PD Effects – extends the previous example by examining the impact of changing the PD
- Example 3: Simple Migration – extends the previous example by including migration risk
- Example 4: Simple Correlations – examines a two-issuer portfolio

The Annex provides a detailed analysis of the impact of these key factors on capital levels before reaching some disquieting conclusions.

EXAMPLE 1: SIMPLE DEFAULT

In the case of measuring default risk for a single issuer with a probability of default of 5% pa, using CLRA produces a higher capitalization requirement than CPA.

Assume we have a single instrument of value 100 with a given issuer. The issuer has a 5% chance of defaulting over the next year (and therefore, 95% chance of surviving). In the event of default, we lose 100% of our investment. Otherwise, the position does not lose anything. This information can be summarized by a loss distribution, as shown in Table A.1.1.

Using CPA, and consistent with current IRB framework settings of one-year and 99.9% we compute: Expected Loss = 5; Loss(99.9%) = 100 and VaR(99.9%) = 95. This implies a capital requirement of 95%, which is clearly punitive and unreasonable. Because positions in the trading book are typically fairly liquid, the idea of a liquidity horizon can be adopted to scale the default probabilities to a horizon shorter than one year, and so (presumably) reduce the capital requirement.

Generally speaking, default probabilities for less than a one-year horizon are difficult to observe directly. Many practitioners scale the one-year probabilities downward to reflect the shorter period. For example, assuming the liquidity horizon for this instrument is one month, we can scale the PD with the following logic:

Not defaulting in one year = surviving 1 year = surviving 12 individual months

Therefore: $ND_{1m} = \sqrt[12]{ND_{1y}}$ and $D_{1m} = 1 - ND_{1m}$

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If the probability of default over one year is 5% then the implied one-month probability of default is 43bp – a reduction of 11.7 times compared to the one-year probability. Table A.1.2 presents the one-month loss distribution. Note that even at one month, the Loss(99.9%) = 100 since 43bp > 10bp.

Table A.1.1: Loss distribution for the one-year time horizon

State	Loss	Probability
ND	0	0.95
D	100	0.05

Table A.1.2: Loss distribution for the one-month time horizon

State	Loss	Probability
ND	0	0.995734681
D	100	0.004265319

However, CLRA requires rolling over this position into positions of equivalent risk (whether risk is defined by VaR, exposure concentration or some other criteria). The implication here is that default is no longer a terminal event: If the position defaults after the first month (or in any month for that matter) then the bank is assumed to re-institute the position in the following month, using capital to recreate the original risk.

This specific point is clearly articulated in footnote 3 of the Guidelines, which states "*For regulatory capital adequacy purposes, it is not appropriate to assume that a bank would reduce its VAR to zero at a short-term horizon in reaction to large trading losses. It also is not appropriate to rely on the prospect that a bank could raise additional Tier 1 capital during stressed market conditions.*" The required behaviour would properly reflect the "going concern" and "constant risk taking" endeavours of the financial institution over the full one-year horizon.

Rolling over positions that may have defaulted leads to the possibility that the financial institution loses as much as 1200 instead of the actual investment of 100. This reflects an extreme case where a default occurs each month and the investment is replenished from capital at the beginning of the next month. Statistically, this is equivalent to convoluting the one-month loss distribution twelve times. The resulting one-year implied loss distribution is shown in Table A.1.3.

The immediate reaction of most readers is to (rightly) point out the extremely low probability of twelve consecutive losses. In fact, as in Table A.1.3, this probability is negligible. However, note that not all probabilities are insignificant: the probability of two losses (i.e., loss = 200) is more than 11bp!

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Comparing this loss distribution to that computed under CPA (see Table A.1.1), we make the following observations concerning the CLRA-implied distribution:

1. The maximum loss is much higher, albeit with very low probability.
2. The expected loss is higher (5.12 instead of 5).
3. Loss(99.9%) = 200.
4. The implied capital requirement is 194.88 – almost double the initial investment.

Thus in the limited case of an exposure of 100 to an issuer with PD = 5% pa and LGD = 100%, the CPA yields a capitalization requirement of 95%. The proposed solution to this problematic result, the CLRA, in fact yields an even higher capitalization requirement: 194.9%. This clearly undesirable result might stem from a variety of causes, the most immediate being that PD = 5% is considerable in the trading book. The next example addresses this issue.

Table A.1.3: Loss distribution at one year assuming monthly roll-over[2]

# of Losses	Loss	Probability	Cumulative Probability
12	1200	3.62592E-29	3.62592E-29
11	1100	1.01576E-25	1.01612E-25
10	1000	1.30421E-22	1.30522E-22
9	900	1.01489E-19	1.01619E-19
8	800	5.33079E-17	5.34095E-17
7	700	1.99115E-14	1.99649E-14
6	600	5.42304E-12	5.443E-12
5	500	1.08515E-09	1.09059E-09
4	400	1.58329E-07	1.5942E-07
3	300	1.64274E-05	1.65869E-05
2	200	0.001150491	0.001167078
1	100	0.048832922	0.05
0	0	0.95	1.00

[2] The probabilities in this table were calculated using the Binomial expansion given a two-state problem with inter-temporal independence (i.e. a default event in a previous month is assumed independent to what happens in the next month). In this case, the probabilities are calculated as follows:

$$P(loss_n) = \frac{12!}{n!(12-n)!} D^n \cdot (1-D)^{(12-n)}$$

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EXAMPLE 2: PD EFFECTS

CLRA yields a capitalization requirement at least as high as CPA across a broad range of realistic trading book probabilities of default.

As in Example #1, we assume a 100 investment with LGD=100%. Rather than assuming a one-year PD, we take the PD from the one-year transition matrix shown in Table A.3.2. Conducting the tests outlined in Example #1 for each initial credit rating, we can compute the corresponding loss distributions under both CLRA and CPA. The results for Loss(99.9%) are shown in Table A.2.1.

We note that CLRA produces the same Loss(99.9%) as CPA for ratings BB and above, but exceeds CPA at lower ratings. **Thus, we conclude that when measuring default risk for a single instrument, CLRA does not provide capital relief compared to CPA.**

Table A.2.1: Loss (99.9%) for CPA and CLRA

Original State	Loss (99.9%) – CPA	Loss (99.9%) – CLRA
AAA	0	0
AA	0	0
A	0	0
BBB	100	100
BB	100	100
B	100	200
CCC	100	300

EXAMPLE 3: SIMPLE MIGRATION

CLRA provides no capital relief compared to CPA when considering both defaults and migrations. As PD increases, the benefit derived from CPA increases.

Extending Example #2 to migration risk allows for a more realistic calculation. It requires two essential elements: a statement concerning spreads and the estimation of a one-month transition matrix (upon which CLRA calculations will be based).

While spread risks are covered under the Guidelines, for the purposes of this example, we assume static (constant) spreads and use them to compute losses from migration. As in the previous examples, we confine ourselves to a single instrument with a certain issuer. However, rather than assuming a 100 investment with LGD = 100% and corresponding 100 loss, we assume an initial investment based on the spread-implied value at the initial rating, and a fixed recovery

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amount for all ratings. We adopt the spreads used in the ISDA/IACPM paper[3] leading to the initial values under each credit state shown in Table A.3.1.

Table A.3.1: Value of investment based on initial credit quality

Credit State	Value
AAA	100.00
AA	99.98
A	99.97
BBB	99.74
BB	98.78
B	95.94
CCC	89.90
Default	47.99

Thus, for example, when analyzing an issuer with initial rating BBB, the investment is 99.74, the loss on a downgrade to B is 3.8 = 99.74 – 95.94 and the loss on default is 51.75 = 99.74 – 47.99. The transition probabilities for CPA remain as shown in Table A.3.2. For CLRA, a one-month transition matrix is required. The scaling is done in the manner proposed by Kreinin and Sidelnikova[4] and results in the one-month transition matrix shown in Table A.3.3.

Table A.3.2: One-year transition matrix (%)

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	92.18	7.06	0.73	0.00	0.02	0.00	0.00	0.01
AA	1.17	90.84	7.63	0.26	0.07	0.01	0.00	0.02
A	0.05	2.39	91.83	5.07	0.50	0.13	0.01	0.02
BBB	0.05	0.24	5.20	88.49	4.88	0.80	0.16	0.18
BB	0.01	0.05	0.50	5.45	85.12	7.05	0.55	1.27
B	0.01	0.03	0.13	0.43	6.52	83.20	3.04	6.64
CCC	0.00	0.00	0.00	0.58	1.74	4.18	68.00	25.50
Default	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

[3] Source: "Convergence of Credit Capital Models", Rutter Associates LLC, sponsored by International Association of Credit Portfolio Managers (IACPM) and International Swaps and Derivatives Association (ISDA), February 2006.

[4] Source: "Regularization Algorithms for Transition Matrices", Kreinin A & Sidelnikova M, Algo Research Quarterly, Volume 4, Numbers 1&2, March/June 2001. Available from www.algorithmics.com.

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Table A.3.3: One-month transition matrix (%)

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	99.32	0.638	0.041	0.000	0.001	0.000	0.000	0.0004
AA	0.106	99.19	0.691	0.006	0.005	0.0003	0.0000	0.002
A	0.003	0.216	99.27	0.464	0.034	0.0091	0.0003	0.001
BBB	0.004	0.016	0.476	98.96	0.461	0.0586	0.0143	0.001
BB	0.001	0.004	0.033	0.516	98.63	0.6870	0.0454	0.086
B	0.001	0.003	0.010	0.022	0.634	98.45	0.3270	0.557
CCC	0.000	0.000	0.000	0.055	0.168	0.4446	96.83	2.506
Default	0.000	0.000	0.000	0.000	0.000	0.000	0.000	100.00

Table A.3.4: Loss(99.9%) including default and migration risks

Original State	Loss (99.9%) – CPA	Loss (99.9%) - CLRA
AAA	0.03	0.04
AA	0.24	0.24
A	4.03	4.03
BBB	51.75	51.75
BB	50.79	50.79
B	47.95	95.90
CCC	41.91	132.91

Proceeding to calculate the capital requirement under the two approaches and changing the initial credit rating, results in the measures shown in Table A.3.4. This table is analogous to Table A.2.1, presenting Loss(99.9%) for each combination of method and credit state. Once again, we observe that the CLRA capitalization levels equal or exceed those of CPA. This leads to the conclusion that, as in the case of default risk only, for the joint measurement of default and migration risks, CLRA does not provide capital relief compared to CPA for the single issuer case.

EXAMPLE 4: SIMPLE CORRELATIONS

CLRA leads to higher or lower capital requirements than CPA based on a complex interaction between probability of default, quantile measured and correlation structure.

We now extend the analysis to a portfolio of names to assess the impact of correlations on the relative size of capital requirements from CLRA and CPA. We are thus attempting to determine if the results obtained for individual positions carry over to portfolios. To maximize tractability, we consider only two instruments, each with its own (different) issuer. Each instrument has an initial value based on spreads (see Table A.3.1) and the issuers share the same credit state and transition matrix (Tables A.3.2 and A.3.3).

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Since the trigger for these investigations was a correlation of 0%, we begin with this assumption for the two issuers. This assumption also provides for some mathematical shortcuts, facilitating the mechanics of the model in each approach.[5] Table A.4.1 presents the Loss(99.9%) results.

In the case of perfect correlation between issuers, we can calculate the final distributions simply.[6] Table A.4.1 presents the Loss(99.9%) results. The relationship between CPA and CLRA observed earlier holds in both cases: at higher levels of PD, the CLRA yields significantly higher capital requirements while at lower levels of PD, the two models are fairly consistent.

Seeing similar behaviour at both extremes of correlation (0,100) might lead us to conclude that correlation has little impact on the relative positioning of the capital requirements under CPA and CLRA. Before reaching this conclusion however, we consider one last correlation example: 50% correlation[7]. We adopt a simulation approach, using one million scenarios to generate the loss distribution at one month and at one year. In the final step for CLRA, the one-month loss distribution is convoluted 12 times, as before. Table A.4.1 shows the Loss(99.9%) measures for each of the three assumed levels of correlation.

Table A.4.1: Comparing Loss(99.9%) measures: various correlation assumptions

Correlation Method	0% CPA	0% CLRA	100% CPA	100% CLRA	50% CPA	50% CLRA
AAA	0.05	0.06	0.06	0.08	0.06	0.07
AA	1.20	1.20	0.48	0.48	1.21	1.21
A	4.03	4.03	8.05	8.06	4.26	4.04
BBB	51.75	51.75	103.50	103.50	52.71	51.76
BB	53.63	53.63	101.57	101.58	101.58	59.68
B	95.90	95.90	95.90	191.80	95.90	107.99
CCC	83.82	167.64	83.82	251.46	83.82	200.68

[5] Specifically, we can take the single-period loss distribution calculated for a single issuer and convolute it with itself to derive the corresponding loss distribution for both issuers together. The calculations under CPA are fairly straightforward based on a convolution of the one-year distribution for the individual name shown in Table A.3.4. For CLRA, we recomputed the one-month distribution, convolute it with itself, then convolute that distribution 12 times.
[6] We double the values for the one-year distribution for CPA (since both issuers move perfectly in tandem) and, for CLRA we double the one-month distribution and convolute it 12 times.
[7] Specifically, this correlation refers to an asset return correlation in a structural model. This is consistent with a Merton model and the model underlying the Basel II approach.

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Notice some similarities and differences across the three correlation assumptions. In all three cases, high credit states (BBB and above) result in comparable results between the two methods. For the lower credit states (B and CCC) CLRA produces capitalization requirements significantly larger than CPA.

However, the most telling number in Table A.4.1 is 59.68 = Loss(99.9%, 50% correlation, CLRA). In words, this value was calculated under CLRA at the 99.9% quantile from a loss distribution resulting from a 50% correlation between names in the portfolio. This is the only result which shows what we expected to see before beginning the analysis: a significant capital reduction in CLRA as compared to CPA (101.58).

From this example we conclude that it is difficult to predict whether CLRA will provide capital relief or increase the capital requirement. Patterns between CPA and CLRA capital levels appear to change based on a combination of the level of probability of default and the level of correlation.

ANALYSIS OF THE EXAMPLES

Based on the examples presented and other related work, we find that there are two competing forces within the CLRA: the propensity to experience multiple losses (increases capital) and the correlation-breaking impact of management intervention (decreases capital). We explore these two forces in turn.

The likelihood of multiple losses is determined by a combination of three factors: (1) probability of default, (2) the number of liquidity horizons and (3) the measure: one-year, 99.9%. Ignoring correlation effects, generally speaking, **the lower the credit quality (higher PD) the greater the impact of a one-month roll-over assumption relative to a one-year holding assumption**. Even though it is scaled back for a shorter liquidity horizon, the scaled PD is still large enough to create the possibility of default (at the 99.9% level) within a single liquidity horizon. Ultimately, it is multiple defaults that cause CLRA to produce results significantly higher than those from CPA. Higher credit qualities (lower PDs) result in equivalence between the two approaches since the scaling of a smaller PD to a one-month liquidity horizon rarely creates multiple defaults at the 99.9% level.

We have focussed on the common one-year, 99.9% standard, as adopted by the Committee for IRC. We note that the level of PD creating the possibility of multiple defaults varies with both the selected confidence level and the relationship between the overall time horizon and the liquidity horizon.

Management intervention in the form of portfolio rebalancing is modelled using roll-overs. Its impact is therefore a function of the liquidity horizon (in relation to

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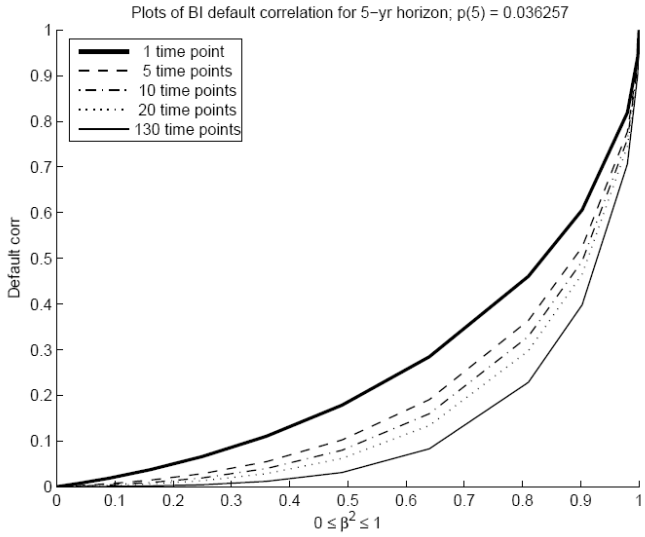
the measurement horizon) and the basis upon which the roll-over is conducted. In isolation, the **roll-over assumption will generally serve to reduce correlations between issuers**. In Example #4, where the correlation between two issuers over one year was assumed to be 50%, the implied default correlation under CLRA would be significantly smaller than that under CPA. This would reduce the concentration risk significantly and thereby the capital requirement. Intuitively, the constant rolling-over and resetting back to an original risk level introduces noise which serves to disrupt the correlated movements of issuers. This same effect has other implications and so has been explored by authors in industry literature[8].

For example, Algorithmics conducted a long series of related tests when constructing multi-period models. Figure 3 is drawn from one particularly relevant paper[9]. In this case the "BI model" is very similar to CLRA[10]. Figure 3 plots asset correlation (horizontal axis) in relation to the default correlation implied (vertical axis).

Figure 3: Correlation losses in multi-period models



[8] For example: (1) "Time for multi-period capital models", Gupta et al, RISK, October 2005.
[9] "Losing Default Correlations, Step by Step", working paper by Algorithmics, enclosed in the Supporting Documents.
[10] The primary difference is that the "BI model" resets the creditworthiness of the issuer to the middle of its credit state band (i.e. the model resets a "good" BB and "bad" BB back to the mid level of the BB band; where "good" signifies a high creditworthiness value within the BB band). In contrast, CLRA resets the creditworthiness to the original credit state of the issuer. This larger change will lead to an even greater watering down of correlation effects under CLRA compared to those shown for the "BI Model".

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The solid dark line shows the relationship for a one-year holding assumption (i.e., CPA). The other lines illustrate the impact of multi-period roll-overs across different periods. It is clear from the graph that in only two cases are default correlations preserved under roll-overs: 0% and 100%. In all other cases, the default correlation for a given, imputed asset correlation is significantly less than expected. As an example, for an asset correlation of 50% and ten assumed roll-overs during the year (slightly less than the 12 brought about by a monthly liquidity horizon) the graph indicates that the default correlation drops from the expected 20% (under CPA / one time point) to only 8% (under CLRA / ten time points).

This explains our observations in Example #4, Table A.4.1. Applying these observations to the original portfolio, Figure 1 shows quite clearly that the correlation effect has a bigger impact than the PD effect on the chosen portfolio at the one-year, 99.9% confidence level. However, if the correlation assumption were reduced to zero then the distributions would be as shown in Figure 2, and we see that the PD effect dominates.

If the quantile of interest interacts with the probability of default to increase capital requirements under CLRA while implicit decreases in correlation lower those requirements, which direction do the capital requirements move? Whether or not CLRA leads to capital relief or excess capital will ultimately depend on a combination of these effects. ***It is impossible to determine apriori whether the capital requirement will be higher or lower under CLRA compared to CPA***. This is an unpalatable situation for financial institutions and regulators alike, and we believe, not an intended consequence of the Guidelines.

CONCLUSIONS

It is troubling that predicting the tipping point (i.e., the point at which decreases from correlation overtake increases from multiple defaults) is not possible under general circumstances. This uncertainty may create issues for regulators during model review and for banks in adopting measures for practical, business purposes.

Perhaps even more troubling than the uncertainty is the reason for the capital relief derived under CLRA: a breakdown in correlation. Given the purpose of capital and the current market environment, allowing modeling techniques that significantly reduce correlations implicitly within the model (compared to those input to it) seems anachronous.

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Annex B: Equity Risks

October 2008

Algorithmics

Content

Summary **3**

Relevant Regulation **3**

Analysis: Standardized Method Comparison **4**

Analysis: Banking Book Comparison **5**

Analysis: Constant Level of Risk **6**

Single Equity Example 6

Portfolio of Equities Example 8

Conclusions **9**

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SUMMARY

This Annex provides supporting evidence for three claims made in our response to the Committee's request for comments on the Guidelines. Specifically:

1. The fall-back feature of the regulations, requiring the standardized method to be used, results in the lowest capital requirements across all trading and banking book models.

2. The total capital requirements under the trading book rules produce capital requirement far above the IRB credit risk framework while making assumptions on the equity holding period that is at best one fifth of the five year minimum in the IRB standard.

3. The constant level of risk approach produces higher capital requirements than the constant position approach, counter to the stated objective of the Guidelines in introducing the option of assuming a constant level of risk.

Our analysis focuses on each point individually before drawing overall conclusions.

RELEVANT REGULATION

This document references three separate documents published by the Committee:

- "Guidelines for Computing Capital for Incremental Risk in the Trading Book", July 2008, referred to as simply "the Guidelines".
- "Proposed revisions to the Basel II market risk framework", July 2008, referred to as "the Proposed Revisions".
- "Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version", June 2006, referred to as simply "the Accord".

We make extensive use of the following abbreviations:

- CLRA – constant level of risk approach
- CPA – constant position approach

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ANALYSIS: STANDARDIZED METHOD COMPARISON

The Standardized Method for the trading book is risk insensitive. In the case of an outright equity holding, the capital requirement is 16%.

To compare we must consider not only the IRC itself, but also the charges for general and specific risks. For simplicity, we assume under the IMM that only general market risks and the IRC are computed for equity risks. We consider both a single equity and a portfolio of fifty liquid, tradable equity positions.

Key inputs to the lognormal simulation model include the drift of the equity price through time (assumed to be 0%), volatility of equity prices (assumed to be 30%) and the level of correlations. Comparisons are drawn at two levels of correlation: 16% and 64%. Table B.1 presents results based on simulation.

Table B.1: Trading Book Regulatory Capital Requirements for Equity Positions

	Portfolio ($\rho = 0.16$)	Portfolio ($\rho = 0.64$)	Single Equity
Standardized Method	**16.0%**	**16.0%**	**16.0%**
IMM - General Market Risk	14.3%	26.6%	38.7%
IRC (Constant Position)	30.2%	51.3%	60.4%
IMM – Total (Constant Position)	**44.6%**	**77.9%**	**99.2%**
IMM - General Market Risk	14.3%	26.6%	38.7%
IRC (Constant Level of Risk)	34.9%	68.0%	86.0%
IMM – Total (Constant Level of Risk)	**49.2%**	**94.7%**	**124.7%**

It is clear from Table B.1 is that the Standardized Method produces significantly lower capital requirements than either of the IMM approaches. It is not uncommon for IMM approaches to yield higher capital requirements than less sophisticated methods under the right circumstances. We would not, however, expect to observe IMM resulting in substantially higher capital requirements under almost all circumstances. This observation is counter to the stated objective of the regulations to facilitate a more risk-sensitive capital framework and the alignment of the IRC to the current market realities.

We compare the two Internal Models Methods in a later section.

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ANALYSIS: BANKING BOOK COMPARISON

To compare capital requirements across the banking and trading books on an even footing would have been considered excessively conservative two years ago. Given market events in the interim, the liquidity of trading book positions has been questioned making the comparison more natural. However, it remains unlikely that the banking book would be considered more liquid than the trading book. Presumably, this means that while capital standards in the trading book are expected to be more conservative under the IRC rules, they are not expected to exceed those of the banking book.

To make the comparison, we revisit the portfolio used earlier to compare the different trading book methods. The models for the banking book vary in risk-sensitivity particularly with regards to the holding period. The IMM method assumes a quarterly simulation horizon while the PD/LGD method assumes a five year maturity adjustment. However, all require an additional piece of information: probability of default, which we assume to be 2%.

Table B.2 compares results from the different methods. Mathematically, the IRC could be calibrated using the correlation parameters to match the IRB framework; however, the overall IMM trading book charge will still be higher. We also observe that higher asset correlations will increase the total capital requirements. This creates an opportunity for regulatory arbitrage.

Table B.2: Comparing Regulatory Capital Requirements for Equity Positions

	Portfolio ($\rho = 0.16$)	Portfolio ($\rho = 0.64$)	Single Equity
Trading Book			
Standardized Method	16.0%	16.0%	16.0%
IMM – Total (Constant Position)	44.6%	77.9%	99.2%
IMM – Total (Constant Level of Risk)	49.2%	94.7%	124.7%
Banking Book			
PD/LGD	35.2%	35.2%	35.2%
Market Based - Simple Method	24.0%	24.0%	24.0%
Market Based – IMM	16.0%	24.3%	29.5%

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ANALYSIS: CONSTANT LEVEL OF RISK

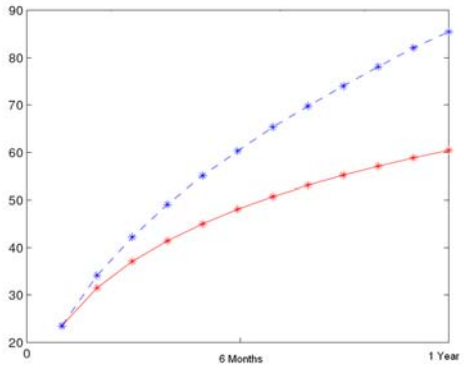
Our analysis shows that the use of the CLRA in the context of equities always leads to higher capital requirements than using the CPA. To gain insight into this phenomenon, we begin by examining a single equity then consider a broader portfolio of fifty equity holdings. Two observations resonate deeply: (1) CLRA creates larger capital requirements by implicitly allowing negative prices for equities and (2) correlation effects observed in the default and migration risks studies[1] do not materialize for equity holdings.

A summary of our findings and experiments is discussed below. Further analysis, including detailed descriptions of the derivations of the figures in this section, is presented in the white paper entitled "Measuring Equity VaR with the Constant Level of Risk Approach" enclosed in the Supporting Documents.

Single Equity Example

We begin with an initial calculation of VaR(99.9%) under both approaches: CPA and CLRA[2]. The results for a single equity with initial price 100 and volatility 30% are shown in Figure B.1. The dashed blue line represents CLRA while the solid red line represents CPA. To understand why CLRA leads to such a large VaR(99.9%) result, we return to the basic assumptions underlying typical equity price models.

Figure B.1: Comparing Loss(99.9%) under CPA and CLRA over one year[3]



[1] Please refer to Annex 1.
[2] The one-month distribution for the equity was calculated via Monte Carlo simulation and then convoluted 12 times to reflect the roll-over under CLRA.
[3] Loss(99.9%) is computed as a quantile of the loss distribution without reference to the mean. The comparison is based on a single equity with an initial value of 100 and volatility of 30% p.a.

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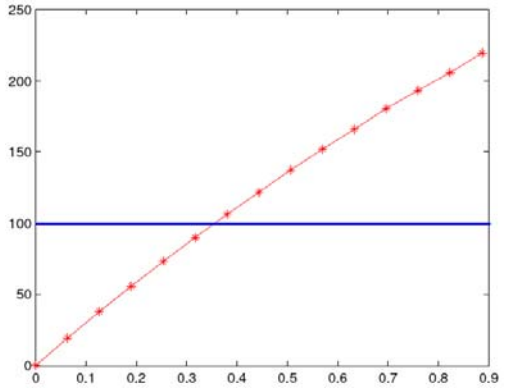
Traditionally, equity prices are modelled through time as a Geometric Brownian Motion (GBM) process which implies that equity prices are log-normally distributed and ensures that the equity price can never fall below zero. In other words, an investor can never lose more than the value of the equity regardless of the time horizon or quantile of the risk measure. This model is consistent with reality: by definition, equity investors cannot lose more than their investment[4].

However, under CLRA the implicit floor of zero for equity prices disappears. The repeated rolling-over of the equity position, combined with capital support to reset back to the original investment, creates the situation where the investor can lose much more than the original investment. When the recapitalization is done at a high quantile and the equity is sufficiently volatile, CLRA implies that banks continue to re-invest heavily in markets after repeated extreme movements.

The question becomes: For the 99.9% quantile mandated for IRC, what is a sufficiently high volatility to create a significant problem? The answer, unfortunately, is that a realistic volatility such as 35% p.a. is sufficient. Figure B.2 plots the relationship between VaR computed using the CLRA (vertical axis) and equity price volatility (horizontal axis). The horizontal blue line depicts the original equity price of 100.

Figure B.2: VaR(99.9%) under CLRA[5] compared to equity price volatility



Based on this analysis of a single equity, the Guidelines imply the possibility of requiring investors to hold capital in excess of 100% of their initial investment. While this might be reasonable for some derivative positions (e.g., forwards, short

[4] We note that this is not true for investors in some equity derivatives. For example, short options, swaps, futures and short selling may lead to losses in excess of the original investment.

[5] For a single equity with initial value 100, assuming a liquidity horizon of one month compounded to one year.

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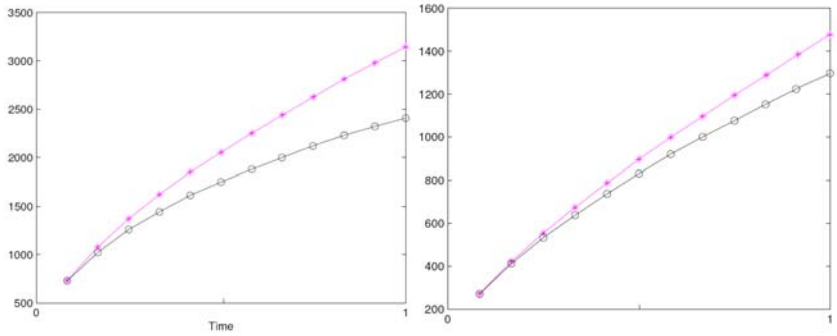
positions), it is unlikely to be realistic for the direct equity position analyzed in this example.

Portfolio of Equities Example

We extend the analysis above to a portfolio of fifty equities. Once again, numerical experiments indicate that the CLRA always produces VaR(99.9%) that is significantly greater than the corresponding CPA result. Perhaps surprisingly, this observation is independent of the assumed correlation between the equities, although the extent of the excess does increase with correlation. This is illustrated in Figure B.3.

Figure B.3: VaR(99.9%) under CPA and CLRA at 64% and 4% correlations



64% Correlation 4% Correlation

Unlike the credit-related examples[6], the correlation between equities does not diminish as a result of rolling-over the positions. Intuitively, equity correlations are reflected directly in the joint behaviour of equity prices, implying a linear transformation from equity prices to losses. In this case, the final distribution adequately preserves the correlation structure. In the credit case, the correlation reflects the joint behaviour of issuer creditworthiness. In general, there is a non-linear relationship between the creditworthiness and the losses suffered. So, in convoluting the credit losses directly, we fail to preserve the original correlations.

This raises an interesting question: “Does the correlation actually used in the model reflect the correlation assumption in a portfolio of equity derivatives?” Since options (by definition) are non-linear with respect to equity price, it is possible that a portfolio of equity options would effectively lose correlation with every roll-over. We did not have the opportunity to investigate whether this is the case.

[6] Please refer to Annex 1.

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CONCLUSIONS

Our analysis leads us to believe that any capital relief provided by the CLRA as compared to the CPA is an artefact of the model implementation rather than the result of considered business practice and assumptions. This leads to potential confusion and misapplication of the risk measures for business purposes, amongst enumerable other problems.

We also conclude that the proposed IRC creates a significant capital burden compared to other methodologies used in the Accord. The inevitable conclusion is that without significant clarification, banks will universally adopt the Standardized Method for measuring specific risk and incremental risks in all traded equity portfolios. We believe this runs contrary to the spirit of the Accord and the Internal Models Approach, and so urge the Committee to reconsider the one-year holding period assumption and/or the extent of the equity risks included in the incremental risk charge.

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Annex C: Integrated Risk

October 2008

Algorithmics

Content

Summary **3**

Relevant Regulation **3**

Introduction **3**

Discussion **4**

Case Study **5**

Conclusions **6**

Appendix: Details of the Event Risk Model **7**

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SUMMARY

This annex first discusses issues relating to integrated risk measurement in the trading book. Next, it presents a case study illustrating the impact of integrated risk measurement on a particular trading portfolio. The portfolio and measures are real; graciously provided by one of our clients based on their current systems and holdings.

The case study supports our assertion that integrated risk measurement is possible, at least under the constant position approach. Further, expanding the scope of the regulations to capture all price risks including commodity prices, foreign exchange and default-free interest rates is feasible as a replacement to the current proposed combination of general, specific and incremental risks.

RELEVANT REGULATION

This document references three separate documents published by the Committee:

- "Guidelines for Computing Capital for Incremental Risk in the Trading Book", July 2008, referred to as simply "the Guidelines".
- "Proposed revisions to the Basel II market risk framework", July 2008, referred to as "the Proposed Revisions".
- "Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version", June 2006, referred to as simply "the Accord".

The Guidelines present information about the constant position approach ("CPA") and the constant level of risk approach ("CLRA") and liquidity horizons in several sections.

INTRODUCTION

In this context, we loosely define event risks as default and migration risks arising from issuers. Actual counterparty credit risk is not discussed, nor presented in the case study. General and specific market risks are discussed, but not presented in the case study.

Several statements in our covering response relate back to the issue of integrated risk measurement in the trading book:

- The extension of the incremental charge from default risk to default, migration, spread and equity risks – including correlations within and

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across those risks – provides simultaneously for realism and appropriate conservatism in capitalization standards for the trading book.

- Incorporating correlation across the modeled risk factors is essential in capturing all material components of price risk.
- Expanding the scope of the regulations to capture all price risks including commodity prices, foreign exchange and default-free interest rates captures the full risk profile of a trading book portfolio.

All of these statements derive from our firm belief that integrated risk management provides many advantages to firms including comprehensiveness and clarity.

DISCUSSION

Capturing the full risk profile of a modern day trading book portfolio is essential in a sound capital measurement and management process. The Guidelines move regulations towards such a model by extending incremental risks to include not only more risk types, but correlations between them. Specifically, the extension from default risk to default, migration, spread and equity risks includes correlations within and across those risks.

Many large, internationally active financial institutions are already capable of calculating an integrated risk measure within their risk systems using sophisticated, generally available approaches. They have moved in this direction for management purposes, seeing the benefits that it provides.

A fully integrated measure of (market) risk adds clarity and removes the necessity of artificially defining and categorizing events and risks. It enables consistent, comprehensive modeling and measurement programmes, and is both easier to interpret and less subject to manipulation. By supporting more effective decision-making within firms adopting integrated measures, compliance issues such as the 'use test' and effective alignment of capital requirements with business practice are more naturally addressed. The gradual merging of regulatory and economic capital frameworks promotes focus on modeling risks holistically and realistically.

Technically, the adoption of integrated risk measures, including commodity, foreign exchange and default-free interest rate risks limits the chance that risks will go uncapitalized or be double-counted. Effectively, the single integrated measure on a compromise, single measurement horizon could replace the current general, specific and incremental market risk definitions, simultaneously simplifying compliance and promoting risk management best practice.

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The case study illustrates measures of the different market risks based on a portfolio provided by one of our clients.

CASE STUDY

The portfolio is roughly 50% bonds and 50% basket products, spanning the financial and corporate sectors. It consists of 675 individual obligors in multiple sovereign jurisdictions with an average rating between A1 and A3. The focus is on interest-bearing products, and so equity risks are not considered further in the case study. Table D.1 provides a breakdown of exposures across three key dimensions. Note in particular the hedging role of the CDS.

Table D.1: Breakdown of exposures

Product Type	%	Sector	%	Rating Range	%
Bond	58%	Corporate	35%	Aaa…Aa3	28%
CDS	-7%	Covered	14%	A1…A3	43%
Index	2%	Emerging	5%	Baa1…Baa3	24%
Basket	47%	Financial	46%	Ba1…Ba3	3%
		Government	0%	B1…B3	1%
				Caa1…N\A	0%

Our analysis focuses on three risk measures consisting of credit spread risk only, event risk only and integrated risk. The results of simulations under each of the three assumptions are summarized in Table D.2.

The spread risk only simulation generates a return distribution based on the volatility of credit spreads. This simulation is typically extended to other, general market risk factors, but those risk factors are omitted in this study to provide clearer insight into the definition of IRC in the Guidelines.

The event risk only simulation uses nominal values for all market rates used in valuing the portfolio and simulates the drivers of event risk ("credit drivers"). The return distribution is generated by measuring only the impact of default and rating migration changes due to the credit driver volatility. The average asset correlation (from which default correlations are derived) was 37%. Other modelling issues are discussed briefly in the Appendix.

The integrated approach simulates all risk factors jointly in a consistent and correlated fashion. The return distribution is a product of credit state migrations and credit spread volatility of all portfolio components.

The Loss(99.9%) measured assuming integrated risks is 25% smaller than the simple summation of the measures for the constituent risk types. The inherent

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correlation between the risk factors is less than perfect, and so decreases the overall risk level of the portfolio, resulting in more realistic capital requirements.

Table D.2: Risk measures at one year

Measure	Spread Risk	Event Risks	Integrated Risk
Mean Exposure	3,758,907,189	3,758,907,189	3,758,907,189
Mean Loss	3,996,115	6,924,101	10,920,216
Loss(99.9%)	94,500,000	171,250,000	203,000,000
Unexpected Loss(99.9%)	90,503,885	164,325,899	192,079,784

CONCLUSIONS

Integrated risk measurement is feasible, and desirable. The extension of the incremental charge to default, migration, spread and equity risks, is a significant step forward in modelling risks inherent in the trading book portfolio coherently and comprehensively.

Integrated risk models correctly reflect the risks and capital requirements by capturing the correlation between risk factors and the portfolio concentration and diversification effects not found in additive or piece-meal approaches.

Replacing the current three-tier risk measurement process for market risks with a single coherent approach under the Internal Models Method is worthy of consideration. In general, it should be possible. Details of regulations in this regards, such as measurement horizon, quantile and ‘exceptions’, need to be carefully considered.

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APPENDIX: DETAILS OF THE EVENT RISK MODEL

The event risk was calculated based on the Merton model (migration adjusted) where the probability of default is equal to the probability that the issuer's assets value becomes lower than the value of its liabilities at a given time horizon.

Drivers of systemic risk are used in a multi-factor Gaussian model to imply asset (and consequentially default) correlations amongst obligors. We opted to use a collection of market indices and CDS credit spreads as the drivers – both in practice and herein. The credit drivers are classified as follows:

- 25 stock exchange indices.
- 273 sector or country indices.
- 12 single name survival probability indices

The contribution to integrated risk given by change in market spreads (market risk component) is assessed by joint simulation of single name credit spreads with the above credit drivers.

The factors are related to individual counterparties using a simple linear model, known as the creditworthiness index (CWI) of the counterparty. The systemic and idiosyncratic components of each CWI are weighted as a result of a regression analysis, the details of which change with the class of the reference asset. In all cases, three years of weekly time series data (approximately 150 observations) are used in the regression analysis. We believe that this provides a meaningful estimation of correlation and will be sustainable and robust over of time.

Events were simulated in Algorithmics' engines using a two-stage sampling technique. In everyday practice 33,000 systemic scenarios (including spreads and systemic credit drivers) and 300 idiosyncratic samples (based on independent, normal distributions) are used to create a total of approximately ten million (10,000,000) scenarios of default & migration events. For the purposes of this exposition, we reduced the scope of the simulation to three million (3,000,000) scenarios by reducing the number of systemic scenarios to 10,000.

Consistent with the Guidelines, the risk factors were simulated over a one-year horizon assuming constant positions and losses calculated at 99.9% confidence level.

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Annex D: About Algorithmics

October 2008

Algorithmics

About Algorithmics

Algorithmics is the world's leading provider of enterprise risk solutions. More than 300 financial institutions rely on Algorithmics' software, analytics, and advisory services to make risk-aware business decisions, maximize shareholder value, and meet regulatory requirements. Supported by one of the largest teams of risk experts in the industry, Algorithmics offers proven, award-winning solutions for market, credit, and operational risk, as well as collateral and capital management.

Algorithmics is committed to helping its clients view and manage risk as a core source of business value. This intelligent risk taking, supported by Algorithmics' software and services, offers firms numerous competitive advantages including increased returns, lower capital requirements, and improved brand reputation.

Founded in 1989, Algorithmics' client base has grown to include global banks, insurance companies, asset management firms, hedge funds, corporations, regulators, and central banks, among others. Many industry awards have recognized the company's achievements, including Risk magazine's 2007 Technology Rankings, which cited Algorithmics as the leader in enterprise risk management.

Algorithmics is a sister company to Fitch Ratings Inc. under a holding company, Fitch Group Inc. Fimalac, a publicly traded French company, owns 80% of the Fitch Group, and The Hearst Corporation, a large media conglomerate, owns the remaining 20%. Until December 2005, Algorithmics operated as a private company and did not release annual reports. Annual reports for Fimalac, which contain a discussion of Algorithmics' performance from January 2006 and forward, are available on Fimalac's website.

About Fitch Group

Fitch is the parent company of Fitch Ratings, a leading global rating agency committed to providing the world's credit markets with accurate, timely, and prospective credit opinions. Fitch Ratings is dual-headquartered in New York and London, operating offices and joint ventures in more than 50 locations and covering entities in more than 80 countries.

About Fimalac

FIMALAC is a publicly traded Paris based company, focused exclusively on financial services with ratings and financial risk management businesses. Fimalac operates two companies, Fitch Ratings and Algorithmics, which have been combined under the Fitch Group banner. Mr. Marc Ladreit de Lacharriere is the founder and CEO of FIMALAC and is also Chairman of Fitch. Fimalac is listed on the first market of Euronext Paris.

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