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Liquidity risk in markets with trading frictions: What can swing pricing achieve?

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Abstract

Open-end mutual funds expose themselves to liquidity risk by granting their investors the right to daily redemptions at the fund's net asset value. We assess how swing pricing can dampen such risks by allowing the fund to settle investor orders at a price below the fund's net asset value. This reduces investors' incentive to redeem shares and mitigates the risk of large destabilising outflows. Optimal swing pricing balances this risk with the benefit of providing liquidity to cash-constrained investors. We derive bounds, depending on trading costs and the share of liquidity-constrained investors, within which a fund chooses to swing the settlement price. We also show how the optimal settlement price responds to unanticipated shocks. Finally, we discuss whether swing pricing can help mitigate the risk of self-fulfilling runs on funds.

Keywords: Financial stability, mutual funds, regulation, liquidity insurance, trading frictions

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1 Introduction

A key role of financial intermediaries is to provide liquidity – essentially, on-demand access to cash – to their investors. Typically, financial intermediaries that provide liquidity also engage in maturity transformation. For example, banks issue long-term loans but grant their depositors the right to withdraw their funds on demand. Similarly, open-end mutual funds ("funds") that invest in comparatively illiquid securities, such as corporate bonds, give their investors the option of redeeming their shares in cash every day. Daily redemptions allow fund investors to insure against their liquidity needs while participating in the higher return their fund earns on less liquid assets. At the same time, funds need to adequately insure the residual liquidity risk that they incur.[1] Insufficiently insured liquidity risk can trigger and amplify financial crises.

In this paper, we assess the effect of swing pricing – a tool for managing liquidity risk in funds, which several types of US funds will be able to use from November 2018 onwards.[2] Swing pricing permits a fund to pay out less than net asset value (NAV[3]) per share when net redemptions are large. It thereby alleviates a situation in which the fund, in the absence of swing pricing, would have to sell assets at a large discount to generate sufficient cash to pay out its redeeming shareholders. Symmetrically, swing pricing allows the fund to raise the price per share above the NAV per share when the fund experiences large net inflows. It can thus help to ensure that the costs associated with purchasing additional assets are borne by incoming investors. If investors anticipate that the fund will settle share transactions above the NAV when net demand for its shares is high, and below the NAV when net demand is negative, swing pricing can help reduce the volatility of flows into and out of the fund.

Our model builds on Diamond and Dybvig (1983) and Jacklin (1997). We derive an optimal rule for adjusting the single price at which a fund settles both share redemptions and subscriptions (the "settlement price"). The settlement price in our model summarises a range of contract features that in practice the fund can offer to its investors: for example, a higher settlement price could also correspond to faster settlement of redemption requests or more generous maximum daily redemption amounts.

In our model, we assume that investors can either purchase assets or invest indirectly in those assets by purchasing fund shares. There is a large number of small funds whose managers maximise their investors' utility. At the time the fund is set up, fund managers commit to a rule whereby they set the settlement price. (In practice, the fund discloses the general terms of its swing pricing policy in its prospectus.) Fund managers take into account that investors might become cash constrained, prompting them to redeem their shares earlier than they expected and independently of the settlement price. The higher the settlement price, the more investors obtain if they turn out to be cash constrained, at the expense of those who stay with the fund.

In this context, three parameters determine the optimal settlement price: the cost of trading in the asset market; the likelihood with which investors become cash constrained; and the degree of investor risk aversion. Their impact is best understood against the background of the optimal investment contract. This contract entails investors receiving less when they need to redeem their shares early than if they hold onto their shares until the fund's assets mature. Thus, even if the return of the fund's assets was certain, the return of an investment in fund shares is risky because the investor is uncertain whether he will have to redeem his shares early. Risk-averse investors dislike this uncertainty. The benefit of raising the settlement price for early redemptions is that investors' payoffs become less volatile. The fund manager balances these benefits with the larger costs he incurs when selling assets to serve redemption requests.

[1] See Financial Stability Board (2017) for policy recommendations to address liquidity risk in asset management activites.
[2] Securities and Exchange Commission (2016).
[3] The net asset value is the market value of the fund's assets net of the fund's liabilities.

Against this background, consider first the impact of trading frictions on the optimal settlement price.[4] The settlement price is lower when trading frictions rise because the fund manager aims to reduce the costs associated with asset sales. Higher trading costs raise the marginal cost of raising the settlement price for early redemptions without altering the benefits. However, the fund does not pass on the entire marginal costs of selling assets, thereby raising the return of cash-constrained investors and reducing the variability of investors' returns. Second, the optimal settlement price falls the more likely investors are to become cash constrained: ceteris paribus, more cash-constrained investors imply greater redemptions, so the marginal cost of raising the settlement price increases. Finally, the settlement price increases if investors become more risk averse. This follows because the more risk-averse investors, the more they dislike variations in the return of their investment. The marginal benefit of raising the settlement price increases, without altering the costs.

We also show that the optimal settlement price lies within a bound that is determined by the trading frictions. If investors can anticipate the fund's settlement price, they may move to arbitrage any difference between the settlement price and the market value of the fund's shares: for example, by redeeming fund shares and purchasing directly the assets the fund holds. If a fund manager settled net redemption requests too far below the market value of the shares, investors would start rushing into the fund, diluting the value of the fund's assets per share. Correspondingly, if the settlement price was too far above the market value of the fund's shares, its existing investors would redeem their shares and cause the fund to be liquidated early. A fund manager wants to avoid both situations because they would reduce the payoff for the fund's existing shareholders. He therefore only varies the settlement price to the extent that it does not generate arbitrage-driven flows in and out of the fund and holds sufficient liquid assets to pay out cash-constrained investors.

Finally, we discuss whether swing pricing might help mitigate the risk of self-fulfilling runs on the fund. Within our theoretical framework, we show that swing pricing can prevent self-fulfilling runs: the fund manager can commit to setting a sufficiently low settlement price to discourage redemptions by investors that are not cash constrained. In practice, however, swing pricing might be less effective, primarily because of uncertainty about how low the settlement price would need to be set without unduly penalising cash-constrained investors that need to redeem shares.

The remainder of the paper is structured as follows. Section 2 discusses how the paper relates to the literature. Section 3 describes the modeling framework. To provide a benchmark for the fund's choice of settlement prices, Section 4 derives the welfare-optimal solution from the view of a planner who is able to directly allocate consumption goods to households. Section 5 compares the planner's solution with the decentralised one and derives the rule for setting the settlement price and the fund's optimal liquidity buffer. Section 6 discusses comparative static properties of optimal settlement prices and asks whether swing pricing might help mitigate the risks of self-fulfilling runs on funds. Section 7 discusses the assumptions made and their influence on the results. Section 8 concludes.

2 Related literature

The paper is related to the literature that studies liquidity insurance provided by financial intermediaries and markets.[5] A key question is whether the existence of financial intermediaries can be explained by their ability to insure risk-averse households against idiosyncratic liquidity shocks. Diamond and Dybvig (1983) show that a well-designed deposit contract can provide households with the welfare-optimal degree

[4] In practice, a multitude of market frictions create costs for trading in asset markets. See Madhavan (2000) for a discussion.

[5] More broadly, insights from the literature on optimal taxation can also be relevant for swing pricing in that the trading costs caused by share redemptions or subscriptions impose a negative externality on existing shareholders. For a discussion of how taxation can be used to internalise externalities, see eg Sandmo (1975).

of insurance: an outcome that households would be unable to achieve if they invested directly in financial markets. The deposit payout corresponds to the settlement price of fund shares in our model. The optimal deposit payout itself depends on parameters such as households' risk aversion and the likelihood of them requiring immediate access to their deposits. The same factors influence the optimal settlement price in our model. The key difference between the two contracts is that the settlement price is allowed to depend on net redemption requests. The fund manager can commit to setting the settlement price sufficiently low to discourage redemptions even if each fund investor believes that a large share of other fund investors will redeem their shares. In contrast, promised deposit payouts are independent of the amount of deposits withdrawn, making a bank vulnerable to self-fulfilling runs.

Jacklin (1987) showed that equity contracts can achieve the same liquidity risk sharing as deposit contracts while being immune to runs. Sales of equity shares correspond to redemptions in our model, and the dividend payout plays a similar role to the settlement price. Similar to deposit contracts, equity contracts promise a dividend payout that is independent of aggregate liquidity needs. However, the price at which shareholders can sell their shares, and hence their level of consumption, depends on the aggregate amount of dividend payouts relative to the aggregate amount of shares sold. The more shares that are sold (the larger redemptions, in our model), the lower the price, and the lower the incentive to sell equity shares (to redeem fund shares, in our model). These welfare-optimal deposit and equity contracts are only feasible if households cannot invest directly in the assets the intermediary holds. Otherwise, as eg Jacklin (1987) shows, financial intermediaries cannot offer contracts that provide their investors with a higher level of utility than if households invested directly in financial markets. The reason is that any difference in payoffs would be exploited by arbitrage trades. Correspondingly, we show that if there are no trading frictions in the asset market, the fund optimally sets the settlement price equal to the market value of its assets per share (the fund's NAV per share).

Von Thadden (1998, 1999) derives an optimal deposit contract in the presence of investment frictions. The influence of such frictions is also the main interest of this paper. Von Thadden's context is different: in his model, households cannot trade assets but can liquidate them, against a loss, and then start a new investment project. Despite this difference, we find that the same factors that matter for the optimal deposit contract also determine optimal settlement prices and the optimal size of the fund's liquidity buffer in our model: the size of the frictions, the likelihood with which investors become cash constrained, and the degree of investor risk aversion.

Lewrick and Schanz (2017) empirically investigate the impact of swing pricing on the flows, the liquidity buffer, and the profitability of open-end mutual funds. In that paper, we also provide a stylised partial-equilibrium model to motivate our estimation hypotheses. Other than that, we are not aware of any other papers that model swing pricing. However, the risk of runs on mutual funds, which swing pricing might mitigate, has been investigated in the recent theoretical and empirical literature. Chen et al (2010), Goldstein et al (forthcoming) and Zheng (2016) show how an incomplete allocation of liquidation costs to those investors redeeming their shares can give rise to a run on an open-end investment fund. Malik and Lindner (2017) discuss whether swing pricing might reduce systemic risk. They suggest measuring the effect of swing pricing by its ability to dampen the impact of large outflows on the fund's NAV. Specifically, they compare changes in the NAV with outflows during normal and stressed periods of funds that implemented swing pricing with those that did not. They find some suggestive evidence for swing pricing to be effective in a small sample of funds. In Lewrick and Schanz (2017), we employ a related method to compare the performance of funds that were allowed to use swing pricing with that of funds not permitted to swing settlement prices. We show that swing pricing dampens outflows in response to weak fund performance, but has a limited effect during stress episodes. Furthermore, swing pricing supports fund returns while raising the volatility of fund share prices, and may incentivise funds to hold less cash. We compare some of these findings with our predictions in section 6.

3 Framework

Our modeling framework is based on Diamond and Dybvig (1983), extended by an asset market, in which households can directly trade the assets the fund invests in. Our economy has three periods, $t = 0, 1, 2$ and two types of agents: households and open-end investment funds.

At date 0, each household is endowed with one unit of a physical good. There are no other goods or endowments. The good, which serves as the numeraire, can be stored for one period in both periods 0 and 1, yielding a gross return of 1 after one period. In this sense, it can be thought of as cash. But it is a real good that also serves as an input to a long-term investment opportunity at date 0. This investment has constant returns to scale, is arbitrarily divisible and yields a certain gross return of R at date 2. It cannot be liquidated early, but agents can issue and trade claims (equity contracts) on their investment's payoff at date 1 in a competitive "asset" market at price p. These claims are risk-free and pay R per unit of investment at date 2.

There are initially two types of households: a fraction μ of households is risk neutral; the remainder is risk averse. We will focus on an equilibrium in which at date 0 only risk-averse households invest in funds: these households value the fund's smoothing of investment returns. Risk-neutral households are potential trading partners for the funds and may decide to subscribe to fund shares at date 1. They have identical preferences over future consumption of the good given by $u_N = c_2$. Risk-averse households' preferences, u_A, are given by

$$u_A(c_1, c_2) = \begin{cases} u(c_1) & \text{with probability} \quad \lambda \\ u(c_2) & \text{with probability} \quad 1-\lambda \end{cases} \tag{1}$$

where the utility function u is twice continuously differentiable in consumption, increasing, strictly concave, and satisfies Inada conditions. λ represents the probability of being subject to a liquidity shock. This shock determines whether the household is "impatient" and needs to consume at date 1, or "patient" and consumes at date 2. The households' individual shocks materialise at date 1, are identically distributed, and satisfy the Law of Large Numbers. Hence, there is no uncertainty about aggregate consumption needs nor about prices. As standard in this literature (eg von Thadden (1999)), the utility function represents the simplifying case where agents consume only once in their lives. Individual consumption needs are private information. Therefore, if agents interact, type-dependent consumption allocations must be incentive compatible. Households decide in period zero how to invest their endowment (storage or the long-term technology) and, at date 1, whether to issue claims on the long-term technology or subscribe to or redeem investment fund shares. For simplicity, we allow households to only hold either fund shares or invest in the long-term technology but not both.

Investment funds do not have own endowments but issue shares to households. They are small and take the market price as given when considering whether to trade in the asset market. For simplicity, we do not allow them also to issue debt. Each fund manager aims to maximise the aggregate utility of those households that invest in fund shares at date 0: that is, we abstract from agency conflicts between the fund manager and the fund's shareholders. This simplifies the comparison of our results with those of Diamond and Dybvig (1983) and Jacklin (1987); their justification - that competing fund managers can attract investors only if they best serve investors' interests - also applies here.

Each fund manager chooses an investment contract (the fund's "investment prospectus"). This contract specifies the fraction ω_0 of the proceeds from share issuance, S_0, that the fund invests at date 0 in storage rather than in the long-term technology (the fund's "liquidity buffer"). It also describes how s_1, the price at which the fund settles requests for share issuance and redemptions at date 1, and s_2 the payout per share to investors at date 2, depend on the number of fund shares that households wish to redeem or acquire at date 1. Because the entire wealth of the fund is distributed to its investors at the end of period 2, s_2 is equal to the fund's assets per share at that time and determined by the fund's choice of s_1. Our focus is therefore on the fund's choice of s_1, which we refer to as the fund's settlement price. In practice, a fund's prospectus would express s_1 as a "swing factor" in relation to the mid-market price of the fund's assets, p:

$$\sigma = \frac{\omega_0 + (1-\omega_0)\, p}{s_1} - 1 \tag{2}$$

Accordingly, a positive swing factor corresponds to a settlement price below the unswung net asset value. Both expressions are equivalent in our model because there is no aggregate uncertainty, so the market price is known.

We normalise the number of shares the fund issues at date 0 to one per share. Thus, s_1 is equal to the consumption of a household that purchases a fund share at date 0 and redeems it at date 1, while s_2 is its consumption if it redeems its share at date 2. At date 1, the fund purchases (or sells) claims on the long-term technology to invest proceeds from net share issuance (or to obtain sufficient amounts of the good to settle redemption requests).

Trading claims is costly. These costs are represented by a bid-ask spread $[(1-\gamma)\, p, p/\,(1-\gamma)]$ around the mid-market price p. That is, a seller of one claim receives $(1-\gamma)\, p$, whereas a buyer of a claim pays $p/\,(1-\gamma)$.

Figure 1 summarises the timeline of the model.

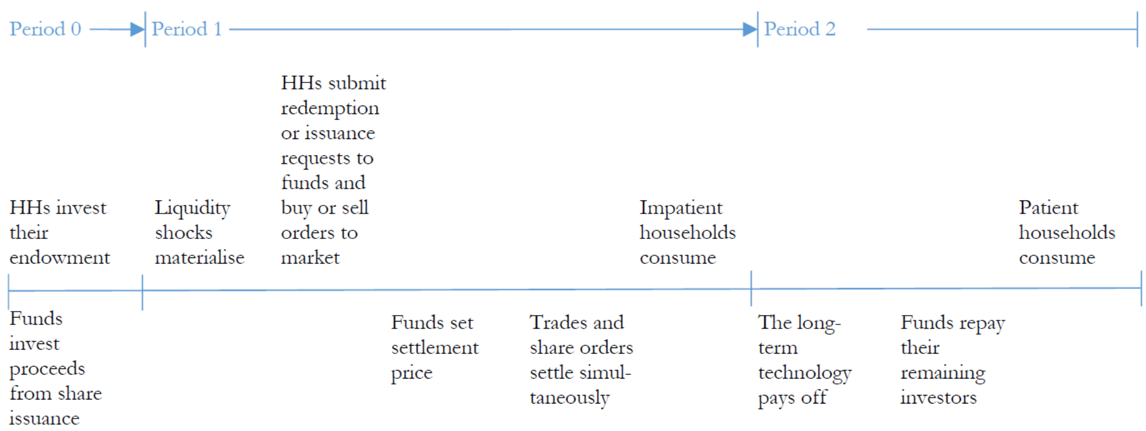

Figure 1: Timeline

4 First best

As a benchmark for the decentralised solution, we present the solution of a social planner's problem in this section. We assume that the planner can distinguish patient from impatient households and is able to allocate the consumption good directly to households without having to trade.

The planner's objective is to maximise the weighted welfare of the two types of households,

$$\begin{aligned} W &= \mu c_N + (1-\mu)\, E\left[u\left(c_A\right)\right] \\ &= \mu c_N + (1-\mu)\left(\lambda u\left(c_A^I\right) + (1-\lambda)\, u\left(c_A^P\right)\right) \end{aligned} \tag{3}$$

where μ is the share of risk-neutral households, c_N their (period-2) consumption, $1-\mu$ the share of risk-averse households, c_A^I the (period-1) consumption of risk-averse impatient households, and c_A^P the (period-2) consumption of risk-averse patient households.

Lemma 1 *(First best) If patient and impatient households are identifiable, and if consumption can be allocated directly to households, the welfare optimal solution is*

$$u'\left(c_A^I\right) = R \tag{4}$$

$$u'\left(c_A^P\right) = 1 \tag{5}$$

$$c_N = \frac{\left(1-\lambda\left(1-\mu\right)c_A^I\right)R-\left(1-\mu\right)\left(1-\lambda\right)c_A^P}{\mu} \tag{6}$$

Proof. To fund consumption of impatient households, the planner needs to store

$$\omega_0 = \lambda\left(1-\mu\right)c_A^I \tag{7}$$

units of the endowment good and invests the remainder in the long-term technology. This leaves $\left(1-\omega_0\right)R$ to be consumed by risk-neutral and patient risk-averse households, that is,

$$\mu c_N + \left(1-\mu\right)\left(1-\lambda\right)c_A^P = \left(1-\lambda\left(1-\mu\right)c_A^I\right)R \tag{8}$$

Using the resource constraint (8) to replace the consumption levels of risk-averse households in the welfare function (3) yields

$$W = \mu c_N + \left(1-\mu\right)\left(\lambda u\left(c_A^I\right)+\left(1-\lambda\right)u\left(\frac{\left(1-\lambda\left(1-\mu\right)c_A^I\right)R-\mu c_N}{\left(1-\mu\right)\left(1-\lambda\right)}\right)\right) \tag{9}$$

The planner's problem can now be written as $\max_{c_N, c_A^I} W$. The first-order constraints are

$$\begin{aligned}\frac{\partial W}{\partial c_N} &= \mu+\left(1-\mu\right)\left(1-\lambda\right)\frac{-\mu}{\left(1-\mu\right)\left(1-\lambda\right)}u'\left(\frac{\left(1-\lambda\left(1-\mu\right)c_A^I\right)R-\mu c_N}{\left(1-\mu\right)\left(1-\lambda\right)}\right) \qquad (10)\\ &= \mu\left(1-u'\left(\frac{\left(1-\lambda c_A^I\right)R-\mu c_N}{\left(1-\mu\right)\left(1-\lambda\right)}\right)\right)=0\\ \frac{\partial W}{\partial c_A^I} &= \left(1-\mu\right)\left(\lambda u'\left(c_A^I\right)+\left(1-\lambda\right)\frac{-\lambda\left(1-\mu\right)R}{\left(1-\mu\right)\left(1-\lambda\right)}u'\left(\frac{\left(1-\lambda\left(1-\mu\right)c_A^I\right)R-\mu c_N}{\left(1-\mu\right)\left(1-\lambda\right)}\right)\right) \qquad (11)\\ &= \left(1-\mu\right)\lambda\left(u'\left(c_A^I\right)-Ru'\left(\frac{\left(1-\lambda c_A^I\right)R-\mu c_N}{\left(1-\mu\right)\left(1-\lambda\right)}\right)\right)=0\end{aligned}$$

Entering the first into the second yields the equilibrium allocation (4) - (6). ■

In the first best, households consume less when impatient than when patient. This is a variant of the standard result that in the first best, liquidity insurance is incomplete (see eg Freixas and Rochet (2008), chapter 2.2). The presence of risk-neutral households means that the marginal utility of risk-averse households when patient, $u'\left(c_A^P\right)$, is equal to that of (patient) risk-neutral households ($=1$).

5 Decentralised solution

We start by presenting the decentralised solution in Section 5.1 and compare it with the first best described in the previous section. Section 5.2 then derives the optimal settlement price in this equilibrium and characterises the proofs for the optimal liquidity buffer and investment decisions at date 0.

5.1 Overview and comparison with the planner's solution

There are two frictions that cause the decentralised solution to differ from the welfare optimum: first, markets are incomplete, because individual liquidity shocks are not publicly observable and securities contingent on these shocks cannot be traded; and second, because consumption needs to be allocated via trading in asset markets and trading is costly.

Definition 1 *A decentralised (Nash) equilibrium consists of the following:*

- *for each household j, investment choices that maximise households' utility (u_N and u_A, respectively)*
- *for each investment fund k, investment and contract choices $\left(\omega_{0,k}^*, s_{1,k}^*, s_{2,k}^*\right)$ that maximise the expected aggregate utility of the households that purchased the fund's shares*
- *A market clearing condition for traded claims on the long-term technology.*

We show existence of an equilibrium in which, at date 0, risk-neutral households invest in the long-term technology whereas risk-averse households invest in fund shares; in which all agents of the same type make the same choices; in which all funds have the same size; and in which there is no trade in the asset market at date 1.

Proposition 1 *(Decentralised equilibrium) There is a decentralised equilibrium with the following properties:*

1. *Risk-neutral households invest their endowment in the long-term technology. Risk-averse households invest their endowment in a fund's shares. If impatient, they redeem their shares at date 1 and consume. If patient, they hold onto their shares.*
2. *Each fund sets the settlement price at date 1 equal to*

$$s_1^* = \begin{cases} s_H & \text{if} \quad \hat{s} \geq s_H \\ \hat{s} & \text{if} \quad \hat{s} \in (s_L, s_H) \\ s_L & \text{if} \quad \hat{s} \leq s_L \end{cases} \tag{12}$$

where

$$\hat{s} = s_1 : u'(s_1) = \frac{R}{(1-\gamma)\,p} u'\left(\frac{1-\lambda s_1}{1-\lambda} R\right) \tag{13}$$

and the bounds s_L, s_H are given by

$$s_H = \frac{p}{(1-\gamma)(1-\lambda)+\lambda p} \tag{14}$$

$$s_L = \frac{p}{(1-\lambda)/(1-\gamma)+\lambda p} \tag{15}$$

3. *Each fund share pays out, at date 2,*

$$s_2^* = \frac{1-\lambda s_1^*}{1-\lambda} R \tag{16}$$

4. *Each fund's liquidity buffer is equal to the payouts to impatient investors at date 1 ($\omega_0^* = \lambda s_1^*$).*

Notice that there is no trade in the asset market in this equilibrium. Nevertheless, the market price at which investors, and the fund, believe they would be able to trade needs to meet a certain condition for this equilibrium to exist. This is stated in Lemma 2. If the market price was lower than $1-\gamma$, all households would prefer to store their endowment at date 0 to purchase claims on the long-term technology at date 1 in the asset market (a contradiction, because there would not be any investment in the long-term technology), despite the trading cost of γ. Correspondingly, if the market price exceeded $1/(1-\gamma)$, all households would prefer to invest at date 0 in the long-term technology; if impatient, they would sell claims at date 1 to obtain more than 1 (a contradiction, because none would have any endowment to purchase those claims).

Lemma 2 *In equilibrium, the mid-market price fulfils*

$$p \in (1-\gamma, 1/(1-\gamma)) \tag{17}$$

To ease the comparison with the first best described in Lemma 1, the following corollary to proposition 1 translates the equilibrium investment contract terms (s_1^*, s_2^*) into consumption levels for the various types of households.

Corollary 1 *In equilibrium, households consume*

$$\begin{aligned} c_N &= R && (18)\\ c_A^I &= s_1^* && (19)\\ c_A^P &= s_2^* = \frac{1-\lambda s_1^*}{1-\lambda} R && (20) \end{aligned}$$

Proposition 1 illustrates how trading frictions affect the fund manager's choice for the settlement price. If the optimal settlement price is sufficiently close to the fund's NAV, trading costs deter investors from exploiting the difference between returns on the fund's share and on a direct investment in the claims on the long-term technology. The fund manager then settles net redemption requests at the unconstrained optimal solution, $\hat{s}$. It differs from the first best only because the fund manager would incur trading costs if he sold or purchased claims on the long-term technology in the asset market. To see this, notice that (13) can be written as $u'\left(c_A^I\right) = R/\left((1-\gamma)\,p\right) u'\left(c_A^P\right)$. Lemma 2 implies that $(1-\gamma)\,p \in \left[(1-\gamma)^2, 1\right]$. Differences between first best consumption, determined by $u'\left(c_A^I\right) = Ru'\left(c_A^P\right)$, and consumption in the decentralised economy when $\hat{s}$ is optimal would disappear if the trading cost was zero.

If the fund settled redemptions above s_H, the fund's shares would be so expensive that patient households with fund shares would prefer to sell those shares and to invest the proceeds in claims on the long-term technology. Similarly, if the fund settled net redemptions below s_L, the fund's shares would be so cheap that households that invested in the long-term technology would sell claims on their investment in order to subscribe to the fund's shares. The fund manager dislikes both situations: the former, because patient households redeeming fund shares unnecessarily incur trading costs; and the latter, because of the dilution of the fund's value induced by additional shares that are issued too cheaply. Lemma 6, below, shows that these flows reduce the payoffs of the fund's investors both if patient and if impatient.

On balance, the fund settles shares at the unconstrained optimal settlement price if this is sufficiently close to the fund's NAV such that trading costs deter arbitrage flows. Otherwise, it chooses the settlement price closest to the unconstrained optimal price that just keeps those arbitrage-driven flows at bay.

Trading frictions have both good and bad effects on the welfare of risk-averse households. They open an interval, $[s_L, s_H]$, in which the fund manager can set the settlement price with the aim of maximising his investors' utility without triggering arbitrage trades. At the same time, trading costs raise the marginal cost of raising the settlement price. Larger trading frictions therefore lower the unconstrained optimal settlement price, $\hat{s}$. The gap widens between risk-averse households' consumption when patient and impatient. The smaller the trading costs γ, the narrower the interval $[s_L, s_H]$. It shrinks to zero in the absence of trading costs. In this case, the fund manager settles shares at a price that ensures that his fund, and claims on the long-term technology, offer the same return ($s_L = s_H = p = 1$). Then $c_A^I = 1$, $c_A^P = R$. This is not generally equal to the first best allocation, which, as shown in Lemma 1, depends on households' utility function.

5.2 Derivation of the optimal settlement price for a given level of a fund's cash buffer

Swing pricing enables the fund manager to set the settlement price as a function of net redemptions. We therefore define net redemptions before deriving the optimal settlement price.

A fund's aggregate redemption requests are the sum of requests of impatient and patient investors. Given that the likelihood of becoming impatient is λ, and the size of the fund (the number of shares it issued at date 0, each against one unit of the endowment) is $S_{0,k}$, its impatient investors withdraw $\lambda S_{0,k}$ shares in equilibrium. We denote patient investors' net redemptions of fund k's shares by ρ_k. If $\rho_k < 0$, the fund experiences net subscriptions by patient investors; if $-\rho_k > \lambda S_{0,k}$, these net subscriptions more than offset redemptions by its impatient investors. We further denote the number of shares redeemed at which the fund's liquidity buffer ($\omega_{0,k}$) just suffices to pay out redeeming shareholders by $\hat{\rho}_k$, ie

$$\hat{\rho}_k = \omega_{0,k} S_{0,k}/s_{1,k} - \lambda S_{0,k} \tag{21}$$

In Section 5.2.1, we derive the "unconstrained" solution, $\hat{s}_k$, which is optimal if only impatient households adjust their portfolios at date 1: that is, the optimal settlement price conditional on $\rho_k = 0$. In Section 5.2.2, we derive the "no-arbitrage bounds", $s_{L,k}$ and $s_{H,k}$. These incentive constraints ensure that patient investors prefer not to alter their portfolio at date 1; hence by construction, $\rho_k = 0$ for any $s_{1,k} \in (s_{L,k}, s_{H,k})$. Lemma 6 argues that the fund manager would not choose a settlement price outside $[s_{L,k}, s_{H,k}]$. This lies behind the result (12) in proposition 1. As mentioned above, we focus on a situation in which only risk-averse households invest in the fund. We derive conditions for the existence of the equilibrium in the annex (these are the date-0 participation constraints for households that make their equilibrium investment decisions optimal) and provide an example of the equilibrium for a CRRA utility function. We omit fund subscripts (k) in the following for ease of exposition.

5.2.1 Unconstrained solution ($\hat{s}$)

$\hat{s}$ is the solution to $\max_{s_1} U$ s.t. $\rho = 0$, where U is the fund's objective function. In an equilibrium in which only risk-averse investors invest in the fund at date 0, $U = E\left[\tilde{u}_A\left(c_1, c_2\right)\right]$. The fund manager takes the size S_0 of his fund as given when maximising U. Given that each risk-averse household purchased one share at date 0, payouts to investors at date 2, s_2, are equal to the value of the fund's assets per share at date 2. These depend on net redemptions at date 1.

- If $\rho > \hat{\rho}$, then net redemption requests, $\lambda S_0 + \rho$, exceed the fund's cash buffer, $\omega_0 S_0$, leaving it short $s_1\left(\lambda S_0 + \rho\right) - \omega_0 S_0 < 0$ units of the endowment good. Because of trading costs, the fund receives not p units of the endowment good per asset sold but only $(1-\gamma)\,p$ units. Assets per share are, at date 2,
$$s_{2|\rho>\hat{\rho}} = \frac{\left(1-\omega_0\right) S_0 - \left(s_1\left(\lambda S_0 + \rho\right) - \omega_0 S_0\right) / \left(\left(1-\gamma\right) p\right)}{\left(1-\lambda\right) S_0 - \rho} R \tag{22}$$
The numerator is the fund's remaining assets after having met net redemption requests: $\left(1-\omega_0\right) S_0$ from its initial investment into the long-term technology, minus date-1 sales. The denominator is the remaining number of shares in issuance after redemption requests have been settled: the number issued at date 0, S_0, minus shares held by its first-period investors who turn out to be impatient, λS_0, minus net redemptions by patient investors, ρ.

- If, in contrast, $\rho < \hat{\rho}$, net redemptions are less than the fund's cash buffer. The fund invests spare cash (a total of $s_1 (\lambda S_0 + \rho) - \omega_0 S_0 > 0$), receiving $1 - \gamma$ claims on the long-term technology for each p units of the endowment good spent. Assets per share are, at date 2,

$$s_{2|\rho<\hat{\rho}} = \frac{(1-\omega_0) S_0 + (1-\gamma)(\omega_0 S_0 - s_1 (\lambda S_0 + \rho))/p}{(1-\lambda) S_0 - \rho} R \tag{23}$$

Notice that assets per share, and hence date-2 payouts to investors, are strictly decreasing in the settlement price if net redemptions are positive (ie if $\rho + \lambda S_0 > 0$): the more paid out per share at date 1, the fewer assets are left to back shares that are redeemed only at date 2. Correspondingly, assets per share are is strictly increasing in the settlement price if net redemptions are negative ($\rho + \lambda S_0 < 0$): the more new subscribers have to pay per share, the larger the fund's resulting assets per share.

Lemma 3 derives an expression for the "unconstrained" optimal settlement price, $\hat{s}$, ie the settlement price that is optimal conditionally on $\rho = 0$.

Lemma 3 *Conditionally on $\rho = 0$, the optimal settlement price, $\hat{s}$, fulfils*

$$\hat{s} = s_1 : \frac{u'(s_1)}{u'(s_2)} = R \cdot \begin{cases} 1/((1-\gamma)p) & \text{if} \quad \omega_0 \leq \lambda s_1 \\ (1-\gamma)/p & \text{if} \quad \omega_0 > \lambda s_1 \end{cases} \tag{24}$$

Proof. If $\rho = 0$, the fund manager's objective function can be written as

$$\begin{aligned} U &= \lambda u(c_1) + (1-\lambda) u(c_2) && (25) \\ &= \lambda u(s_1) + (1-\lambda) u(s_2) && (26) \end{aligned}$$

The solution to the fund manager's problem is given by the first-order condition

$$\frac{\partial U}{\partial s_1} = \lambda u'(s_1) + (1-\lambda) \frac{\partial s_2}{\partial s_1} u'(s_2) = 0 \tag{27}$$

where the response of the fund's date-2 payout (its assets per share at date 2) to a marginal increase in its date-1 payout (the settlement price) is

$$\frac{\partial s_2}{\partial s_1} = -\frac{\lambda}{1-\lambda} R \cdot \begin{cases} 1/((1-\gamma)p) & \text{if} \quad \omega_0 \leq \lambda s_1 \\ (1-\gamma)/p & \text{if} \quad \omega_0 > \lambda s_1 \end{cases} \tag{28}$$

Substituting (28) into (27) yields (24). ■

5.2.2 No-arbitrage bounds (s_L, s_H)

Lemma 4 describes an interval for the return of an investment in fund k's shares at date 1 within which patient investors would not alter their portfolio at date 1. Deriving the interval in terms of gross returns of such an investment, s_2/s_1, is be simpler than deriving the interval directly for the settlement price, s_1. The two expressions are equivalent because s_2 is a function of s_1.

Lemma 4 *If all funds set the same settlement price, then $\rho = 0$ if*

$$\frac{s_2}{s_1} \in \left[(1-\gamma) \frac{R}{p}, \frac{1}{1-\gamma} \frac{R}{p} \right] \tag{29}$$

Proof. If all funds set the same settlement price, households that at date 0 subscribed to fund shares have no incentive to switch funds. The payoff of a patient household that at date 0 subscribed to fund

k's share and does not change its portfolio is s_2. If it redeemed its share and invested the proceeds in the asset market, it would earn $s_1 (1-\gamma) R/p$. Thus, for the investor to remain with the fund,

$$\frac{s_2}{s_1} \geq (1-\gamma) \frac{R}{p} \tag{30}$$

The payoff of a patient household that at date 0 invested in the long-term technology and does not change its portfolio is R. If the household instead issued claims on the investment and used the proceeds, $(1-\gamma) p$, to subscribe to the shares of a fund, it would earn $(1-\gamma) ps_2/s_1$. Thus, for the investor to remain invested in the long-term technology,

$$\frac{s_2}{s_1} \leq \frac{1}{1-\gamma} \frac{R}{p} \tag{31}$$

■

Lemma 5 translates this return interval into an interval for the settlement price for the fund's shares.

Lemma 5 *If the fund sells assets at date 1 (if $\rho \geq \hat{\rho}$), (29) holds if and only if*

$$s_{1|\rho>\hat{\rho}} \in \left[\omega_0 + (1-\omega_0)(1-\gamma) p, \frac{\omega_0 + (1-\omega_0)(1-\gamma) p}{\lambda + (1-\lambda)(1-\gamma)^2 + \left(1-(1-\gamma)^2\right)\rho/S_0}\right] \tag{32}$$

If the fund purchases assets at date 1 (if $\rho < \hat{\rho}$), (29) holds if and only if

$$s_{1|\rho<\hat{\rho}} \in \left[\frac{\omega_0 + (1-\omega_0) p/(1-\gamma)}{\lambda + (1-\lambda)/(1-\gamma)^2 + \left(1-1/(1-\gamma)^2\right)\rho/S_0}, \omega_0 + (1-\omega_0)\frac{p}{1-\gamma}\right] \tag{33}$$

The proof is in the annex; it uses the fact that the fund's assets per share, s_2, are a strictly monotone function of its settlement price, and solves for the settlement prices that apply at the bounds of (29). Just as for assets per share (see (22) and (23)), a positive bid-ask spread implies a discontinuity at $\hat{\rho}$, such that the interval is defined separately for the case in which the fund sells or purchases assets in equilibrium. Notice that $s_{1|\rho>\hat{\rho}}$ exceeds the market value of a share net of transaction costs, $\omega_0 + (1-\omega_0)(1-\gamma) p$. When the fund is not constrained in its choice of the optimal settlement price by arbitrage trades (ie if $s_1^* = \hat{s}$), then it does not charge the entire cost of liquidation to investors redeeming their shares at date 1.

Lemma 6 states that the optimal settlement price falls within that interval.

Lemma 6 *In an equilibrium in which funds are symmetric, the settlement price, s_1^*, fulfils*

$$\frac{s_2^*}{s_1^*} \in \left[(1-\gamma)\frac{R}{p}, \frac{1}{(1-\gamma)}\frac{R}{p}\right] \tag{34}$$

Proof. The proof is by contradiction.

Suppose that $s_2/s_1 < (1-\gamma) R/p$ (implying s_1 is above the upper bound of (32)). Then the fund's patient investors would strictly prefer to redeem their shares because the return on fund k's shares would be less than the return on investments in the market. The fund would liquidate its assets, retrieving $(1-\gamma) p$ per unit sold. Each investor would obtain a payout of $\omega_0 + (1-\omega_0)(1-\gamma) p$ per share. As a result, each household with fund shares would consume, if impatient,

$$c_A^I = \omega_0 + (1-\omega_0)(1-\gamma) p \tag{35}$$

and

$$c_A^P = c_A^I (1-\gamma) R/p \tag{36}$$

if patient, having reinvested its payout in the market. Because s_1 reaches a minimum at $\omega_0 + (1-\omega_0)(1-\gamma) p$ when the fund sells assets (see (32)), these consumption levels are lower than what patient and impatient households, respectively, would obtain if $s_2/s_1 \geq (1-\gamma) R/p$.

Suppose instead that $s_2/s_1 > \frac{1}{1-\gamma} R/p$ (implying s_1 is below the lower bound of (33)). Then households that invested at date 0 in the long-term technology would strictly prefer to sell claims on their investment and use the proceeds to subscribe to fund shares. As a result, $\rho \to -\infty$ (the fund is small relative to the share of households that would flow in). Because s_1 is below the lower bound of (33), $c_A^I < c_A^{I*}$. In addition, patient households would also obtain less (ie $c_A^P < c_A^{P*}$) because of dilution: the fund's assets per share would fall in response to the inflows to a lower level than that implied had the settlement price been set to the lower bound of (33). To see this, notice that for all $s_{1|\rho<\hat{\rho}}$ in the interval (33),

$$\begin{aligned}
\lim_{\rho\to-\infty} s_{2|\rho<\hat{\rho}} &= \lim_{\rho\to-\infty} \frac{(1-\omega_0) S_0 + (1-\gamma)(\omega_0 S_0 - s_1(\lambda S_0 + \rho))/p}{(1-\lambda) S_0 - \rho} R && (37)\\
&< \lim_{\rho\to-\infty} \frac{(1-\omega_0) S_0 + (1-\gamma)\left(\omega_0 S_0 - s_{1|\rho<\hat{\rho}}(\lambda S_0 + \rho)\right)/p}{(1-\lambda) S_0 - \rho} R && (38)\\
&= s_{1|\rho<\hat{\rho}} (1-\gamma) R/p && (39)
\end{aligned}$$

That is, $\lim_{\rho\to-\infty} s_{2|\rho<\hat{\rho}}/s_{1|\rho<\hat{\rho}}$ is less than the lower bound of the range of returns set in equilibrium; see (34). ■

By construction, $\rho = 0$ if s_1 is in the interior of the no-arbitrage interval, $[s_L, s_H]$. In contrast, if s_1 is equal to one of the bounds of $[s_L, s_H]$, ρ is undetermined because patient investors are, by construction, indifferent. We focus on an equilibrium in which $\rho = 0$ for all $s_1 \in [s_L, s_H]$. The rule for the optimal settlement price in (12) then follows because the fund manager's objective function is concave in s_1:

$$\begin{aligned}
\frac{\partial^2 U}{(\partial s_1)^2} &= \lambda u''(s_1) + (1-\lambda)\left(\frac{\partial^2 s_2}{(\partial s_1)^2} u'(s_2) + \left(\frac{\partial s_2}{\partial s_1}\right)^2 u''(s_2)\right)\\
&= \lambda u''(s_1) + (1-\lambda)\left(\frac{\partial s_2}{\partial s_1}\right)^2 u''(s_2) < 0
\end{aligned}$$

5.3 Equilibrium cash buffer and first-period choices

In this section, we provide the intuition for our results for the equilibrium cash buffer and households' first-period choices. The proofs are in the annex.

Consider first the optimal liquidity buffer. In order to settle net redemptions at date 1, the fund can either keep a cash buffer at date 0 or sell assets at date 1. Because the date-1 market price fulfils $p \in (1-\gamma, 1/(1-\gamma))$ (see Lemma 2), the opportunity costs of investing in the cash buffer are lower than the trading costs the fund would incur if it had to sell assets at date 1 ($= 1/(1-\gamma)$). The fund chooses the cash buffer with a view to avoiding transaction costs and holds just enough cash in order to settle date-1 net redemption requests.

Now consider households' first-period choices. We are interested in an equilibrium in which risk-neutral households invest in the long-term technology at date 0 and risk-averse households invest in fund shares. Two conditions need to be met for this equilibrium to exist. The first is that the expected return of investing in the long-term technology from date 0 to date 2 is larger than that of holding fund shares over the same horizon. This ensures that risk-neutral households, all of which are patient, invest in the long-term technology. The condition holds if the fund offers investors that redeem early a return on its shares larger than that of storage, ie if $s_1^* > 1$. This correspondingly reduces the payout of investors staying with the fund until date 2, discouraging risk-neutral households from investing in the fund.[6] The second condition is that the return on investments in fund shares has sufficiently lower volatility than that of investing in the long-term technology. If so, risk-averse households prefer investing in fund shares, despite their lower expected returns. In the annex, we show that these conditions are met for plausible parameterisations of the model.

6 Properties of the optimal settlement price

In this section, we show how the equilibrium described in proposition 1 is affected if the share of liquidity-constrained investors, λ, and the trading costs, γ, rise. In Section 6.1, we assume that the new values of λ and γ are known at date 0, when the investment contract is written and the fund chooses the optimal liquidity buffer: ie we derive the comparative static properties of the equilibrium. In Section 6.2, we assume instead that the share of liquidity-constrained investors or of trading costs unanticipatedly increases at date 1, when it is too late for the fund to build its cash buffer to meet redemptions. This scenario might help understand funds' responses to sudden stress in financial markets. Finally, we ask whether swing pricing can mitigate the risk of self-fulfilling runs.

6.1 Comparative static properties

We express the properties of the optimal settlement price also in terms of the swing factor, which is given by the relative difference between the fund's net asset value and the settlement price. Entering the equilibrium values for the liquidity buffer and the settlement price into (2) yields the optimal swing factor. It is strictly declining in the optimal settlement price:

$$\begin{aligned} \sigma^* &= \frac{\omega_0^* + (1-\omega_0^*)\,p}{s_1^*} - 1 = \frac{\lambda s_1^* + (1-\lambda s_1^*)\,p}{s_1^*} - 1 \\ &= \lambda + \left(\frac{1}{s_1^*} - \lambda\right) p - 1 \end{aligned} \tag{40}$$

We define $\hat{\sigma}$ as the value of (40) when the no-arbitrage bounds do not constrain the fund manager, ie if $s_1^* = \hat{s}$ (see (13)). Lemma 7 summarises key properties of the optimal swing factor. The proofs are in the annex (Section 10.5).

Lemma 7 *(Properties of the optimal swing factor and the optimal settlement price).*

1. The optimal swing factor fulfils

$$\sigma^* \in \left[-\gamma\,(1-\lambda)\,, \frac{\gamma}{1-\gamma}\,(1-\lambda)\right] \tag{41}$$

2. Larger trading costs, γ, widen the interval (41) and raise $\hat{\sigma}$ (reduce $\hat{s}$), and lower the liquidity buffer, ω_0.

3. A greater likelihood of becoming impatient, λ, shrinks the interval (41) and raises $\hat{\sigma}$ (reduces $\hat{s}$).

The first result in Lemma 7 is the no-arbitrage interval which contains the optimal swing factor, (41). It reflects the frictions induced by trading costs. But the interval is also shaped by the share of

[6] A similar condition applies in Diamond and Dybvig (1983): deposits need to offer households that withdraw early a higher return than storage. This condition ensures that households' utility in a world in which banks offer deposit contracts is higher than in a situation in which banks do not exist, forcing households to invest directly in the asset market.

impatient investors. The larger that share, the smaller the interval. This is because, in equilibrium, only impatient investors redeem their shares. The larger their share, λ, the more responsive the fund's return to the settlement price. To see this, notice from (16) that that the fund's payout at date 2, s_2^*, is strictly declining in λ. This means that when λ is large, a small change in the settlement price leads to a substantial change in the return of fund shares from date 1 to date 2. The no-arbitrage interval (41) shrinks, reflecting that the fund manager has less scope to vary the settlement price without triggering flows of patient investors. In practice, funds often commit to a maximum swing factor.[7]

The second result is that trading frictions widen the no-arbitrage interval. If they were zero, the interval would collapse to zero. Trading frictions also raise the marginal cost to raising the settlement price. To see this, notice that a larger γ makes the first derivative of the fund's assets per share at date 2, given by (22), more negative. This translates into a stronger decline of payouts at date 2, s_2, in response to an increase in s_1. As a result, larger trading frictions induce the fund manager to lower the settlement price. He thereby reduces the degree of liquidity insurance the fund provides to its investors.

Finally, notice that the optimal liquidity buffer inherits the properties of the optimal settlement price because the fund manager chooses to hold a sufficiently large buffer to avoid trading in the market, instead funding redemptions from the buffer.

6.2 Optimal settlement price following unanticipated shocks

Some events in financial markets can best be characterised as unanticipated, in the sense that market participants have assigned only a very small, or indeed no, probability to their occurrence. For example, few investors would have thought possible the rapid increase in long-term bond yields during the 2013 "taper tantrum", and the associated large outflows from bond funds.[8] We therefore briefly consider the fund's response to unancitipated increases in the share of cash-constrained investors, λ, and an unanticipated increase in the trading cost, γ, at date 1, for a given value of the fund's cash buffer.

Lemma 8 *The optimal settlement price is declining following an unanticipated increase in the share of cash-constrained investors, λ, and an unanticipated increase in the trading cost, γ, at date 1.*

An unanticipated increase in λ leads to an increase in redemption requests. At the equilibrium settlement price, the fund would not have enough cash to accommodate the redemption requests. It cannot sell assets to raise cash because the increase in λ affects all funds in the same way, and impatient investors consume the endowment good received rather than reinvest it in the asset market. The fund therefore distributes the available cash among redeeming investors. The settlement price falls, raising the date-2 fund's assets per share. As a result, only impatient investors redeem their shares.

The response to an unanticipated increase in γ follows because the fund manager trades off the marginal costs and benefits of raising the settlement price. The marginal costs have increased following the increase in γ, while the marginal benefits (an increase in the consumption of investors that stay with the fund) has remained unchanged. This induces the fund manager to lower the settlement price. The proof is in the annex (Section 10.6).

In Lewrick and Schanz (2017), we find evidence that is consistent with these results. We compare the performance of funds that are permitted to use swing pricing with that of funds that are not, immediately following the taper tantrum. Arguably, the taper tantrum led to both an unanticipated increase in cash-constrained investors and an increase in the costs of trading. We find that the return of funds that are permitted to swing their settlement price on average falls less in response to net redemptions than that of funds constrained to set their settlement price equal to their net asset value.

[7] See ALFI (2015).

[8] In Lewrick and Schanz (forthcoming), we assess the effect of swing pricing on fund profitability and the extent of their outflows during this episode from an empirical perspective.

6.3 Can swing pricing mitigate the risk of self-fulfilling runs?

The model for which Diamond and Dybvig (1983) derived the optimal deposit contract has two equilibria: the efficient one, in which only impatient investors withdraw their funds at date 1, and an inefficient one, in which all investors withdraw and the bank's assets are liquidated at a loss in an attempt to fund those withdrawals. As mentioned in Section 2, a key difference between a deposit contract and the contract offered by an open-end investment fund that uses swing pricing is that the fund manager can fix the settlement price *after* he has collected redemption and subscription requests. In particular, the fund manager can commit to setting a sufficiently low settlement price following large net redemptions, aiming to induce patient investors not to redeem their shares early. Within the context of our model, this can make a fund that uses swing pricing immune to the risk of self-fulfilling runs.

Lemma 9 *There are (sufficiently low) settlement prices such that no patient fund investor redeems its shares at date 1, even if it believes that other patient fund investors will redeem their shares at date 1.*

Proof. Suppose each patient fund investor believes that a share of ρ/S_0 other patient fund investors will redeem their shares, and that the fund manager sets a settlement price of $\dot{s}_1$ should net redemptions be equal to $\lambda S_0 + \rho$. Then each patient investor anticipates that

- if he redeems his share, he obtains a payoff of $\dot{s}_1 (1-\gamma) R/p$ after reinvesting the redemption proceeds in the market
- if he stays with the fund, he obtains a payoff given by (22),

$$s_2(\dot{s}_1) = \frac{(1-\omega_0) S_0 - (\dot{s}_1 (\lambda S_0 + \rho) - \omega_0 S_0) / ((1-\gamma) p)}{(1-\lambda) S_0 - \rho} R$$

If $\dot{s}_1 < s_2$, each patient fund investor strictly prefers to remain invested with the fund. Positive values that satisfy $\dot{s}_1 < s_2$ exist because $s_2(0) > 0$, and $\partial s_2 / \partial s_1 < 0$. ■

In practice, the scope for swing pricing to prevent self-fulfilling runs is more limited than Lemma 9 suggests. For one, the fund manager would need to be able to correctly assess the share of investors that are liquidity constrained (λ). In our framework, the fund manager knows λ, so he can commit to lowering the settlement price whenever net redemptions exceed λ without any negative impact on investors' utility. In practice, however, λ is stochastic. In this case, any commitment to lower the settlement price when net redemptions are unusually large comes at a cost: cash-constrained fund investors would suffer from receiving only a low settlement price if λ is higher than what the fund manager believes it to be.

7 Discussion

In this section, we assess how relaxing some of the main assumptions of the model might affect our results. First, we assume that there is no aggregate uncertainty in the model. All prices, and the share of cash-constrained investors, are known. This implies that fund managers can perfectly insure against redemptions by holding a liquidity buffer that is equal to the amount paid out to redeeming investors. Nevertheless, we can provide an interesting characterisation of optimal swing pricing policies. The reason is that the marginal cost and benefits of varying the settlement price are influenced by the size of redemptions and trading frictions, even if, in equilibrium, there is no need to trade. That said, our framework may underestimate the leeway that funds have in swinging the settlement price (ie the width of the no-arbitrage interval, (34)). If the fund manager had private information about the realisation of a shock to the (aggregate) redemption requests his fund experiences, he would only ensure that no arbitrage opportunities arise *in expectation over those redemption requests other agents believe possible.*

This is evidently a less binding constraint than the one which holds in our case, where the fund manager ensures that no arbitrage opportunities arise *for a known value of redemption requests*. Whether the fund manager would exploit that additional leeway is a different question; for example, he might opt for settlement prices to be less sensitive to redemptions in order to reduce the volatility of consumption of impatient investors.

Second, by assuming that funds are small, we abstract from the influence a fund's trading may have on the market price itself. A fund that internalises its own impact on the market price would swing the settlement price more aggressively than in our solution in order to further reduce the amount it needs to trade and keep its price impact minimal. Allowing the fund to be large enough to impact the market price would therefore work in the same direction as an increase in trading costs: that is, it would likely reduce the unconstrained optimal settlement price, $\hat{s}$, and widen the no-arbitrage interval, (34).

Finally, we assume that trades and share orders are submitted simultaneously and also settle instantly. Depending on the assets the fund invests in, cut-off times for orders and settlement times may differ, and it may take time to find a counterparty in the asset market. These timing differences introduce additional frictions that, in practice, further increase the ability of fund managers to swing the settlement price.

8 Conclusion

Investors rely on financial intermediaries as a source of liquidity. Just like banks, open-end mutual funds offer this liquidity insurance. Even though they invest in comparatively illiquid securities, they typically grant their investors the right to redeem their shares on a daily basis. We assess the scope of swing pricing to manage the associated liquidity risk. Swing pricing, to be introduced in the US from late 2018 onwards, allows funds to settle investor orders at a price different from the fund's net asset value (NAV). Yet in contrast to other redemption charges, the swing pricing adjustment depends on the amount of net redemption requests.

We study swing pricing in a Diamond/Dybvig-style model with costly trading. We show that the fund manager sets a settlement price that passes on some but not all trading costs to its redeeming investors. Trading costs open a window within which a fund manager adjusts (swings) the settlement price to offer better insurance of liquidity risks than the market. The price adjustment remains in a bound which increases with trading costs and in the share of investors that seek liquidity. This result is in line with the observation that in practice, fund managers swing the settlement price only by a few percentage points around the NAV. At the same time, trading costs also justify why the fund manager would consider swinging the settlement price. Trading costs lower the marginal revenue from selling assets and raise the marginal costs of investing funds. As a result, a fund manager lowers the settlement price when redemptions or trading costs are higher, and vice versa when they are lower.

We also derive optimal settlement prices in response to unanticipated shocks. Unanticipated increases in the share of liquidity-constrained investors and in the cost of trading in the asset market both lower the optimal settlement price. The fund's NAV per share falls by less relative to a fund that does not swing its settlement price. In our companion paper, where we compare the performance of funds that are permitted to use swing pricing with that of funds that are not, we find evidence that is consistent with these results.

Finally, we show that, within our theoretical framework, swing pricing can prevent self-fulfilling runs on the fund. However, in practice, the scope for swing pricing to prevent self-fulfilling runs is more limited, primarily because the share of liquidity-constrained investors is difficult to assess.

An interesting avenue for future research is to add a richer set of shocks to the model. This would allow, for example, investigating external effects of a fund's swing pricing policy on the market.

9 References

ALFI, Association of the Luxembourg Fund Industry, 2015. Swing Pricing Update 2015. December.

Chen, Q., Goldstein, I., Jiang, W., 2010. Payoff complementarities and financial fragility: Evidence from mutual fund out flows. Journal of Financial Economics 97, 239-262.

Diamond, D.W., Dybvig, P.H., 1983. Bank runs, deposit insurance, and liquidity. Journal of Political Economy 91 (3), 401-419.

Financial Stability Board, 2017. Policy recommendations to address structural vulnerabilities from asset management activities.

Goldstein, I., Jiang, H., Ng, D. T., forthcoming. Investor Flows and Fragility in Corporate Bond Funds. Journal of Financial Economics.

Jacklin, C., 1987. Demand deposits, trading restrictions, and risk sharing. In: Prescott, E.C., Wallace, N. (Eds.), Contractual Arrangements for Intertemporal Trade. University of Minnesota Press, Minneapolis, MN, 26-47.

Lewrick, U., Schanz, J., 2017. Is the price right? Swing pricing and investor redemption. Bank for International Settlements Working Paper 664.

Madhavan, A., 2000. Market microstructure: A survey. Journal of Financial Markets 3, 205-258.

Malik, S., Lindner, P., 2017. On Swing Pricing and Systemic Risk Mitigation, IMF WorkinPaper, WP/17/159.

Sandmo, A., 1975. Optimal Taxation in the Presence of Externalities. The Swedish Journal of Economics 77 (1), 86-98.

Securities and Exchange Commission, 2016. Investment Company Swing Pricing, RIN 3235-AL61.

von Thadden, E.-L., 1998. Intermediated versus direct investment: Optimal liquidity provision and dynamic incentive compatibility. Journal of Financial Intermediation 7, 177-197.

von Thadden, E.-L., 1999. Liquidity creation through banks and markets: Multiple insurance and limited market access. European Economic Review 43 991-1006.

Zheng, Y., 2017. A dynamic theory of mutual fund runs and liquidity management. European Systemic Risk Board Working Paper Series no 42.

10 Annex

10.1 Notation

ρ_k net redemptions by liquidity-unconstrained households invested in fund k. A negative value means that the fund experiences net subscriptions.

$s_{1,k}$ price at which fund k settles share subscriptions at date 1.

$s_{2,k}$ fund k's payouts per share in the final period

σ_k fund k's swing factor

$\omega_{0,k}$ share of fund k's assets invested at date 0 in storage (the fund's liquid asset buffer)

p market price of claims on the long-term technology at date 1

γ trading cost. Sellers receive $(1-\gamma)\,p$ per unit sold; buyers receive $1-\gamma$ units of the asset per p units of the endowment good.

$S_{0,k}$ number of shares fund k issued at date 0 (equal to the units of the endowment good k collects at date 0)

c_A^I consumption of impatient risk-averse households (which, in equilibrium, invest in the fund)

c_A^P consumption of patient risk-averse households (which, in equilibrium, invest in the fund)

c_N consumption of risk-neutral households (which, in equilibrium, invest in the long-term technology)

μ share of risk-neutral households among all households
λ likelihood of becoming impatient (of having to consume at date 1)
u_A utility function of risk-averse households
u_N utility of risk-neutral households
ρ_k fund k's redemption requests from patient households at date 1

10.2 The no-arbitrage interval in terms of the settlement price (proof of Lemma 5)

Proof. The proof proceeds by equating the returns of date-1 investments in fund shares with those of date-1 investments in claims on the long-term technology. Because the fund's return is a function of its assets per share, whose expression depends on whether the fund purchases or sells assets, the derivations generate two no-arbitrage intervals, one for the case in which the fund sells assets, $\left[s_{L|\rho>\hat{\rho}}, s_{H|\rho>\hat{\rho}}\right]$, and one for the case in which it purchases assets at date 1, $\left[s_{L|\rho<\hat{\rho}}, s_{H|\rho<\hat{\rho}}\right]$.

1. Settlement prices at the lower bound for the fund's return (an upper bound to $s_{1,k}$). At this bound, the returns of both date-1 investments are equal if $s_{2,k}/s_{1,k} = (1-\gamma)\,R/p$ (see Lemma 4). We omit in the following the subscipt k that indicates the fund.

   - If $\rho > \hat{\rho}$, then the returns of the two date-1 investments are equal if $s_1 = s_{H|\rho>\hat{\rho}}$, where $s_{H|\rho>\hat{\rho}}$ is given by

$$s_{H|\rho>\hat{\rho}} = \frac{(1-\omega_0)\,S_0 - \left(s_{H|\rho>\hat{\rho}}\,(\lambda S_0+\rho) - \omega_0 S_0\right)/\left((1-\gamma)\,p\right)}{(1-\lambda)\,S_0-\rho} R \frac{p}{(1-\gamma)\,R} \tag{42}$$

$$= \frac{\omega_0 + (1-\omega_0)\,(1-\gamma)\,p}{\lambda + (1-\lambda)\,(1-\gamma)^2 + \left(1-(1-\gamma)^2\right)\rho/S_0} \tag{43}$$

   where the second line follows after solving (42) for $s_{H|\rho>\hat{\rho}}$.

   - If $\rho < \hat{\rho}$, then the returns of the two date-1 investments are equal if $s_1 = s_{H|\rho<\hat{\rho}}$, where $s_{H|\rho<\hat{\rho}}$ is given by

$$s_{H|\rho<\hat{\rho}} = \frac{(1-\omega_0)\,S_0 - (1-\gamma)\left(s_{H|\rho<\hat{\rho}}\,(\lambda S_0+\rho) - \omega_0 S_0\right)/p}{(1-\lambda)\,S_0-\rho} R \frac{p}{(1-\gamma)\,R} \tag{44}$$

$$= \omega_0 + (1-\omega_0)\,\frac{p}{1-\gamma} \tag{45}$$

   $s_{H|\rho>\hat{\rho}}$ is strictly decreasing in patient investors' net redemptions, ρ, and reaches a maximum for the lowest ρ at which the fund would sell assets in response to a marginal increase in $s_{H|\rho>\hat{\rho}}$ ($\rho = \hat{\rho}$). Entering this value of ρ into (42) yields the maximum of $s_{H|\rho>\hat{\rho}}$,

$$\max_{\rho} s_{H|\rho>\hat{\rho}} = \omega_0 + (1-\omega_0)\,\frac{p}{1-\gamma} \tag{46}$$

   Correspondingly, $s_{H|\rho>\hat{\rho}}$ reaches its minimum when net redemptions are maximal ($\rho = (1-\lambda)\,S_0$) of

$$\min_{\rho} s_{H|\rho>\hat{\rho}} = \omega_0 + (1-\omega_0)\,(1-\gamma)\,p \tag{47}$$

2. Settlement prices at the upper bound for the fund's return (a lower bound to s_1). At this bound, the returns of both date-1 investments are equal if $s_{2,k}/s_{1,k} = R/\left((1-\gamma)\,p\right)$ (see Lemma 4). We omit in the following the subscipt k.

- If $\rho > \hat{\rho}$, then the returns of both date-1 investments are equal if $s_1 = s_{L|\rho>\hat{\rho}}$, where $s_{L|\rho>\hat{\rho}}$ is given by

$$
\begin{aligned}
s_{L|\rho>\hat{\rho}} &= \frac{(1-\omega_0) S_0 - \left(s_{L|\rho>\hat{\rho}} (\lambda S_0 + \rho) - \omega_0 S_0\right) / \left((1-\gamma) p\right)}{(1-\lambda) S_0 - \rho} R \frac{(1-\gamma) p}{R} && (48)\\
&= \omega_0 + (1-\omega_0)(1-\gamma) p && (49)
\end{aligned}
$$

where the second line follows after solving (42) for $s_{L|\rho>\hat{\rho}}$.

- If $\rho < \hat{\rho}$, then the returns of both date-1 investments are equal if $s_1 = s_{L|\rho<\hat{\rho}}$, where $s_{L|\rho<\hat{\rho}}$ is given by

$$
\begin{aligned}
s_{L|\rho<\hat{\rho}} &= \frac{(1-\omega_0) S_0 - (1-\gamma)\left(s_{L|\rho<\hat{\rho}} (\lambda S_0 + \rho) - \omega_0 S_0\right) / p}{(1-\lambda) S_0 - \rho} R \frac{(1-\gamma) p}{R} && (50)\\
&= \frac{\omega_0 + (1-\omega_0) p / (1-\gamma)}{\lambda + (1-\lambda) / (1-\gamma)^2 + \left(1 - 1/(1-\gamma)^2\right) \rho / S_0} && (51)
\end{aligned}
$$

$s_{L|\rho<\hat{\rho}}$ is strictly decreasing in patient investors' net redemptions (strictly increasing in asset purchases), ρ, and reaches a minimum for the smallest ρat which the fund would just purchase assets ($\rho = \hat{\rho}$) of

$$
\min_{\rho} s_{L|\rho<\hat{\rho}} = \omega_0 + (1-\omega_0)(1-\gamma) p \tag{52}
$$

The interval (32) is then constructed using the bounds (49) and (43), while the interval (33) is the result of combining (51) and (45). ■

10.3 Optimal liquidity buffer

Lemma 10 *The fund manager chooses a liquidity buffer equal to equilibrium redemptions:*

$$
\omega_0^* = \lambda s_1^* \tag{53}
$$

Proof. Fund manager k solves $\max_{\omega_{0,k}} U$ where, using the fact that impatient households invested in the fund in equilibrium consume $c^I_{A,k} = s^*_{1,k}$ and patient households $c^P_{A,k} = s^*_{2,k}$,

$$
U = \lambda u\left(s^*_{1,k}\right) + (1-\lambda) u\left(s^*_{2,k}\right) \tag{54}
$$

The proof proceeds by evaluating the first derivatives to the fund's problem. (We omit in the following the subscipt k.) The optimal settlement price s_1^* is defined by intervals (see 12), so we need to compute the first-order conditions separately for each interval, depending on whether s_1^* is at the upper bound of the no-arbitrage interval, at the lower bound, or in the interior. For each interval, we need to consider the case in which the fund sells assets separately from that in which the fund purchases assets. The reason is that the fund's assets per share at date 2 (equal to s_2^*) are defined separately for each case, reflecting the discontinuity of s_2^* introduced by the bid-ask spread.

1. Suppose $\hat{s} < s_H$, where $\hat{s}$ is the "unconstrained" optimal solution for the settlement price given by (24). Then $s_1^* = s_H$. If $s_1^* = s_H$, then $s_2^* = s_1^* (1-\gamma) R/p$ and the fund manager's first-order condition is

   - if $\rho > \hat{\rho}$, such that the fund would sell assets to meet redemption requests, the first derivative is, using the definition of $s_{H|\rho>\hat{\rho}}$ in (43),

$$\frac{\partial U}{\partial \omega_0} = \left(\lambda u'\left(s_{H|\rho>\hat{\rho}}\right) + (1-\lambda)\frac{(1-\gamma)R}{p}u'\left(s_{H|\rho>\hat{\rho}}\frac{(1-\gamma)R}{p}\right)\right)\frac{\partial s_{H|\rho>\hat{\rho}}}{\partial \omega_0} > 0 \tag{55}$$

The sign of $\partial s_{H|\rho>\hat{\rho}}/\partial\omega_0$ follows from $p < 1/(1-\gamma)$. That is, if, for a given ω_0, the fund would sell assets, the fund manager strictly prefers to raise the liquid asset buffer.

- if $\rho < \hat{\rho}$, the first derivative is, using the definition of $s_{H|\rho<\hat{\rho}}$ in (45),

$$\frac{\partial U}{\partial \omega_0} = \left(\lambda u'\left(s_{H|\rho<\hat{\rho}}\right) + (1-\lambda)\frac{(1-\gamma)R}{p}u'\left(s_{H|\rho<\hat{\rho}}\frac{(1-\gamma)R}{p}\right)\right)\frac{\partial s_{H|\rho<\hat{\rho}}}{\partial \omega_0} < 0 \tag{56}$$

The sign of $\partial s_{H|\rho<\hat{\rho}}/\partial\omega_0$ follows from $p > 1-\gamma$. That is if, for a given ω_0, the fund has spare liquidity after servicing net redemptions, the fund manager strictly prefers to lower the liquid asset buffer.

As a result, if $s_1^* = s_H$, then $\omega_0^* = \lambda s_H$.

2. If $s_1^* = s_L$, then $s_2^* = s_1^*(1-\gamma)\frac{R}{(1-\gamma)p}$ and the fund manager's first-order condition is

- if $\rho > \hat{\rho}$, such that the fund would sell assets to meet redemption requests, the first derivative is, using the definition of $s_{L|\rho>\hat{\rho}}$ in (49),

$$\frac{\partial U}{\partial \omega_0} = \left(\lambda u'\left(s_{L|\rho>\hat{\rho}}\right) + (1-\lambda)\frac{R}{(1-\gamma)p}u'\left(s_{L|\rho>\hat{\rho}}\frac{R}{(1-\gamma)p}\right)\right)\frac{\partial s_{L|\rho>\hat{\rho}}}{\partial \omega_0} > 0 \tag{57}$$

The sign of $\partial s_{L|\rho>\hat{\rho}}/\partial\omega_0$ follows from $p < 1/(1-\gamma)$. That is, if, for a given ω_0, the fund would sell assets, the fund manager strictly prefers to raise the liquid asset buffer.

- if $\rho < \hat{\rho}$, the first derivative is, using the definition of $s_{L|\rho<\hat{\rho}}$ in (49),

$$\frac{\partial U}{\partial \omega_0} = \left(\lambda u'\left(s_{L|\rho<\hat{\rho}}\right) + (1-\lambda)\frac{(1-\gamma)R}{p}u'\left(s_{L|\rho<\hat{\rho}}\frac{(1-\gamma)R}{p}\right)\right)\frac{\partial s_{L|\rho<\hat{\rho}}}{\partial \omega_0} < 0 \tag{58}$$

The sign of $\partial s_{H|\rho<\hat{\rho}}/\partial\omega_0$ follows from $p > 1-\gamma$. That is if, for a given ω_0, the fund has spare liquidity after servicing net redemptions, the fund manager strictly prefers to lower the liquid asset buffer.

As a result, if $s_1^* = s_L$, then $\omega_0^* = s_L\lambda$.

3. Suppose that $s_1^* \in (s_L, s_H)$.

- if $\rho > \hat{\rho}$, equivalently $\omega_0 < s_1\lambda$, the fund needs to sell assets in response to a marginal increase in the settlement price, and $s_2 = s_{2|\rho>\hat{\rho}}$. The response of $s_{2|\rho>\hat{\rho}}$, defined in (22), to an increase in the liquidity buffer is

$$\begin{aligned}
\frac{ds_{2|\rho>\hat{\rho}}}{d\omega_0} &= \frac{\partial}{\partial\omega_0}\left(\frac{(1-\omega_0)S_0 - (1-\gamma)\left(s_{1|\rho>\hat{\rho}}\lambda S_0 - \omega_0 S_0\right)/p}{(1-\lambda)S_0}R\right) \\
&= \frac{1}{(1-\lambda)S_0}\left(-S_0 - \left(\lambda S_0\frac{ds_{1|\rho>\hat{\rho}}}{d\omega_0} - S_0\right)(1-\gamma)/p\right)R \\
&= -\frac{1-\gamma}{(1-\lambda)p}\left((p/(1-\gamma)-1) + \lambda\frac{ds_{1|\rho>\hat{\rho}}}{d\omega_0}\right)R
\end{aligned} \tag{59}$$

Entering this into the first derivative of the fund's objective function yields

$$\begin{aligned}
&\lambda u'\left(s_{1|\rho>\hat{\rho}}\right)\frac{ds_{1|\rho>\hat{\rho}}}{d\omega_0}+(1-\lambda)\frac{ds_{2|\rho>\hat{\rho}}}{d\omega_0}u'\left(s_{2|\rho>\hat{\rho}}\right) &(60)\\
=\;&\lambda u'\left(s_{1|\rho>\hat{\rho}}\right)\frac{ds_{1|\rho>\hat{\rho}}}{d\omega_0}+(1-\lambda)\frac{1-\gamma}{(1-\lambda)p}\left((1-p/(1-\gamma))-\lambda\frac{ds_{1|\rho>\hat{\rho}}}{d\omega_0}\right)Ru'\left(s_{2|\rho>\hat{\rho}}\right)\\
=\;&\lambda\left(u'\left(s_{1|\rho>\hat{\rho}}\right)-\frac{1-\gamma}{p}Ru'\left(s_{2|\rho>\hat{\rho}}\right)\right)\frac{ds_{1|\rho>\hat{\rho}}}{d\omega_0}-(p-(1-\gamma))Ru'\left(s_{2|\rho>\hat{\rho}}\right)\\
=\;&-(p-(1-\gamma))Ru'\left(s_{2|\rho>\hat{\rho}}\right)<0
\end{aligned}$$

The last equality follows because the fund manager optimally chooses the settlement price, hence $u'\left(s_{1|\rho>\hat{\rho}}\right)-\frac{1-\gamma}{p}Ru'\left(s_{2|\rho>\hat{\rho}}\right)=0$ (see (13) in proposition 1). The sign follows from $p>1-\gamma$.

- if $\rho<\hat{\rho}$, equivalently $\omega_0>s_1\lambda$, then $s_{2|\rho<\hat{\rho}}$, defined in (23). Its response to an increase in the liquidity buffer is

$$\begin{aligned}
\frac{ds_{2|\rho<\hat{\rho}}}{d\omega_0} &= \frac{\partial}{\partial\omega_0}\left(\frac{(1-\omega_0)S_0-\left(s_{1|\rho<\hat{\rho}}\lambda S_0-\omega_0 S_0\right)/((1-\gamma)p)}{(1-\lambda)S_0}R\right) &(61)\\
&= \frac{1}{(1-\lambda)S_0}\left(-S_0-\left(\lambda S_0\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}-S_0\right)/((1-\gamma)p)\right)R\\
&= \frac{1}{1-\lambda}\left(-1-\left(\lambda\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}-1\right)/((1-\gamma)p)\right)R\\
&= -\frac{1}{(1-\lambda)((1-\gamma)p)}\left(-(1-(1-\gamma)p)+\lambda\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}\right)R
\end{aligned}$$

Entering this into the first derivative of the fund's objective function yields

$$\begin{aligned}
&\lambda u'\left(s_{1|\rho<\hat{\rho}}\right)\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}+(1-\lambda)\frac{ds_{2|\rho<\hat{\rho}}}{d\omega_0}u'\left(s_{2|\rho<\hat{\rho}}\right) &(62)\\
=\;&\lambda u'\left(s_{1|\rho<\hat{\rho}}\right)\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}-(1-\lambda)\frac{1}{(1-\lambda)((1-\gamma)p)}\left(-(1-(1-\gamma)p)+\lambda\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}\right)Ru'\left(s_{2|\rho<\hat{\rho}}\right)\\
=\;&\lambda\left(u'\left(s_{1|\rho<\hat{\rho}}\right)-\frac{1}{(1-\gamma)p}Ru'\left(s_{2|\rho<\hat{\rho}}\right)\right)\frac{ds_{1|\rho<\hat{\rho}}}{d\omega_0}+(1-(1-\gamma)p)Ru'\left(s_{2|\rho<\hat{\rho}}\right)\\
=\;&(1-(1-\gamma)p)Ru'\left(s_{2|\rho<\hat{\rho}}\right)>0
\end{aligned}$$

The last equality follows because the fund manager optimally chooses the settlement price, hence $u'\left(s_{1|\rho<\hat{\rho}}\right)-\frac{1}{(1-\gamma)p}Ru'\left(s_{2|\rho<\hat{\rho}}\right)=0$. The sign follows from $p<1/(1-\gamma)$.

■

10.4 First-period choices and existence of equilibrium

Lemma 11 *A necessary condition for the existence of the equilibrium is* $s_1^*\geq 1$

Proof. Suppose $s_1^*<1$. A risk-averse household's equilibrium utility is

$$\lambda u\left(s_1^*\right)+(1-\lambda)u\left(s_2^*\right)$$

If it deviated to storage at date 0 and invested in fund shares at date 1, its utility would be strictly higher at $\lambda u(1)+(1-\lambda)u\left(s_2^*/s_1^*\right)$: a contradiction to $s_1^*<1$ in an equilibrium in which all risk-averse households subscribe to fund shares at date 0. ■

Lemma 12 shows that $p_S^* \geq 1$ is also a sufficient condition for risk-neutral households to prefer investing in the long-term technology.

Lemma 12 *At date 0, a risk-neutral household prefers to invest in the long-term technology if* $s_1^* > 1$.

Proof. $s_1^* \geq 1$ implies that the risk-neutral household prefers investing in the fund over storage. By construction of s_1^*, if invested in the fund at date 0, it prefers to remain with the fund at date 1 to earn s_2^*. Because $s_1^* \geq 1$, this would be less than its equilibrium payoff, R (see (16)). ■

In contrast, $s_1^* \geq 1$ is not sufficient for risk-averse households to prefer investing in fund shares. A risk-averse household that deviates to investing in the long-term technology earns $(1-\gamma)\,p < s_1^*$ if impatient and $R > s_2^*$ if patient. If $s_1^* \geq 1$, the variance of its equilibrium payoff is therefore smaller than the variance of this deviation payoff. At the same time, the expected equilibrium payoff from investing in fund shares is smaller than the payoff earned from deviating to investing in the long-term technology. To see this, notice that his expected payoff is

$$\begin{aligned}\lambda s_1^* + (1-\lambda)\,s_2^* &= \lambda s_1^* + (1-\lambda)\,\frac{1-\lambda s_1^*}{1-\lambda}R \\ &= \lambda s_1^* + (1-\lambda s_1^*)\,R\end{aligned}$$

where the first equality follows from inserting the equilibrium expression for s_2^*, given by (16). The expected payoff of investing in the long-term technology is

$$\lambda\,(1-\gamma)\,p + (1-\lambda)\,R$$

The expected payoff from investing in shares is smaller if

$$\lambda\,(s_1^* - (1-\gamma)\,p) + ((1-\lambda s_1^*) - (1-\lambda))\,R \leq 0$$

equivalently,

$$s_1^* \geq \frac{R-(1-\gamma)\,p}{R-1}$$

ie whenever $s_1^* \geq 1$. This suggests that for a sufficiently large degree of risk aversion, a risk-averse household will not deviate to investing into the long-term technology at date 0. Lemma 13 shows how the no-deviation condition

$$\lambda u\,(s_1^*) + (1-\lambda)\,u\left(\frac{1-\lambda s_1^*}{1-\lambda}R\right) \geq \lambda u\,((1-\gamma)\,p) + (1-\lambda)\,u\,(R) \tag{63}$$

translates into a condition on households' risk aversion in the context of a CRRA utility function and shows existence for a few plausible parameter calibrations.

Lemma 13 *Suppose* $u\,(c) = c^{1-a}/\,(1-a)$.

1. *The optimal settlement price is*

$$s_1^* = \begin{cases} \frac{p}{(1-\gamma)(1-\lambda)+\lambda p} & \text{if} \quad a \geq \ln\frac{R}{p(1-\gamma)}/\ln R\frac{(1-\gamma)}{p} \\ R/\left(\lambda R + (1-\lambda)\left(\frac{R}{p(1-\gamma)}\right)^{\frac{1}{a}}\right) & \text{if} \quad a \in \left(1, \ln\frac{R}{p(1-\gamma)}/\ln R\frac{(1-\gamma)}{p}\right) \\ \frac{p}{(1-\lambda)/(1-\gamma)+\lambda p} & \text{if} \quad a \leq 1 \end{cases} \tag{64}$$

2. $s_1^* > 1$ *if and only if* $a > \ln\left(\frac{R}{p(1-\gamma)}\right)/\ln R$, *ie if households are sufficiently risk averse.*

Proof. We first derive expression (64) for the optimal settlement price and then consider the case that it lies in the interior region.

1. Using CRRA utility, the optimality condition (13) specialises to

$$\hat{s} = s_1 : (s_1)^{-a} = \frac{R}{(1-\gamma)\,p}\left(\frac{1-\lambda s_1}{1-\lambda}R\right)^{-a}$$

   Re-arranging terms yields

$$\frac{1-\lambda s_1}{s_1\,(1-\lambda)}R = \left(\frac{R}{(1-\gamma)\,p}\right)^{1/a}$$

   Solving for s_1 then yields $\hat{s}$, which is equal to the optimal settlement price in the middle interval of (64). Notice that, in equilibrium,

$$\frac{s_2}{s_1} = \frac{1-\lambda s_1}{s_1\,(1-\lambda)}R > 1$$

   because

$$\left(\frac{R}{(1-\gamma)\,p}\right)^{1/a} > 1$$

   equivalently,

$$R > (1-\gamma)\,p$$

2. Solving $s_1 = s_H$ and $s_1 = s_L$ yields the bounds to a for the different intervals of (64). $\hat{s} \leq s_H$ is equivalent to

$$R/\left(\lambda R + (1-\lambda)\left(\frac{R}{p\,(1-\gamma)}\right)^{\frac{1}{a}}\right) \leq \frac{p}{(1-\gamma)\,(1-\lambda)+\lambda p}$$

   Taking inverses and re-arranging terms yields

$$\begin{aligned} \lambda R + \lambda R\,(1-\lambda)\left(\frac{R}{p\,(1-\gamma)}\right)^{\frac{1}{a}} &\geq \frac{R\,((1-\gamma)\,(1-\lambda)+\lambda p)}{p} \\ \left(\frac{R}{p\,(1-\gamma)}\right)^{\frac{1}{a}} &\geq R\frac{(1-\gamma)}{p} \end{aligned}$$

   Taking logs then yields the optimal settlement price in the top interval of (64). The proof for the optimal settlement price in the bottom interval of (64) is analogous.

3. Existence: we only show here that there are plausible parameter values such that $p_S^* > 1$ and that risk-averse households prefer subscribing to fund shares at date 0 instead of investing in the long-term technology, ie that the no-deviation condition (63) holds. For example, for a return of the long-term technology of $R = 1.1$, a mid-market price of $p = 1$, a coefficient of relative risk aversion of $a = 2$, trading costs of $\gamma = 0.05$, and a share of liquidity-constrained fund investors of $\lambda = 0.1$,

$$\begin{aligned} \hat{s} &= R/\left(\lambda R + (1-\lambda)\left(\frac{R}{p\,(1-\gamma)}\right)^{\frac{1}{a}}\right) = 1.02 > 1 \\ s_H &= \frac{p}{(1-\gamma)\,(1-\lambda)+\lambda p} = 1.047 > \hat{s} \end{aligned}$$

   such that the optimal settlement price is given by the unconstrained solution, $s_1^* = \hat{s}$. Risk-averse households' expected utilities are, if they subscribe at date 0 to fund shares,

$$\frac{1}{1-a}\left(\lambda\left(s_1^*\right)^{1-a}+(1-\lambda)\left(\frac{1-\lambda s_1^*}{1-\lambda}R\right)^{1-a}\right)=-0.918\,04$$

greater than what they would earn if they invest at date 0 in the long-term technology,

$$\frac{1}{1-a}\left(\lambda\left((1-\gamma)\,p\right)^{1-a}+(1-\lambda)\,R^{1-a}\right)=-0.923\,44$$

Another parameter combination for which the same type of equilibrium exists is: $R=1.05$, $p=1$, $a=2$, $\gamma=0.03$, and $\lambda=0.05$. For larger degrees of risk aversion, an equilibrium exists in which the settlement price is at the upper bound of the no-arbitrage interval, $s_1^*=s_H$.

■

10.5 Properties of the equilibrium settlement price and the swing factor

10.5.1 Payouts to impatient vs payouts to patient investors

If $s_1^*=\hat{s}$, (13) states that

$$\frac{u'\left(s_1\right)}{u'\left(s_2\right)}=\frac{R}{(1-\gamma)\,p} \tag{65}$$

Then $u''<0$ implies $s_1<s_2$ because $R>(1-\gamma)\,p$.

10.5.2 Bounds for the optimal swing factor

Inserting the corresponding bounds for the settlement price, s_L and s_H, into the expression for the swing factor, (2), yields

$$\begin{aligned}\sigma_L &= \sigma^*\left(s_H\right)=\lambda+\left(\frac{1}{\frac{p}{(1-\gamma)(1-\lambda)+\lambda p}}-\lambda\right)p-1=-\gamma\,(1-\lambda)\\ \sigma_H &= \sigma^*\left(\sigma_L\right)=\lambda+\left(\frac{1}{\frac{p}{(1-\lambda)/(1-\gamma)+\lambda p}}-\lambda\right)p-1=\frac{\gamma}{1-\gamma}\,(1-\lambda)\end{aligned}$$

For illustration, using the same calibration as in the proof of Lemma 13, $\gamma=0.05$ and $\lambda=0.1$, we have $\sigma^*\in[-.045,+0.047]$.

10.5.3 Larger trading costs

The response of the bounds of the no-arbitrage intervals to larger trading costs ($\partial s_H/\partial\gamma>0$, $\partial s_L/\partial\gamma<0$) can be seen directly by inspecting (14) and (15). If $\gamma=0$, both collapse to a single value, the fund's net asset value per share, $\omega_0+(1-\omega_0)\,(1-\gamma)\,p$.

The response of $\hat{s}$ can be derived by totally differentiating the fund manager's first-order condition, (13). Denote (13) by G. Using the equilibrium value of s_2^*, given by (16)

$$G=(1-\gamma)\,pu'\left(s_1^*\right)-Ru'\left(\frac{1-\lambda s_1^*}{1-\lambda}R\right)=0 \tag{66}$$

Then

$$\frac{d\hat{s}}{d\gamma}=-\frac{\partial G}{\partial\gamma}/\frac{\partial G}{\partial s_1}<0 \tag{67}$$

because

$$\frac{\partial G}{\partial s_1^*} = (1-\gamma)\, pu''(s_1^*) + R\frac{\lambda}{1-\lambda} u''(s_2^*) < 0 \tag{68}$$

and

$$\frac{\partial G}{\partial \gamma} = -pu'(s_1^*) < 0 \tag{69}$$

10.5.4 A greater likelihood of becoming impatient

The response of the bounds of the no-arbitrage intervals to a greater likelihood of becoming impatient, λ, is

$$\frac{\partial s_H}{\partial \lambda} = \frac{\partial \left(\frac{p}{(1-\gamma)(1-\lambda)+\lambda p}\right)}{\partial \lambda} = -p\frac{p-(1-\gamma)}{((1-\gamma)(1-\lambda)+\lambda p)^2} < 0 \tag{70}$$

and

$$\frac{\partial s_L}{\partial \lambda} = \frac{\partial \left(\frac{p}{(1-\lambda)/(1-\gamma)+\lambda p}\right)}{\partial \lambda} = \frac{p}{(1-\gamma)} \frac{1-(1-\gamma)\,p}{((1-\lambda)/(1-\gamma)+\lambda p)^2} > 0 \tag{71}$$

The response of $\hat{s}$ can be derived by totally differentiating the fund manager's first order condition, (66), as above, using

$$\frac{\partial G}{\partial \lambda} = -R\left(-R\frac{p_S^*-1}{(1-\lambda)^2}\right) u''\left(c_A^P\right) < 0 \tag{72}$$

10.6 Response to an unanticipated increase in trading frictions

The unanticipated increase in trading frictions is public information at the start of date 1. The fund manager maximises

$$\begin{aligned} U &= E\left[u_A(c_1, c_2)\right] && (73)\\ &= \lambda u(c_1) + (1-\lambda)\, u(c_2) && (74)\\ &= \lambda u(s_1) + (1-\lambda)\, u(s_2) && (75) \end{aligned}$$

where the last line follows under the assumption that patient investors remain with the fund (we show below that this is the case). The optimal settlement price following the unanticipated change in γ is given by the first-order constraint

$$\frac{\partial U}{\partial s_1} = \lambda u'(s_1) + (1-\lambda)\frac{\partial s_2}{\partial s_1} u'(s_2) = 0$$

Assets per share at date 2 are

- if $\lambda s_1 \geq \omega_0^*$, the fund sells assets in response to a marginally increasing settlement price, and s_2 is given by (22) evaluated at $\rho = 0$. Then

$$\begin{aligned} s_{2|\lambda s_1 \geq \omega_0^*} &= \frac{(1-\omega_0^*)\, S_0 - (s_1 \lambda S_0 - \omega_0^* S_0)/((1-\gamma)\,p)}{(1-\lambda)\, S_0} R \\ \frac{\partial s_{2|\lambda s_1 \geq \omega_0^*}}{\partial s_1} &= \frac{-\lambda S_0/((1-\gamma)\,p)}{(1-\lambda)\, S_0} R = -\frac{R\lambda}{p(1-\lambda)(1-\gamma)} \end{aligned}$$

- if $\lambda s_1 < \omega_0^*$, the fund purchases fewer assets in response to a marginally increasing settlement price, and s_2 is given by (23) evaluated at $\rho = 0$. Then

$$\begin{aligned} s_{2|\lambda s_1 < \omega_0^*} &= \frac{(1-\omega_0^*) S_0 - (1-\gamma)(s_1 \lambda S_0 - \omega_0^* S_0)/p}{(1-\lambda) S_0} R \\ \frac{\partial s_{2|\lambda s_1 < \omega_0^*}}{\partial s_1} &= -\frac{R\lambda (1-\gamma)}{p(1-\lambda)} \end{aligned}$$

1. Suppose $\lambda s_1 \geq \omega_0^*$. Then the solution to the fund manager's first-order condition is $\check{s}_1$, where

$$\check{s}_1 = s_1 : \lambda u'(s_1) + (1-\lambda)\left(-\frac{R\lambda}{p(1-\lambda)(1-\gamma)}\right) u'(s_2) = 0$$

   Let

$$G = u'(s_1) - \frac{R}{p(1-\gamma)} u'(s_2) = 0$$

   Then

$$\frac{d\check{s}_1}{d\gamma} = -\frac{\partial G}{\partial \gamma} / \frac{\partial G}{\partial s_1}$$

   The partial derivates are:

$$\begin{aligned} \frac{\partial G}{\partial s_1} &= u''(s_1) - \frac{R}{p(1-\gamma)} \frac{\partial s_2}{\partial s_1} u''(s_2) \\ &= u''(s_1) - \frac{R}{p(1-\gamma)}\left(-\frac{R\lambda}{p(1-\lambda)(1-\gamma)}\right) u''(s_2) < 0 \end{aligned}$$

   and

$$\frac{\partial G}{\partial \gamma} = -\left(\frac{R}{p(1-\gamma)^2} u'(s_2) + \frac{R}{(1-\gamma)p} \frac{\partial s_2}{\partial \gamma} u''(s_2)\right)$$

   where

$$\frac{\partial s_2}{\partial \gamma} = \frac{\partial \left(\frac{(1-\omega_0^*)S_0 - (s_1\lambda S_0 - \omega_0^* S_0)/((1-\gamma)p)}{(1-\lambda)S_0} R\right)}{\partial \gamma} = R\frac{\omega_0^* - \lambda s_1}{p(1-\lambda)(1-\gamma)^2} \leq 0$$

   where the sign follows because $\lambda s_1 \geq \omega_0^*$. Then $\partial G/\partial \gamma < 0$ and $d\check{s}_1/d\gamma < 0$: a contradiction to $\lambda s_1 \geq \omega_0^*$.

2. Suppose instead that $\lambda s_1 < \omega_0^*$ and that $\check{s}_1 \geq s_L$, where s_L is given by (15). Then the solution to the fund manager's first-order condition is $\check{s}_1$, where

$$\check{s}_1 = s_1 : \lambda u'(s_1) + (1-\lambda)\left(-\frac{R\lambda(1-\gamma)}{p(1-\lambda)}\right) u'(s_2) = 0$$

   Let

$$G = u'(s_1) - \frac{(1-\gamma)R}{p} u'(s_2) = 0$$

   Then

$$\frac{d\check{s}_1}{d\gamma} = -\frac{\partial G}{\partial \gamma} / \frac{\partial G}{\partial s_1}$$

   The partial derivates are:

$$\begin{aligned} \frac{\partial G}{\partial s_1} &= u''(s_1) - \frac{(1-\gamma)R}{p} \frac{\partial s_2}{\partial s_1} u''(s_2) \\ &= u''(s_1) - \frac{(1-\gamma)R}{p}\left(-\frac{R\lambda(1-\gamma)}{p(1-\lambda)}\right) u''(s_2) < 0 \end{aligned}$$

and

$$\frac{\partial G}{\partial \gamma} = -\left(\frac{R}{p\left(1-\gamma\right)^2}u'\left(s_2\right) + \frac{R}{\left(1-\gamma\right)p}\frac{\partial s_2}{\partial \gamma}u''\left(s_2\right)\right)$$

where

$$\frac{\partial s_2}{\partial \gamma} = \frac{\partial\left(\frac{(1-\omega_0^*)S_0-(1-\gamma)(s_1\lambda S_0-\omega_0^* S_0)/p}{(1-\lambda)S_0}R\right)}{\partial \gamma} = -R\frac{\omega_0^* - \lambda s_1}{p\left(1-\lambda\right)} \leq 0$$

where the sign follows because $\lambda s_1 < \omega_0^*$. Then $\partial G/\partial\gamma < 0$ and $ds_1/d\gamma < 0$. As a result, following an unanticipated increase in γ, $s_1 < s_1^*$. This implies that the fund retains $\omega_0^* - \lambda s_1$ in cash. This raises patient investors' payoff above what they would obtain in equilibrium, in which they do not trade. Patient fund investors therefore prefer to stay with the fund. Correspondingly, investors outside the fund have no incentive to purchase if $s_1 \geq s_L$.

3. Suppose instead that $\lambda s_1 < \omega_0^*$ and that $\check{s}_1 < s_L$, where s_L is given by (15). Then, for the same reason as argued in the proof of proposition 1, the optimal settlement price is equal to s_L. Given that $\partial s_L/\partial\gamma < 0$, we again have $ds_1/d\gamma < 0$.

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