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by

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Abstract

This paper presents an approach to portfolio selection using fuzzy decision theory. The approach is such that a given target rate of return is achieved for an assumed market scenario. If the assumed market scenario turns out to be incorrect, the portfolio is guaranteed to secure a given minimum rate of return. The methodology is useful in the management of assets against given liabilities or in forming structured portfolios that guarantee a minimum rate of return.

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Contents

Introduction

Fund managers are constantly faced with the dilemma of guessing the direction of market moves in order to meet the return target for assets under management. Given the uncertainty inherent in financial markets, fund managers are very cautious in expressing their market views. The information content in such cautious views can be best described as being “fuzzy” or vague, in terms of both the direction and the size of market moves. Nevertheless, such fuzzy views are the ones needed to structure portfolios so that the target return, which is assumed to be higher than the risk-free rate, is met. In general, achieving returns higher than the risk-free rate requires taking either market or credit risk. In this paper we will propose a methodology using fuzzy decision theory to select optimal portfolios that target returns above the risk-free rate by taking only market risk.

As suggested above, generating returns higher than the risk-free rate requires fund managers to hold a portfolio of risky assets. Such portfolios may be structured around imprecise and potentially incorrect views of asset managers regarding the size and direction of market moves. Additionally, fund managers may operate under strict guidelines requiring a minimum rate of return for the assets being managed. Under such a constraint, fund managers are forced to examine the performance of the risky portfolio they hold under various possible market scenarios. The risky portfolio frequently has to be structured so that the return from holding the portfolio meets the target rate of return for those scenarios deemed most probable by the fund manager and the minimum rate of return is achieved for other scenarios.

In general, the target rate of return and the minimum rate of return for the portfolio will be a function of the investment horizon, the risk preference of the investor and the nature of assets that can be included in the portfolio. In this paper we will assume that the investment horizon is typically one to three months and the investment universe consists of government debt securities and plain vanilla options on these securities. The target rate of return will be assumed to be a certain number of basis points above Libor over the given investment horizon. Such an investment horizon and security selection could be ideally suited for central banks managing their short-term liquidity portfolio. This is particularly true for those central banks that try to target returns above the money market rate for the liquidity portfolio without taking additional credit risk. Seen in a broader perspective, the target audience for the investment objective stated above could be asset liability management groups who seek to meet a given target return by choosing an optimal portfolio that best replicates their liabilities. Further, since liabilities can be interpreted as a short position on assets, the proposed methodology is also applicable to proprietary trading groups who hold risky portfolios funded at Libor in order to generate returns above the risk-free rate.

Structuring a portfolio along the lines described above is a difficult exercise using existing techniques for portfolio selection. For instance, the approach using mean-variance optimisation to structure the portfolio may be inappropriate. This is because the joint distribution of returns for the assets in the investment universe considered here is usually not computable with any degree of confidence. For the same reason many other techniques for portfolio selection, such as the average shortfall approach (J.P. Morgan (1993)) or asymmetric risk measures for portfolio optimisation under uncertainty (King (1993)) are also not useful because they assume that the return distribution of the portfolio can be computed. In order to solve the portfolio selection problem described here, one has to apply techniques that do not require the estimation of the joint distribution of asset returns. In this paper, the problem is addressed using fuzzy decision theory since it satisfies this requirement. The numerical results suggest that when holding the optimal portfolio consisting of US treasury securities and plain vanilla options on them, it is possible to generate above 100 basis points (bps) excess return over Libor on an annualised basis with a 99.5% confidence level. This is confirmed by carrying out a Monte Carlo simulation of potential yield curve scenarios and computing the annualised excess return over Libor for the given investment period. However, it is important to note that in general the excess return distribution will depend on the shape of the yield curve, repo rates and bond market volatility.

The rest of the paper is organised in the following manner. Section 1 provides the motivation for posing the portfolio selection problem in the fuzzy decision theory framework. Section 2 gives an overview of fuzzy decision theory and fuzzy multi-criteria optimisation. Section 3 describes the formulation of the portfolio selection problem under multiple scenarios as a multi-objective linear programming problem. It is shown that a satisfactory solution to this problem can be easily determined using fuzzy decision theory. Section 4 presents a numerical example to illustrate the various issues. It is assumed in Section 4 that the allowable asset class for the structured portfolio consists of government bonds and plain vanilla options on them. The final section concludes.

1. Motivation

In a capitalist economy, individuals and business firms make portfolio and investment decisions with the objective of maximising the expected income over a given time horizon. Such decisions are based on the subjective evaluation of income expectations over the chosen time horizon and the risk preferences of the individuals or institutions taking these decisions. However, when one is faced with uncertainty in the sense of Keynes (see Minsky (1975)), there is no scientific basis on which one could compute the relevant probabilities for various propositions that determine the income distribution. The subjective estimates of the relevant probabilities and the confidence with which they are held are themselves subject to substantial changes during emerging events such as crises. Nevertheless, individuals and investment firms must take decisions under imperfect knowledge by forming subjective views that help to forecast changes to those factors that influence their end-of-period wealth.

In the classical approach to portfolio selection, one often applies the theory of expected utility that is derived from a set of axioms concerning investor behaviour as regards the ordering relationship for deterministic and random events in the choice set. The specific nature of the axioms that characterise the utility function is based on the assumption that a probability measure can be defined on the random outcomes. However, if we assume that the origins of these random events are not well known, then the theory of probability proves inadequate because of a lack of experimental information. In such instances, one has to approach the decision theory problem under uncertainty using different mathematical tools. Further, the preference function that describes the utility of the investor may itself be changing with the degree of uncertainty. Moreover, one could postulate that the investor has multiple preference functions each of which corresponds to a particular view on various factors that influence the future state of the economy and the confidence with which it is held. Under these conditions, the existing literature in the field of economic theory does not provide the investor with sufficient tools to address the portfolio selection problem.

The discussion above highlighted potential difficulties one would encounter when addressing the portfolio selection problem under uncertainty. It was postulated that under uncertainty the investor would be confronted with multiple utility functions. Each one of these utility functions may be attributed to a particular market view being held and can be broadly described as capturing the investor’s level of satisfaction if it turns out to be true. For instance, a fund manager structuring a fixed-income portfolio may have only vague views regarding future interest rate scenarios and these can broadly be described as being “bullish”, “bearish” or “neutral”. Such views may arise out of the subjective and/or intuitive opinion of the decision-maker on the basis of information available at the given point in time. Under these circumstances, one might try to characterise the range of acceptable solutions to the portfolio selection problem as a fuzzy set (see Bellman and Zadeh (1970)). In simple terms, a fuzzy set is a class of objects in which there is no clear distinction between those objects that belong to the class and those that do not. Further, associated with each object is a membership function that defines the degree of membership of the object in the set. In this respect, fuzzy set theory provides a framework to deal with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables. This provides the point of departure from probability theory, where the uncertainty arises from the random nature of the environment rather than from any vagueness of human reasoning.

In the context of choosing optimal portfolios that target returns above the risk-free rate for certain market scenarios while at the same time guaranteeing a minimum rate of return, fuzzy decision theory provides an excellent framework for analysis. This is because the nature of the problem requires one to examine various market scenarios, and each such scenario will in turn give rise to an objective function. In the face of uncertainty, one will not be able to assign a numerical value to the probability of these scenarios occurring. Under this constraint, it is not clear how a suitable weighting vector can be determined to solve the multi-objective optimisation problem. One way to overcome this difficulty is to use the membership function that arises in fuzzy decision theory to serve as a suitable preference function for finding an ordering relation for the uncertain events. In fact, one can describe the membership function as the fuzzy utility of the investor, which describes the behaviour of indifference, preference or aversion towards uncertainty (Mathieu-Nicot (1990)). The advantage of using the membership function is that it does not rely necessarily on the existence of a probability measure but rather on the existence of relative preference between the uncertain events.

The above arguments show how the portfolio selection problem under uncertainty can be transformed into a problem of decision-making in a fuzzy environment (see Bellman and Zadeh (1970)). To do this, one has to model the aspirations of the investor on the basis of the strength of the views held on various market scenarios through suitable membership functions of a fuzzy set. For instance, a fund manager structuring a fixed-income portfolio may have aspiration levels as to what the portfolio's acceptable excess return over the risk-free rate should be for those scenarios he/she considers more likely. For other scenarios deemed less likely in the fund manager's view, returns lower than the risk-free rate may be considered acceptable for the structured portfolio. In such circumstances, it is rather easy for the fund manager to describe the goals so as to target 50-100 bps excess return over the liabilities or risk-free rate for the more likely market scenarios. Such a characterisation of the aspiration levels will allow one to model the fuzzy preferences of the fund manager through a suitable membership function of the fuzzy subset. Once this is done, the portfolio selection problem can be easily tackled in the framework of fuzzy decision theory. This will be described in Section 3. In the section that follows, the concepts of fuzzy sets, fuzzy goals and fuzzy decision will be introduced and a fuzzy multi-criteria optimisation problem will be formulated.

2. Fuzzy decision theory

Many decision-making problems in the real world take place in a framework where the goals and/or constraints are imprecisely defined. The source of imprecision in such problems is the absence of sharply defined criteria of class membership rather than the presence of random variables. In order to deal quantitatively with imprecision, the notion of fuzzy sets was introduced by Zadeh (1965). This section presents the concepts of fuzzy sets, fuzzy goals and fuzzy decision and formulates a fuzzy multi-criteria optimisation problem. We refer the reader to Zimmermann (1985) for a more detailed description of these concepts.

In simple terms a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not. If, for instance, $X = \{x\}$ denotes a collection of objects, then a fuzzy set *A* in *X* is a set of ordered pairs

$$A = \{(x, \mu_A(x))\}, \quad x \in X \tag{1}$$

where $\mu_A(x)$ is called the grade of membership of *x* in *A*, and $\mu_A : X \to Z$ is a function from *X* to a space *Z* called the membership space. Note that when *Z* contains only two points, 0 and 1, then the set *A* is non-fuzzy and its membership function is identical to the characteristic function of a non-fuzzy set. Again, if $X = \{x\}$ denotes a set of alternatives to a decision-making problem, then a fuzzy goal *G* in *X* will be identified with a given fuzzy set *G* in *X*. For example, if one considers *X* to denote the set of real numbers, then the fuzzy goal expressed in words as "*x* should be considerably larger than 10" can be represented as a fuzzy set whose membership function is subjectively given by

$$\begin{aligned} \mu_G(x) &= 0, \quad x < 10 \\ &= \left(1 + (x-10)^{-1}\right)^{-1}, \quad x \geq 10 \end{aligned} \tag{2}$$

Similarly, the goal "*x* should be in the vicinity of 15" can be interpreted as a fuzzy set with a membership function of the form

$$\mu_G(x) = \left(1 + (x-15)^2\right)^{-1} \tag{3}$$

In a similar manner, one can define a fuzzy constraint *C* in *X* as a fuzzy set in *X*.

Given the above notion of fuzzy goals and fuzzy constraints, one can define the decision-making problem in a fuzzy environment as the intersection of goals and constraints. Specifically, if we are given a space of alternatives *X*, then the fuzzy decision *D* is defined as a fuzzy set in *X* given by $D = G \cap C$ where the symbol $\cap$ denotes the intersection of sets. The corresponding membership function of the fuzzy set *D* is given by

$$\mu_D(x) = Min\left[\mu_G(x), \mu_C(x)\right] \tag{4}$$

More generally, if there are *n* goals $G_1, \cdots, G_n$ and *m* constraints $C_1, \cdots, C_m$, then the decision *D* is a fuzzy set given by

$$D = G_1 \cap G_2 \cdots \cap G_n \cap C_1 \cdots \cap C_m \tag{5}$$

with the membership function

$$\mu_D(x) = Min\left[\mu_{G_1}(x), \cdots, \mu_{G_n}(x), \mu_{C_1}(x), \cdots \mu_{C_m}(x)\right] \tag{6}$$

Further, if *D* is a fuzzy decision represented by its membership function $\mu_D(x)$, then the non-fuzzy subset D^G of *D* defined by

$$\mu_{D^G}(x) = Max\ \mu_D(x) \qquad (7)$$

is said to be the optimal decision and the corresponding state $x \in X$ will be the maximising decision.

To illustrate the above concepts, let us consider the simple example with two fuzzy goals G_1 and G_2 and one fuzzy constraint C_1 characterised by the fuzzy sets given in Table 1. The fuzzy decision D is characterised by the fuzzy set arising from the intersection of the goals and constraints and is again given in Table 1 by the membership function $\mu_D(x)$. The maximising decision in this case is the state x = 4 which maximises the membership function in the fuzzy decision set.

Table 1

Fuzzy membership function

x	**1**	**2**	**3**	**4**	**5**
$\mu_{G_1}(x)$	0.0	0.1	0.3	0.6	1.0
$\mu_{G_2}(x)$	0.1	0.4	1.0	0.8	0.6
$\mu_{C_1}(x)$	0.2	0.6	0.7	1.0	0.5
$\mu_D(x)$	0.0	0.1	0.3	0.6	0.5

2.1 Fuzzy multi-criteria optimisation

In any decision-making framework, the term optimisation is associated with the process of choosing a specific set of actions from a class of possible actions such that a predefined utility or preference function of the decision-maker is maximised. The solution to this problem involves ordering the scalar values arising from the mapping of the set of possible actions using the utility function and choosing the one that maximises the utility of the decision-maker. However, on a number of occasions the decision-maker may not be able to define a utility function crisply. Such a problem arises in particular when the decision-maker has multiple utilities or objectives to maximise. Under the assumption that the utility function is not crisply defined, it is not obvious how the set of possible actions can be mapped to find an ordering relation for the set of alternatives. In such instances, fuzzy decision theory provides a framework to find an ordering relation for the set of possible actions using a fuzzy membership function that characterises the degree of satisfaction of the decision-maker. Such a membership function can be interpreted as the fuzzy utility of the decision-maker (Mathieu-Nicot (1990)). Moreover, multiple objectives are easy to handle in this framework, since finding the optimal solution to such a problem only involves computing the intersection of fuzzy sets and choosing the state that maximises the fuzzy membership function. In this section, we will describe a multi-objective linear programming problem formulation where the objective functions are considered to be fuzzy. The transformation of the fuzzy objectives using a suitable membership function in order to characterise the degree of satisfaction of the decision-maker with respect to the given objective will also be described.

In general, any optimisation problem in which the objective function and the constraints in the space of the decision variables are linear is said to be a linear programming problem. In addition, if there are multiple objectives, then the optimisation problem will be called a multi-objective linear programming problem. Such an optimisation problem can be characterised by:

$$\text{Maximize} \quad f_k(x) = \sum_{i=1}^{N} c_{ik} x_i \,, \quad k = 1,2,\cdots,M \tag{8}$$

subject to

$$b_j^{\min} \le \sum_{i=1}^{N} a_{ij} x_i \le b_j^{\max} \,, \quad j = 1,2,\cdots,P \tag{9}$$

In the context of decision-making in a fuzzy environment, several modifications to the above optimisation problem formulation are possible. For instance, the decision-maker may not really be interested in maximising the objective function in a strict sense. Rather, the decision-maker may only be interested in reaching certain aspiration levels for the various objective functions. If, for instance, one of the objective functions represents the net profit of the firm, the decision-maker may be interested in achieving “good” profits above a given minimum level. Another possible modification to the classical linear programming problem is to consider the constraints as soft constraints so that one may wish to attach different degrees of importance to violations of various constraints.

In this paper, we will consider the case where the decision-maker has multiple objectives each of which is formulated so as to achieve certain goals that are not crisply defined. To solve problems of this nature, each of the objectives has to be transformed using a membership function so as to characterise the degree of satisfaction to the decision-maker with respect to the particular objective. There are many possible choices to such a membership function depending on the nature of the decision-maker's objective (see Sakawa (1993)). We will consider in this paper linear membership functions which are appropriate to transform goals expressed in the form “the objective function should be considerably larger than a given value”. Such linear membership functions will also preserve the linearity of the original problem and the resulting optimisation problem can be solved using linear programming methods. Moreover, when drawing comparisons with utility functions, a linear membership function can be interpreted as the fuzzy utility of a risk-neutral investor. By choosing a membership function that is either concave or convex, one could model the fuzzy utility of a risk-averse or risk-seeking investor, respectively.

Let us now consider the case of a decision-maker who has a fuzzy goal such as “the objective function $f_k(x)$ should be much greater than $p_k^{\min}$”. Further, let us assume that the degree of satisfaction of the decision-maker with respect to achieving the objective does not change beyond the level $p_k^{\max}$. Then the corresponding linear membership function that characterises the fuzzy goal of the decision-maker is given by:

$$\mu_k(f_k(x)) = \begin{cases} 0 & ; \quad f_k(x) \le p_k^{\min} \\ \dfrac{f_k(x) - p_k^{\min}}{p_k^{\max} - p_k^{\min}} & ; \quad p_k^{\min} < f_k(x) \le p_k^{\max} \\ 1 & ; \quad f_k(x) > p_k^{\max} \end{cases} \tag{10}$$

Given the membership functions for the various objectives of the decision-maker, the maximising decision can be computed by solving the following optimisation problem:

$$\mathit{Maximize} \quad \underset{k=1,2,\cdots,M}{\mathit{Min}} \quad \mu_k(f_k(x)) \tag{11}$$

subject to

$$b_j^{\min} \le \sum_{i=1}^{N} a_{ij} x_i \le b_j^{\max}, \quad j = 1,2,\cdots,P \tag{12}$$

By introducing the auxiliary variable λ, the above optimisation problem can be reduced to the following conventional linear programming problem:

$$\mathit{Maximize} \quad \lambda \tag{13}$$

subject to

$$\mu_k(f_k(x)) \ge \lambda, \quad k = 1,2,\cdots,M \tag{14}$$

$$b_j^{\min} \le \sum_{i=1}^{N} a_{ij} x_i \le b_j^{\max}, \quad j = 1,2,\cdots,P \tag{15}$$

In the next section we will describe how fuzzy decision theory can be used to structure portfolios that meet multiple objectives of the investor arising from the uncertainty inherent in quantifying future market moves. In particular, we will pose an optimal portfolio selection problem with the asset class comprising bonds and plain vanilla options on them as a multi-objective optimisation problem. Each of the objective functions in this optimisation problem will correspond to a particular membership function of a fuzzy set for a given yield curve scenario.

3. Structured portfolio

In an efficient market, investors who hold a risky portfolio are on average compensated with returns above the risk-free rate. However, over the short to medium-term horizon (typically less than a couple of years) there is no guarantee that holding a risky portfolio of assets will produce returns above the risk-free rate. Investors who have to meet liabilities over the short term (assumed here to be above the risk-free rate) are usually interested in structuring their risky portfolio of assets such that the downside risk of holding the portfolio is relatively small in adverse market conditions. Further, under favourable market conditions the investor would like to target returns well in excess of the liabilities. Clearly, with respect to the portfolio of assets being held, one needs to define what characterises such adverse and favourable market conditions. In the context of the optimal portfolio selection problem of interest in this paper, such adverse and favourable market conditions will give rise to multiple objectives. In this section, we will illustrate how the multiple scenario portfolio optimisation problem can be formulated so that the resulting structured portfolio meets the multiple goals of the investor.

Let us consider the case of a fund manager who has to choose a structured portfolio from an investment universe of *N* assets with $X_i^{\min}$ and $X_i^{\max}$ being the minimum and maximum weight of the *i*th asset in the portfolio. In order to select the structured portfolio, the fund manager may examine *M* potential market scenarios, and for each of these scenarios he/she may wish to maximise the portfolio return. To achieve the return objective the fund manager could formulate the following optimisation problem:

$$Maximize \quad R_k(x) = \sum_{i=1}^{N} r_{ik} x_i, \quad k = 1,2,\cdots,M \tag{16}$$

subject to

$$\sum_{i=1}^{N} x_i = 1 \tag{17}$$

$$X_i^{\min} \le x_i \le X_i^{\max}, \quad i = 1,2,\cdots,N \tag{18}$$

In equation (16), r_{ik} denotes the return from the *i*th asset for the *k*th market scenario at the end of the investment period and $R_k(x)$ the portfolio return for the *k*th scenario. Since the above optimisation problem has multiple objective functions, one has to compute a Pareto optimal solution for the problem (see Sakawa (1993)). For instance, one could characterise the set of Pareto optimal solutions using the weighted minimax method and select one solution from this set. The set of Pareto optimal solutions to the above optimisation problem is characterised by:

$$Maximize \quad \lambda \tag{19}$$

subject to

$$w_k R_k(x) \ge \lambda, \quad k = 1,2,\cdots,M \tag{20}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{21}$$

$$X_i^{\min} \le x_i \le X_i^{\max}, \quad i = 1,2,\cdots,N \tag{22}$$

In the above relations, λ is an auxiliary variable and $w_k > 0,\ k = 1,2 \cdots, M$ are any arbitrarily chosen weights. Given any suitable weighting vector, one can determine the Pareto optimal solution. Here, we assume without loss of generality that $R_k(x) > 0,\quad X^{\min} \le x \le X^{\max}$. If this is not the case, the objective functions can be rewritten as

$$\hat{R}_k(x) = R_k(x) - C,\quad k = 1,2,\cdots, M \tag{23}$$

where C is a suitable constant that ensures $\hat{R}_k(x) > 0, \forall k$. Incorporating this change in equation (20), one can compute the Pareto optimal solution.

The optimisation problem formulated above is a linear programming problem and can be easily solved using standard algorithms. However, finding a satisfactory Pareto optimal solution requires one to define the a priori probabilities of various scenarios that incorporate the market views. In the face of uncertainty these a priori probabilities are not computable, and hence it is difficult to compute a Pareto optimal solution that can be characterised as being satisfactory. Moreover, the fund manager may like to structure the portfolio such that the return targets are different for each market scenario, for instance with those scenarios that he/she considers more likely to occur (although no experimental evidence is available) being targeted to achieve greater return. Transforming such goals into suitable weights $w_k > 0,\ k = 1,2 \cdots, M$ for the various scenarios is not obvious from the fund manager's perspective.

The notion of uncertainty and the process of decision-making under uncertainty were long-standing intellectual interests of Keynes. In Keynes's view, when investors make decisions under imperfect knowledge they are faced with changing preference functions. Moreover, the relevant probabilistic propositions and the weight attached to such propositions also change in response to emerging events. One way to characterise such subjective probability assignments to emerging events and the confidence with which they are held is through the use of fuzzy sets. For instance, let us consider the case of an investor structuring a fixed-income portfolio who defines the objectives vaguely such as to target a "good" return above the risk-free rate for "bullish" yield curve scenarios. In this case the investor expresses a subjective view that the market will rally and targets a "good" return that describes his/her preference function if the view turns out to be true. Since such an objective is not crisply defined, it is appropriate to address the portfolio selection problem in the fuzzy decision theory framework. In other words, it is assumed here that the motivation for solving the optimal portfolio selection problem in the fuzzy decision theory framework is the absence of crisply defined objectives when the investor's knowledge of emerging events is uncertain. Such an assumption is quite realistic given that investors usually have only an intuitive feeling as to how the market will perform, and hence to define crisply the return objectives for such market scenarios may be unrealistic.

Let us now consider a fund manager structuring a portfolio based on *M* potential market scenarios. For each such scenario, the fund manager may have a target range for the expected return over the investment period. We will denote by $p_k^{\min}$ and $p_k^{\max}$ the minimum and maximum expected return for the *k*th market scenario. Note that it is quite easy for the fund manager to provide information on the expected target range of return for various scenarios rather than to define the a priori probabilities for different scenarios. Using the linear membership function given in equation (10) it is possible to compute the degree of satisfaction $\mu_k(R_k(x))$ for any given portfolio *x* for the *k*th market scenario. Given that the degree of satisfaction to the fund manager for the *k*th market scenario is $\mu_k(R_k(x))$, the structured portfolio can be computed by solving the following optimisation problem:

$$\text{Maximize} \quad \lambda \tag{24}$$

subject to

$$\mu_k\left(R_k(x)\right) \geq \lambda, \quad k = 1,2,\cdots,M \tag{25}$$

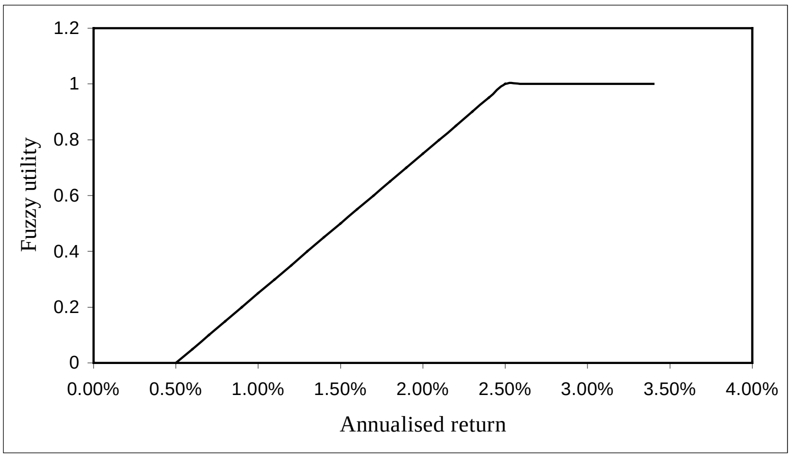
$$\sum_{i=1}^{N} x_i = 1 \tag{26}$$

$$X_i^{\min} \leq x_i \leq X_i^{\max}, \quad i = 1,2,\cdots,N \tag{27}$$

It is easy to show that the solution to the above optimisation problem (if one exists) will be Pareto optimal (see Sakawa (1993)). It is again useful to remind the reader that one can interpret the membership function $\mu_k(R_k(x))$ for the kth market scenario in (25) as modelling the fuzzy utility of the investor for the given scenario. In this case, the structured portfolio computed by solving the above optimisation problem maximises the fuzzy utility of the investor. For a choice of the membership function as in equation (10), Figure 1 shows the fuzzy utility of the investor for the *k*th scenario under

Figure 1

Fuzzy utility of risk-neutral investor



the assumption $p_k^{\min} = 0.5\%$ and $p_k^{\max} = 2.5\%$. As pointed out in Section 2.1, the linear membership function can be interpreted as the fuzzy utility of a risk-neutral investor. In the next section we will consider a numerical example to illustrate the portfolio selection problem using fuzzy decision theory.

4. Numerical example

In this section we will illustrate the use of fuzzy decision theory to structure portfolios by considering the case of a fund manager who is allowed to hold only government bonds and plain vanilla options on them. Further, we will assume that borrowing at Libor finances both the bond position and any option premium paid for a given investment period and that there is no foreign exchange risk. The objective of the fund manager is to select a structured portfolio in the given asset class such that, under a variety of yield curve scenarios, the portfolio return is greater than a given minimum. For yield curve scenarios deemed more likely in the fund manager's view, the portfolio is structured to target a return considerably higher than the Libor rate of return. The details of formulating such a multiple scenario optimisation problem are given below.

4.1 Yield curve scenarios

In order to formulate a suitable optimisation problem for selecting the structured bond portfolio, one needs to consider potential market scenarios and their impact on the portfolio being held. Let us again consider the case of a fund manager holding a portfolio of bonds. In order to hedge against a potential capital loss due to the yield curve shifting up, the fund manager could buy a put option on the bond portfolio over the time horizon of interest. However, the premium paid for buying this put option will result in a loss if the market rallies or if the yield curve is unchanged at the end of the investment horizon. Although in a rallying market there is capital appreciation of the bond portfolio which would compensate for the premium lost on the put option, this is not the case if the yield curve remains unchanged. For such a market scenario, the investor could finance part or most of the put option premium by selling call options on the bonds. Clearly, this simple analysis shows that a well-structured portfolio for the example considered would consist of holding a portfolio of bonds and buying put options and selling call options on the same portfolio in some suitable proportion. The individual weights for the various assets in such a structured portfolio can be computed by formulating a suitable optimisation problem. In this section we will illustrate the formulation of such an optimisation problem that meets the multiple goals of the fund manager.

It will be assumed here for the purposes of illustrating the methodology of structuring portfolios based on fuzzy decision theory that the fund manager holds a portfolio of US treasuries consisting of the current two, five and ten-year bonds. Using the current yield to maturity of these bonds and the yield of the one-year T-bill it is possible to construct the linearly interpolated par yield curve. Under the assumption that the par yield curve remains constant over the investment period, it is possible to compute the total return from holding the bond portfolio. This return comprises two parts, one arising from the roll-down along the yield curve and the other from the accrued interest over the investment period. In case the par yield curve shifts up or down, the total return would include another component which is the capital loss or gain resulting from the yield curve move. It is important to note that, for any given yield curve scenario, it is possible to compute the price of a coupon bond, and hence the pay-off from an option on the bond at the maturity date. Clearly, the total return over the investment period from a portfolio of bonds and plain vanilla options on them can be computed for any given yield curve scenario.

Let us now consider the portfolio of current two, five and ten-year bonds and plain vanilla options with strike prices both at-the-money and out-of-the-money on these securities. We will denote by x_1, x_2 and x_3 the weights of the two, five and ten-year bonds in the portfolio. The portfolio weights of at-the-money put options bought on these bonds will be denoted x_4, x_5 and x_6, and the portfolio weights of out-of-the-money put options bought will be denoted x_7, x_8 and x_9. Similarly, let x_{10}, x_{11} and x_{12} denote the portfolio weights of at-the-money call options sold on these bonds and x_{13}, x_{14} and x_{15} denote the portfolio weights of out-of-the-money call options sold. For any given par yield curve on the maturity date of the investment, it is possible to compute the return from each asset in the portfolio by repricing the bonds using the par yield curve. We will denote the annualised return from the *i*th asset in the portfolio for the *k*th yield curve scenario as r_{ik} The various yield curve scenarios will be described as "bullish", "neutral" or "bearish". For each such scenario, we will consider three potential par yield curves that describe the scenario; these are shown graphically in Figures 2-4. For the constraint functions of the optimisation problem given by (25), $k = 1, 2, 3$ will denote bullish scenarios, $k = 4, 5, 6$ will denote neutral scenarios and $k = 7, 8, 9$ will denote bearish scenarios. We note here that there are in total $M = 9$ market scenarios to be considered.

The minimum and maximum yield curve shifts given by $\Delta y_i^{\min}$ and $\Delta y_i^{\max}$ respectively in the above figures were computed from constant maturity yields over the previous two years for the given investment horizon. For instance, assuming that the investment horizon is one month, $\Delta y_2^{\max}$ is computed from the constant maturity two-year yields using a one-month rolling window and choosing the worst case absolute yield change over the previous two years. In the numerical study presented here, the investment horizon of interest was assumed to be one month, and Table 2 shows the values used for $\Delta y_i^{\min}$ and $\Delta y_i^{\max}$ in the study.

Figure 2

Bullish yield curve scenarios

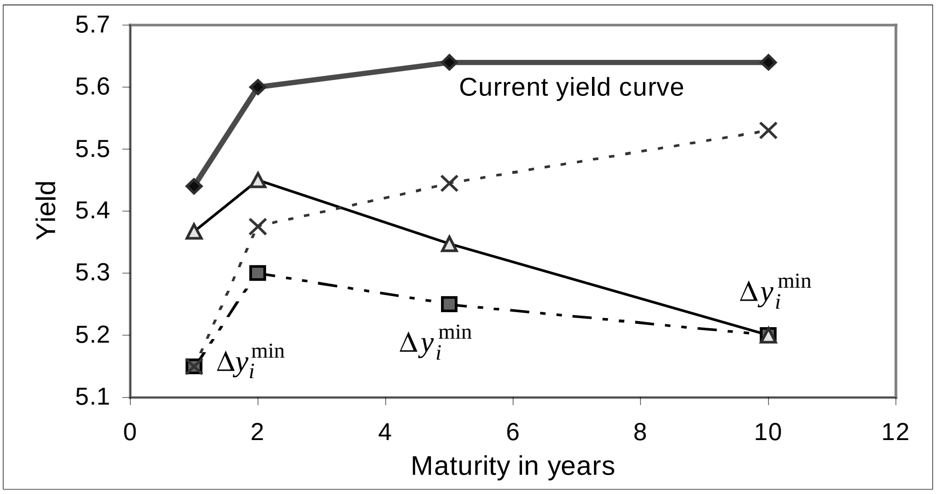


Figure 3

Neutral yield curve scenarios

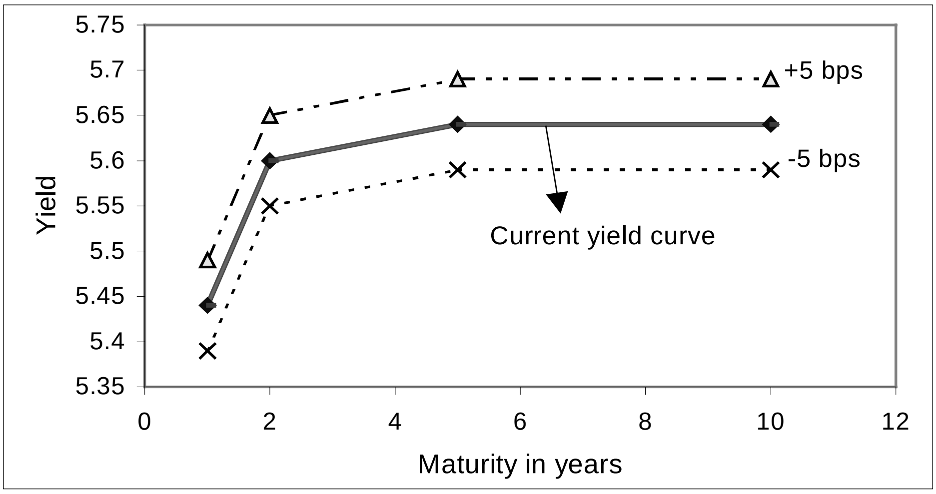


Figure 4

Bearish yield curve scenarios

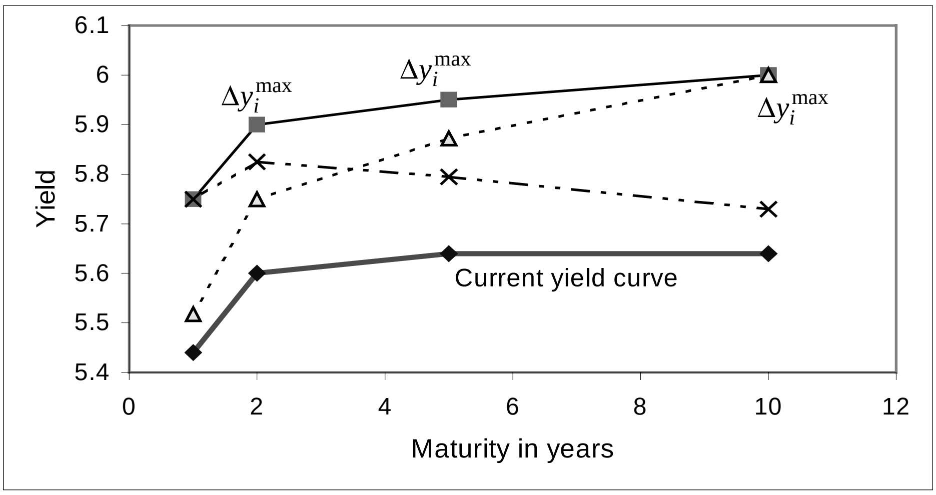


Table 2

Worst case yield changes

Maturity	$\Delta y_i^{\min}$ **(bps)**	$\Delta y_i^{\max}$ **(bps)**
1-year	-53.0	53.0
2-year	-67.0	67.0
5-year	-72.0	72.0
10-year	-63.0	63.0

In the numerical study we will present two market views, one considered "bullish" and the other "bearish" by the fund manager, and for each view find a structured portfolio that meets the fuzzy goals of the fund manager. It is assumed that for the perceived market view of the fund manager, the structured portfolio should yield at least 200 bps annualised excess return over Libor. Further, the fund manager is assumed to be indifferent to annualised return in excess of 500 bps above Libor. For all other market scenarios, it is assumed that the fund manager expects to target an annualised excess return of only 100 bps above Libor and is indifferent to excess return above 400 bps over Libor. Such an assumption implies that $p_k^{\min} = R_L + 2.0$ and $p_k^{\max} = R_L + 5.0$ for $k = 1, 2, 3$ and $p_k^{\min} = R_L + 1.0$ and $p_k^{\max} = R_L + 4.0$ for $k = 4, 5, \cdots, 9$ to compute the structured portfolio with a bullish market view. For structuring the portfolio with a bearish market view, the membership functions can be computed by setting $p_k^{\min} = R_L + 1.0$ and $p_k^{\max} = R_L + 4.0$ for $k = 1, 2, \cdots, 6$ and $p_k^{\min} = R_L + 2.0$ and $p_k^{\max} = R_L + 5.0$ for $k = 7, 8, 9$. Here, R_L denotes the quoted Libor rate, which was equal to 5.60% when this study was carried out (14th September 1998). The constraints on the portfolio weights are assumed to be given by:

$$0 \le x_i \le 0.5, \quad i = 1, 2, 3 \tag{28}$$

$$0 \le x_i \le 0.5 P_i, \quad i = 4, 5, \cdots, 9 \tag{29}$$

$$-0.5 C_i \le x_i \le 0, \quad i = 10, 11, \cdots, 15 \tag{30}$$

In the above inequalities, P_i and C_i denote the prices of the put and call options respectively for a bond with a one dollar nominal amount. The constraints given by (29) and (30) will ensure that for a portfolio size of 100 million the nominal amount on which the put options are bought and call options are sold for a given strike price will not exceed 50 million. The negative weights in the inequality (30) arise because we assume that the call options are sold, and hence this will result in an inflow of capital at the beginning of the investment period.

Using the fuzzy goals of the fund manager given above, it is easy to compute the membership functions $\mu_k(R_k(x))$ for each yield curve scenario. The resulting optimisation problem given by (24)-(27) in Section 3 can then be used to compute the optimal portfolio that meets the goals of the fund manager. Table 3 gives the details of the bonds in the portfolio including the one-year T-bill used to construct the par yield curve. Table 4 shows the prices of put and call options maturing in one month on these bonds. The optimal structured portfolio for a "bullish" market view for a one-month investment period is given in Table 5. In this table, all figures refer to a nominal amount of bond holdings where the size of the portfolio is assumed to be 100 million. The amounts shown against the put and call options refer to the nominal amount on which the put options should be bought and call options sold respectively. Since the call options are sold, the nominal amounts against call

Table 3

Details of bond and T-bill prices

Instrument	Maturity	Coupon rate	Quoted price
1-year T-bill	19.08.99	–	95.082
2-year note	31.08.00	5.125	100.813
5-year note	15.08.03	5.250	102.625
10-year note	15.05.08	5.625	106.125

Table 4

Details of one-month option prices

Underlying	Option type	Strike price	Option price
2-year note	Put	100.813	0.15625
2-year note	Put	100.547	0.05468
2-year note	Call	100.813	0.26560
2-year note	Call	101.313	0.07810
2-year note	Put	102.625	0.40625
5-year note	Put	101.938	0.13280
5-year note	Call	102.625	0.55470
5-year note	Call	103.625	0.20310
10-year note	Put	106.125	0.67187
10-year note	Put	105.000	0.22660
10-year note	Call	106.125	1.03125
10-year note	Call	107.969	0.37500

options are given with a negative sign. Table 6 shows the optimal structured portfolio assuming that the fund manager has a "bearish" market view.

Table 5

Structured portfolio for bullish market view

Bond issue	2-year	5-year	10-year
Nominal bond holding (mn)	0.28	48.51	46.29
At-the-money put on nominal (mn)	50.0	8.91	50.0
Out-of-money put on nominal (mn)	0.0	6.52	7.61
At-the-money call on nominal (mn)	-50.0	-15.60	-50.0
Out-of-money call on nominal (mn)	0.0	0.0	-7.33

Table 6

Structured portfolio for bearish market view

Bond issue	2-year	5-year	10-year
Nominal bond holding (mn)	0.24	48.51	46.29
At-the-money put on nominal (mn)	50.0	29.95	50.0
Out-of-money put on nominal (mn)	0.0	0.0	0.0
At-the-money call on nominal (mn)	-50.0	-19.77	-50.0
Out-of-money call on nominal (mn)	0.0	0.0	-5.30

In order to investigate whether the portfolio returns do guarantee the minimum return targeted by the fund manager, a Monte Carlo simulation was carried out to generate a total of 5,000 yield curve scenarios. The various yield curve scenarios considered were a parallel shift, downward-sloping and upward-sloping yield curves with respect to the current par yield curve. The distributions of the annualised excess returns against Libor are shown in Figures 5 and 6 for the structured portfolios with bullish and bearish market views respectively. It can be inferred from these figures that although only nine yield curve scenarios were modelled to structure the portfolio, there is practically no downside risk for the portfolios constructed on the basis of either a bearish or bullish market view for the scenarios simulated. Using the distribution of excess returns over Libor generated by the Monte Carlo simulation, the excess return over Libor of the structured portfolio constructed using a bullish market view is found to lie above 100 bps with a 99.5% confidence level. In the case of the portfolio structured using the bearish market view, there is a 100 bps excess return over Libor with a 95% confidence level. However, it is important to stress that the results presented here are based on one particular yield curve environment. A different yield curve shape, repo rate or implied volatility for the bond options could change the distribution of expected excess returns. Nevertheless, the results from the Monte Carlo simulation suggest that fuzzy decision theory provides an interesting framework for structuring portfolios on the basis of the views expressed by market participants.

Figure 5

Distribution of excess return over Libor for bullish portfolio

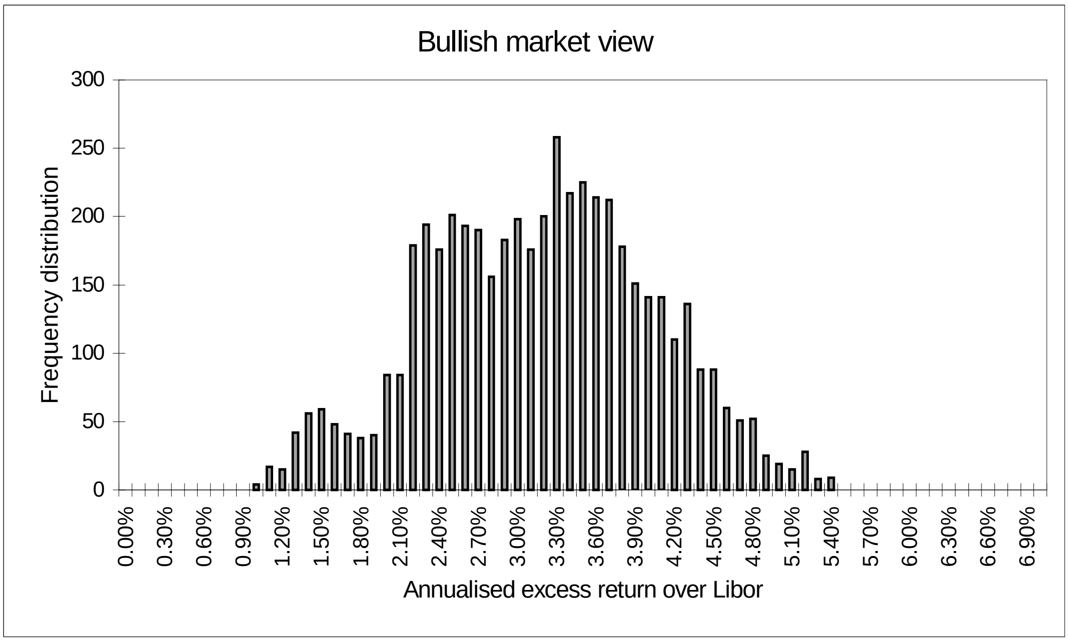
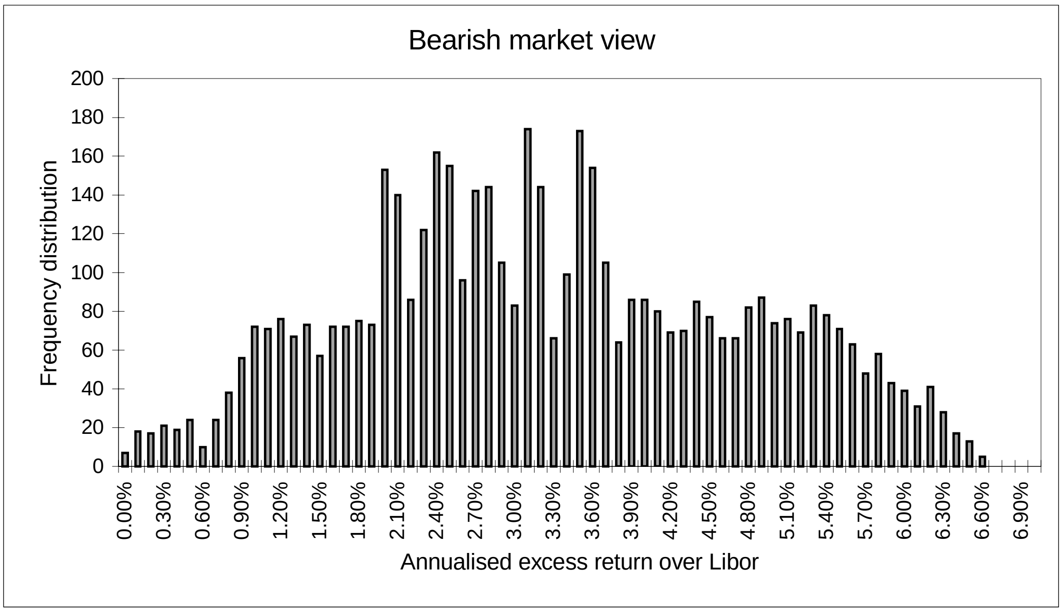


Figure 6

Distribution of excess return over Libor for bearish portfolio



Conclusions

In this paper we examined the problem of forming structured portfolios under uncertainty in order to meet a given target return. It was shown that this could be posed as a fuzzy multi-criteria optimisation problem. Using concepts from fuzzy decision theory, it was shown how one could compute a membership function that describes the subjective views of the fund manager under uncertainty in a quantitative manner. The advantage of the problem formulation presented here is that it does not entail computing the return distribution of the assets. For purposes of illustration, a linear membership function was used to compute a structured portfolio that meets the fund manager's objectives. It was argued that the membership function can be interpreted as modelling the fuzzy utility of the investor, and hence the chosen optimal portfolio will maximise the fuzzy utility. However, it is important to point out that for some choices of the membership function there may be no solution to the portfolio selection problem. In such instances, one will have to modify the aspiration levels for the various scenarios to find a satisfactory solution. Structuring portfolios using the optimisation framework presented in this paper will be useful in the context of asset liability management where fund managers have specific return targets. In such cases, it is easy to express the goals in terms of an expected range for the target return for a variety of market scenarios. Moreover, this methodology could also be potentially useful to proprietary trading groups who can finance portfolios at Libor to target return in excess of the borrowing costs.

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