

# Liquidity risk and positive feedback

by

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## Abstract

This paper reviews some of the literature on market liquidity and feedback trading in the context of a simple multi-asset rational expectations general equilibrium model. The determinants of market liquidity and feedback trading are discussed, and the effect of feedback trading on price volatility and on market liquidity is examined in the context of the model.

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## 1. Introduction

This paper provides a brief summary of some of the relevant issues raised in the literature on feedback and market liquidity for our research on market stress. Rather than presenting an extensive review of all of the literature on this topic, I provide a static model which contains what I believe are the most important features of models of liquidity and feedback. I discuss the relevant literature and the issues that are raised in the context of the model. A bibliography provides a more extensive list of readings than is discussed in this summary.<sup>1</sup> Finally, the last section of this summary discusses areas for further research.

## 2. A basic model

In this section of the paper we model how the prices of assets in an economy are determined by the trading strategies of its market participants, and by the informational structure of the market. Specifically, we will consider an economy endowed with a fixed supply of  $N$  fundamental assets  $A_1, \dots, A_N$ , where  $P$  and  $V$  denote  $N \times 1$  vectors of their prices and long run fundamental values. The economy also contains derivative securities that are in zero net supply; the value of these securities is determined by the prices of the underlying fundamental assets.

Although derivative securities are in zero net supply, we will assume that some participants buy and hold derivative securities while others hedge the risk associated with their derivatives position. I will make the simplifying assumption that derivatives dealers hedge their net derivatives exposures, while other participants do not.  $H(P)$  denotes the value of dealers' net derivative securities positions as a function of underlying asset prices.

There are three types of participants in the markets for the underlying assets: noise traders, derivatives dealers and value investors. The trades of noise traders are uncorrelated with other participants objectives. We will denote their net trades by  $\varepsilon$ , and make the additional simplifying assumption that

$$\varepsilon \sim N(O, \Sigma_\varepsilon). \quad (1)$$

The second type of traders are derivatives dealers. Derivatives dealers maintain positions in the underlying assets to hedge their net positions in the derivatives market. For simplicity we will maintain the assumption that derivatives dealers choose their positions in the underlying assets to remain delta-neutral overall. To derive the implications of this assumption for derivative dealers net asset demands, let  $X_{DD}$  denote the net position of derivatives dealers in the underlying assets at time  $t$ .

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<sup>1</sup> The article by Hebner (1996) provides a good review of the market microstructure liquidity literature. Damodaran and Subrahmanyam (1992) provide a summary of the literature on the effects of introducing options and futures markets.

The value of the derivatives dealers positions in the fundamental assets and the derivative securities is given by

$$X_{DD}^I P + H(P).$$

If the current time is time period 0, and P is initially  $P_0$  but changes to  $P_1$  at time 1, then the change in the value of the derivative dealers position is approximately:

$$\left[ X_{DD} + H_P(P_0) \right] (P_1 - P_0).$$

To minimise the change in the value of the derivatives dealers position,  $X_{DD}$  should be chosen so that  $X_{DD} = -H_{P_0}(P)$ . As P goes from  $P_0$  to  $P_1$ , the hedges will have to be readjusted to maintain delta neutrality. In this case, dynamic hedging requires that

$$\Delta X_{DD} = -H_{P,P}(P_0)(P_1 - P_0). \quad (2)$$

The third type of traders are value investors. Value investors are modelled as long-term players in the securities markets who are willing to take positions to exploit deviations of price P from long run fundamental value V. More specifically, we assume that the net position of value investors in the underlying assets can be described by the equation:

$$X_{VI} = X_{VI}(0) + K(V^* - P)$$

where  $X_{VI}$  is the value investors net position,  $X_{VI}(0)$  is the desired position conditional on  $V = P_0, V^*$  is the  $E(V|P, I)$  based on the observed price P and the information set (I) of the value investors. As  $E(V|P, I)$  and as P changes, the desired position of value investors also changes. The equation for the change is

$$\Delta X_{VI} = K(V_1^* - V_0^* - (P_1 - P_0)). \quad (3)$$

Finally, assume that the change in fundamental value V conditional on P is distributed normally as follows:

$$V_1^* - V_0^* \sim N(0, \Sigma_v).$$

Equilibrium prices in the model are those prices which make the net changes in all three participants positions sum to 0, i.e. the equilibrium condition requires that:

$$\Delta X_{VI} + \Delta X_{DD} + \varepsilon = 0.$$

Solving for  $P_I - P_0$  yields:

$$P_I - P_0 = [H_{PP}(P_0) + K]^{-1} [K(V_I - V_0) + \varepsilon]. \quad (4)$$

This implies that  $\Sigma_P$  the variance of P is given by:

$$\Sigma_P = [H_{PP}(P_0) + K]^{-1} (K \Sigma_V K^T + \Sigma_\varepsilon) [H_{PP}(P_0) + K]^{-1}. \quad (5)$$

The liquidity of the market is measured by its ability to accommodate liquidity shocks without prices being driven far away from fundamentals. One measure of the deviation of prices from fundamentals is the variance of price relative to the variance of fundamentals. It is useful to illuminate what determines this variance since it is these same variables that need to be and to some extent have been modelled on papers on liquidity.

The above model is very simple and stylised. Its main contribution is that it contains important features of feedback and liquidity models that appear in the academic literature. The next sections discuss the liquidity and feedback aspects of the model in more detail.

### 3. Liquidity

Aspects of market liquidity include the time involved in acquiring or liquidating a position and the price impact of this action. Beyond these aspects market liquidity is difficult to define in practice. However, it is sensible to talk about it in the confines of specific models.

#### Liquidity of the derivatives market

Many OTC derivatives markets are not highly liquid which is part of why dealers in these markets make profits. In markets which are not highly liquid, the main determinants of liquidity are dealers' willingness to bear risk, and their ability to hedge risk. This latter ability depends on the liquidity of the underlying markets which is the focus of most of our analysis.

#### Liquidity of the underlying markets

There are two main sources of liquidity in the underlying markets. The first source of liquidity is the liquidity provided by market-makers. Market-makers quote prices at which they are willing to buy and sell fixed, typically small quantities of underlying assets. The spread between these prices (Bid-Ask spread), or the price impact associated with making a trade (often denoted  $\lambda$ ) with a market-maker is an appropriate measure of liquidity in normal market conditions. However, it is probably not reasonable in abnormal conditions.

I make this distinction between normal and abnormal conditions because I view market-makers as providers of immediacy; i.e. market-makers provide immediate temporary liquidity to the market to absorb short-term order imbalances which they believe will disappear when the other side of the market eventually (relatively soon) emerges. In abnormal market conditions, this other side of the market may be small or non-existent; in these abnormal circumstances market-makers will provide very little liquidity to the market. More importantly, any analysis of liquidity that is based on market-makers ability and willingness to absorb volume in normal markets will undoubtedly generate erroneous implications about market liquidity in abnormal market conditions.

The most important determinant of market liquidity in the event of abnormal market conditions is not market-makers, but value investors who will presumably be willing to eventually take the other side of the positions that market-makers hold temporarily. Value investors willingness to provide liquidity is represented by the matrix  $K$  in our stylised model because  $K$  measures the propensity of value based investors to push prices back towards what they perceive as fundamental values when prices appear to deviate from fundamentals. Equation (5) shows that as  $K$  goes to infinity, the variance of asset prices goes to  $\Sigma_v$ , i.e. asset prices do not deviate from fundamentals as  $K$  becomes large. This corresponds to the case of infinitely liquid markets.

As  $K$  becomes small, there is a distinct absence of value investors. In this case, variations in price are due to noise traders and derivatives dealers, but are not due to long-run fundamentals. This creates a scenario where prices could wander far from long-run asset value.

### **Determinants of $K$**

In noisy rational expectations models, the standard functional form for  $K$  is

$$K = \tau \Pi(V|I, P)$$

where  $\Pi(V|I, P) = (\Sigma_v^{-1}|I, P)$ .

$\Pi(V|I, P)$  is known as the precision of value investors estimates of  $V$  given their information set  $I$  and observed asset prices  $P$ , and  $\tau$  is the average value of value investors coefficient of risk tolerance. The formula for  $K$  shows that markets will be more liquid the greater is value investors tolerance for risk or the more precise are value investors estimates of  $V$ .

The precision of value investors information is partially determined ex-ante and partially determined ex-post. From an ex-ante perspective, value investors have an incentive to expend effort learning about  $V$  because this will be useful in exploiting differences between realised securities prices and value investors estimates of  $V$ .

The desire to learn about  $V$  also depends on the amount of ex-ante expected dynamic hedging in the market by dealers, and on the ex-ante expected amount of noise trading. The larger are these latter two, the greater is the incentive to learn about  $V$  to exploit potential mispricing. However, if value investors do not ex-ante know the amount of dynamic hedging interest in a market, or set of

markets, then they will choose their amount of information gathering based on the unconditional average amount of dynamic hedging that takes place in a market. This means they will gather too little information in some markets and too much information in others. This may lead to suboptimal risk-sharing and too much price volatility. Grossman (1988) makes a similar argument (mine is based on his) to illustrate the distinction between exchange traded and OTC options. The open interest in exchange traded options is publicly known which implies that demands for dynamic hedging are known as well. This generates the right incentives to engage in liquidity provision. By contrast, if an option is OTC, then the amount outstanding is not known, and thus incentives for liquidity provision may be suboptimal.

In a framework with asymmetric information,  $K$  could potentially be determined by ex-post as well as ex-ante factors, although modelling this is very difficult. For example, suppose there are two types of value investors. One type only has public information about  $V$  and the other has private information. Under these circumstances, if a value investor with public information observes a change in  $P$ , he knows it could be because of dynamic hedging, noise traders, or news about  $V$  that other value investors received. Under these circumstances, a large change in  $P$  may cause the value investor to revise downward the precision of his assessment of  $V$ .

#### **Determinants of $E(V|I, P)$**

The other important determinant of liquidity demand is  $E(V|I, P)$ . For value investors with only public information, if  $E(V|I, P)$  declines for sufficiently large decreases in  $P$ , then these value investors will tend to buy less as prices decline. Moreover, their beliefs may place them in a position where once price declines are steep enough, it looks better to sell into a decline than to buy. This is the scenario for the crash in Genotte and Leland (1990).

Put slightly differently, after a large decrease in  $P$ , value investors with public information may be very hesitant to purchase stock because they do not know whether the change in  $P$  is due to a change in  $V$  or not. If it is due to a downward move in  $V$ , they should not want to purchase, and may want to sell. But, if it is due to a liquidity shock, they should want to purchase.

The determinants of  $E(V|I, P)$  depend on the signal extraction problem solved by publicly informed value investors. If publicly informed value investors are not aware of liquidity trades, or hedging trades by derivatives dealers, they may mistakenly attribute these price movements to information. The key to avoiding this particular problem is the ability of value investors to distinguish to some extent among various reasons for trade. This will be discussed more below.

#### 4. Feedback and liquidity demand

Liquidity demand comes from two sources, noise traders and derivatives dealers. The demands from noise traders, as modelled here, are not price sensitive which means they represent liquidity demand, but they do not have potential feedback effects on prices. By contrast, derivatives dealers hedging trades are contingent on price. Thus, they have a feedback effect into the market. The strength of this feedback is measured by  $H_{pp}(P_0)$ . Roughly speaking, this is the slope of dynamic hedgers excess demand curve for the underlying assets. If the diagonal elements of  $H_{pp}$  are negative this would imply that the dynamic hedgers (on net) positive feedback trade by selling as prices go down and buying as prices go up. Alternatively, if the diagonal elements are positive, then the dynamic hedgers (on net) sell as prices rise and buy as prices fall.  $K + H_{pp}(P_0)$  is the slope of the excess demand curve for dynamic hedgers and value investors together. Since value investors tend to buy if the asset is underpriced, it is probably safe to assume that the elements of  $K$  (or at least its diagonals) are positive. This implies that those markets which are the most illiquid, i.e. those with the greatest amount of price volatility will be those with  $K + H_{pp}(P_0)$  close to zero; i.e. it will be those where there is negative feedback trading. If this negative feedback trading is so severe that the slope of the excess demand curve approaches 0, then a shock from noise traders or a shock from liquidity traders will generate price volatility that is nearly infinite.

##### **Sunshine trading**

Dynamic hedgers (derivative dealers) and value investors have much to gain by trade with each other since value investors are generally liquidity providers and (derivatives dealers) are liquidity demanders. One way that hedgers can minimise their own price impact is by taking steps to increase  $K$ . This can be done by making some aspect of their trades or trading intentions known to the market in advance. They can also indicate that the trades are not informationally motivated. This practice is known as Sunshine Trading and is discussed extensively in Admati and Pfleiderer (1991). The advantage of preannouncing planned large trades is twofold. First, it gives value investors some time to investigate market conditions (acquire information) before they provide liquidity. Second, since the trades are informationless, preannouncing the trades reduces the chance that the ensuing price movement will be misconstrued as information, potentially triggering a very large price move. Admati and Pfleiderer (1991) show that preannouncing trades can have an important effect in reducing price volatility. Gennotte and Leland (1990) take this reasoning a step further. They show that in a rational expectations model, with asymmetric information, if some participants are not aware of the presence of hedging trades, then very large price drops may be misconstrued as information. This can cause prices to drop much further, i.e. this can cause the market to crash. Furthermore, they show fairly convincingly that this type of reasoning is probably needed to explain the crashes of 1929 and 1987.

## Simulation results on liquidity and feedback

To gain a better understanding of the model presented in section II and expanded upon in the appendix, two types of model simulations were conducted. In the first we assumed that there is only one type of value investor in the market, that these represent a fixed proportion of market participants, and that these value investors have a common private signal of the asset's underlying value. We refer to this as the symmetric information case since all value investors have the same information. We then examined how varying the intensity of dynamic hedging (as measured by  $H_{pp}$  or Derivatives Dealers Net Gamma) affected price volatility in the underlying asset market (shown in Figure 1), and how it affected the sensitivity of asset prices to liquidity trades (shown in Figure 2).<sup>2</sup> In the second type of simulation we kept the same number of value investors, but divided them into two types, informed value investors who observe a common signal about  $v$ , and publicly informed value investors who base their decisions on the information about  $v$  revealed by market price. We refer to this as the asymmetric information case since value investors do not have the same information. The result for the second type of simulation is also contained in figures 1 and 2.

The vertical axes in figures 1 and 2 are the natural logarithm of the variance of price and the natural logarithm of liquidity sensitivity respectively. This scaling was chosen because when gamma becomes negative enough, the market's excess demand curve becomes nearly vertical leading to near price indeterminacy and major price volatility which is too large to appear in a graph on a different scale. That said, Figures 1 and 2 display two striking features. The first is that when gamma becomes negative enough price volatility increases very sharply. For example Figure 1 shows that in the symmetric information case, when net gamma is near -2200, a small decrease in gamma increases price volatility by a factor of about  $e^8$ , which is a 3000-fold increase in price volatility. The same feature appears for the asymmetric information case, but it occurs much earlier, i.e. in the asymmetric information case price volatility increases far faster as gamma increases. This occurs for two reasons, first, value investors with lower quality information are more hesitant to provide liquidity to the market, i.e. they are less willing to provide an offset to liquidity trades. Secondly, and more importantly, publicly informed value investors are making inferences about value from prices, thus when prices go down, they are more likely not to purchase because it may indicate a decline in assets underlying value and not a good time to buy underpriced assets. Figure 2 shows essentially the same story as measured by the sensitivity of price to noise traders demands. The figure shows this sensitivity is increasing as derivative dealers gamma positions become more negative and also as we move from the symmetric to asymmetric information cases.

A single-asset static model lacks the richness associated with multi-asset models. Figure 3 illustrates some of the richness of the multi-asset setting. Specifically, in Figure 3 we examine a six asset economy in which derivative dealers net gamma position in one of the assets (asset one) becomes progressively negative. This has implications for the price volatility of asset 1, but also has

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<sup>2</sup> Sensitivity is measured as the change in asset price due to a change in noise trading ( $\epsilon$ ).

major general equilibrium spillover effects that occur as value investors adjust their entire portfolios to accommodate the large amounts of dynamic hedging that take place in asset 1. More specifically, the Figure shows that as derivative dealers net gammas become more negative in asset 1, the volatility of the other assets (shown in markets 2 through 6) rise as well. This effect appears to be more pronounced the greater is the asymmetric information in the market, as shown by the dashed line.

A feature of Figure 3 that is puzzling at first is that price volatility appears to begin falling again as derivative dealers net gamma becomes sufficiently negative. This is because the economy's excess demand curve for these assets goes from near vertical to backward sloping as gamma becomes negative enough. This is a very perverse case which I do not believe should be taken seriously but does illustrate some of the weaknesses involved in using a Walrasian model.<sup>3</sup>

## **5. Items for further study**

Many important details have been left out of this broad-brush treatment. The purpose of this section is to highlight additional areas for further research.

The first important item is further study of hedging behaviour. While one of our primary feedback concerns is dynamic hedging by dealers, little is known about how this hedging is actually done in practice. This is very important for our analysis on liquidity. I think this can be studied in two ways. First, as part of this effort some of us should go to dealers and ask them details about how and where they hedge their risk, the frequency with which hedges are adjusted, etc. Second, our simulations exercises should impose a variety of assumptions about hedging behaviour. Then we can test the sensitivity of our results to these assumptions.

A second important area for more research involves an examination of the role of risk sharing among dealers. The basic model that I have presented is based on net aggregate positions of all derivative dealers. However, the distribution of these positions across the dealer community is probably very important for determining hedging needs. For example, if derivative dealers net position is equivalent to one position in a put option with a huge open interest, then if this is distributed evenly across derivatives dealers, the need for each dealer to dynamically hedge may be very small. However, if the net position is concentrated with one or a very small number of dealers, the needs for dynamic hedging become larger. We need to inquire more about how these risks are shared.

A third area for study involves transparency of option positions. More specifically, most of the papers that I have discussed make the point that knowledge of the amount of liquidity motivated trade (or trades for hedging purposes) may reduce price volatility. To some extent, market forces

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<sup>3</sup> To illustrate the perversity, in this case the market's response to a liquidity induced purchase requires asset prices to fall so that derivative dealers will sell enough assets to clear the market. This is probably an unrealistic representation of how the market clears.

already provide this transparency via sunshine trading. We should study: what determines the prevalence of sunshine trading, and why do not we observe it more often? Second, even without sunshine trading, knowledge of open interest in options may provide some information on hedging demands. This may explain why the introduction of option markets does not tend to increase volatility in the underlying assets. The role of information in option open interest can be studied empirically. This should be an intermediate goal of this project. Finally, we should study the role of disclosure of some aggregates of risk exposure. In particular, we should study the extent to which this potential disclosure substitutes for knowledge of open interest or other hedging demand proxies in OTC markets. We also need to study the potential for a moral hazard problem if it is perceived that the government will monitor market liquidity and take steps to maintain it.

A fourth area for additional effort involves studying the behaviour of value investors. Two tasks need to be carried out. First, we need to figure out who these investors are. My guess is they are pension funds, life insurers, and others (?). Second, we should get additional information on their demands. Part of this could involve studying which investors are perceived to have long horizons. A small, so-so, literature exists on this subject.

Last, but not least, the relationship between market stress and market liquidity should be studied more. With disclosures about stress in various scenarios, central banks will have better measures of markets in stress even if the markets themselves have not broken down. However, we do not know how conventional measures of liquidity and market function are related to these measures of stress. Acquiring the data on stress, but not releasing it, may provide an avenue to study these relationships further.

FIGURE 1: SINGLE ASSET MARKET  
LOG PRICE VOLATILITY AS A FUNCTION OF GAMMA  
SYMMETRIC INFO: SOLID LINE, ASYMMETRIC INFO: DASHED LINE

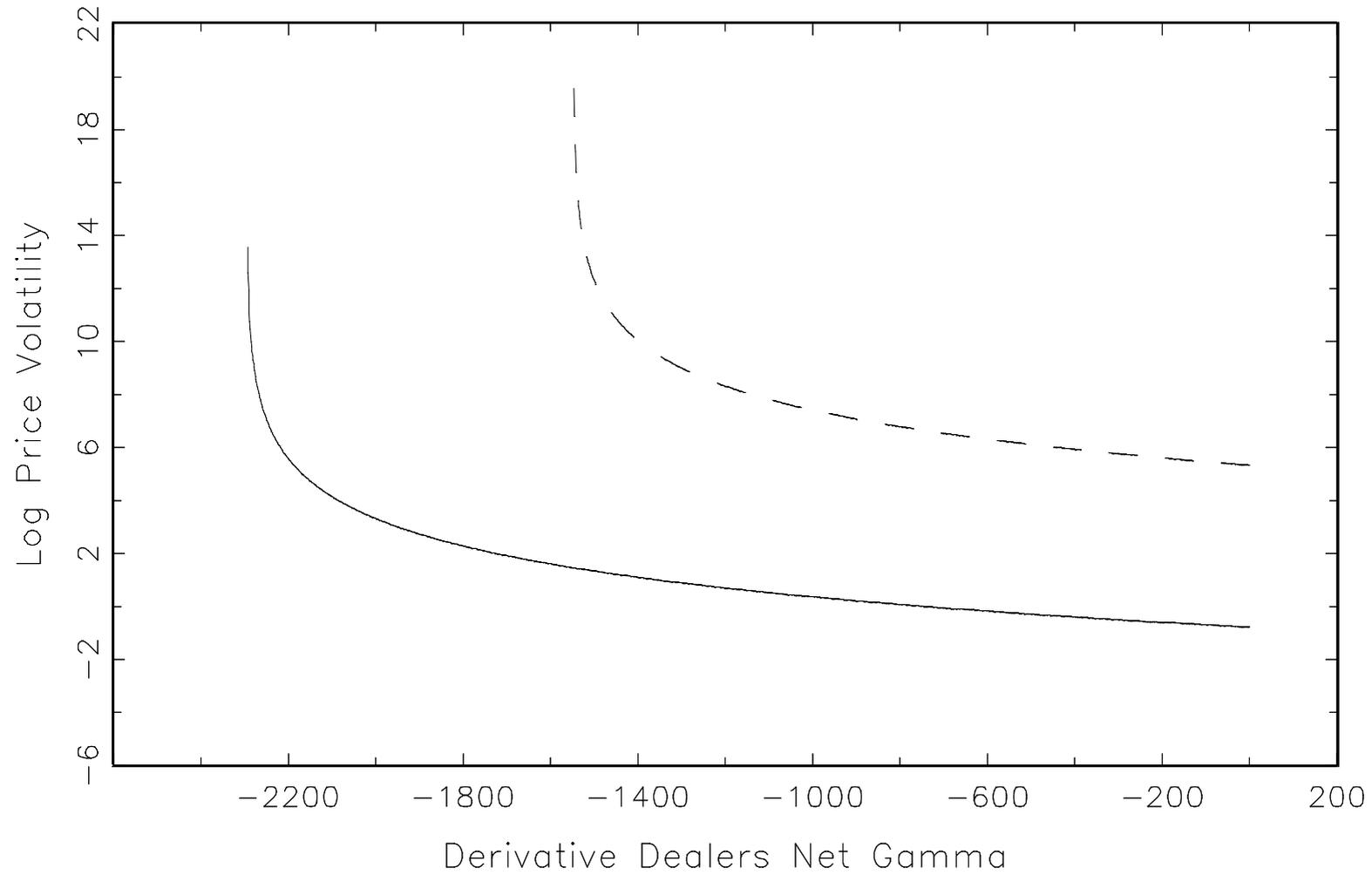


FIGURE 2: SINGLE ASSET MARKET  
LOG LIQUIDITY SENSITIVITY AS A FUNCTION OF GAMMA  
SYMMETRIC INFO: SOLID LINE, ASYMMETRIC INFO: DASHED LINE

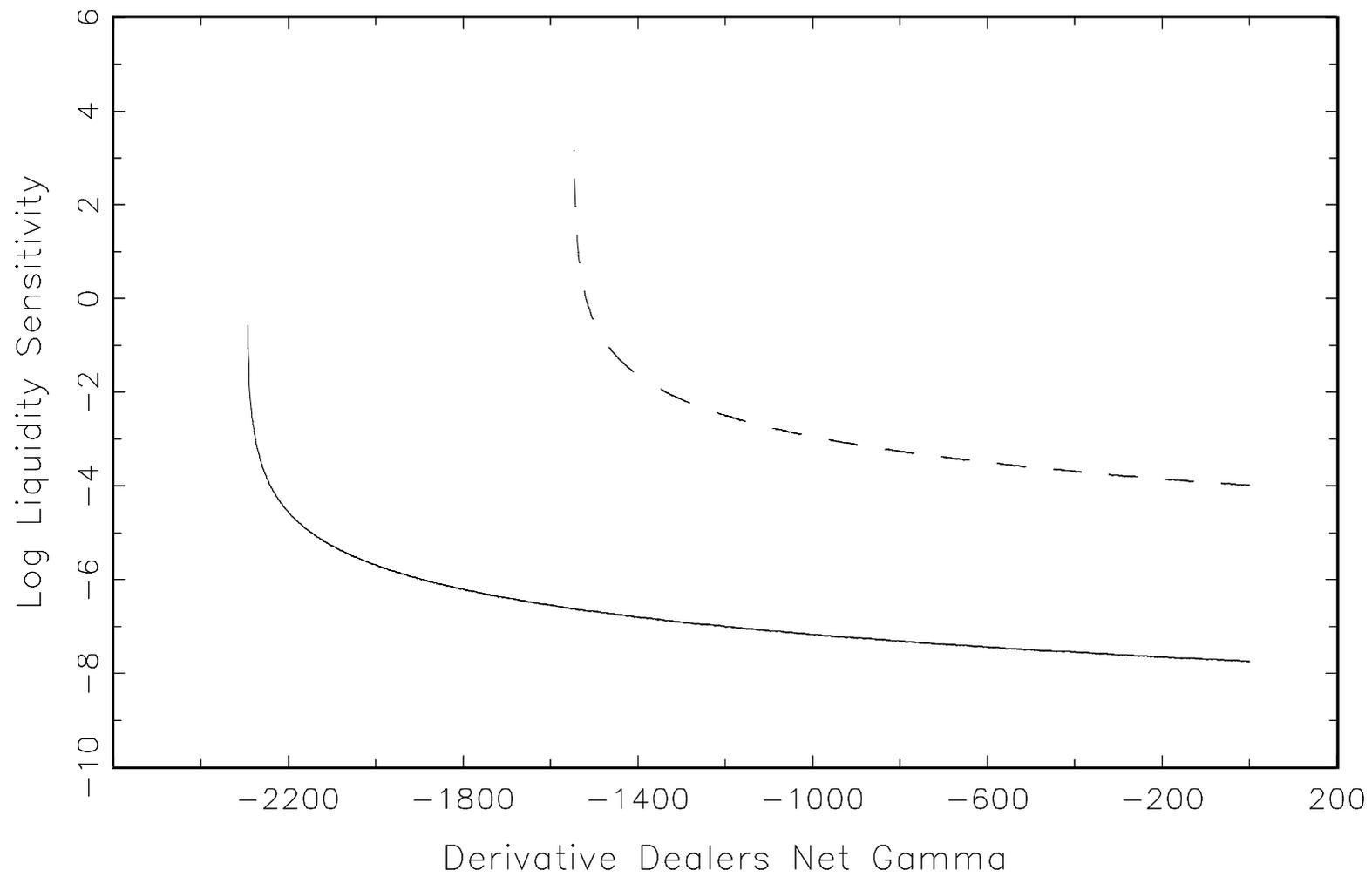
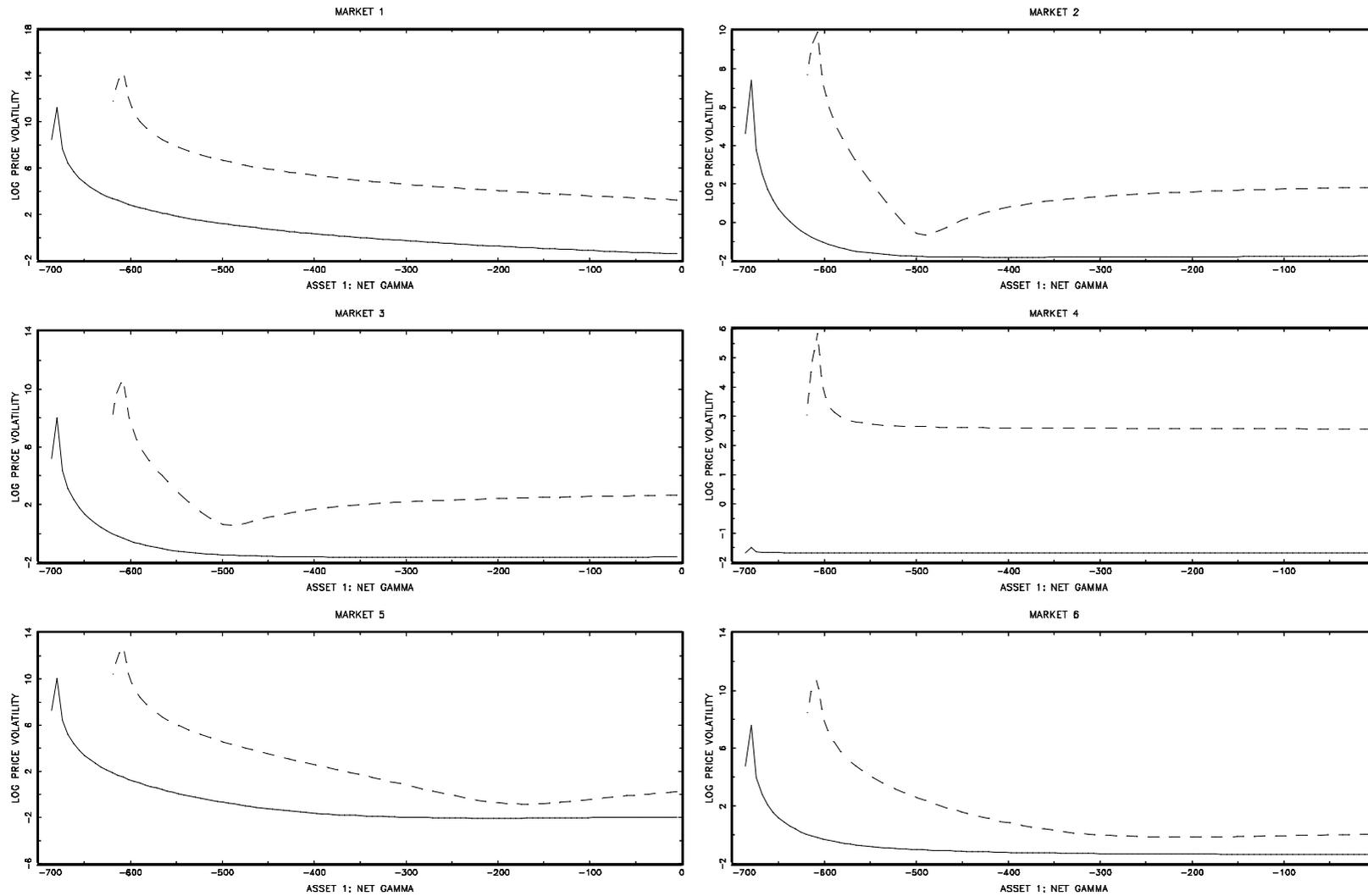


FIGURE 3: MULTI-ASSET MARKET MODEL  
LOG MARKET PRICE VOLATILITY VS. GAMMA IN MARKET 1  
SYMMETRIC INFO: SOLID LINE, ASYMMETRIC INFO: DASHED LINE



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## Appendix

The purpose of this appendix is to further extend the model presented in the text and to provide additional details on the model's derivation and results.

### 1. An extended model

#### The assets

Assume there are  $N$  financial assets indexed  $A_1, \dots, A_N$ , represented by the  $N \times 1$  vector  $A$ . The net supply of the assets is represented by the  $N \times 1$  vector  $X_T$ . The price of the assets is represented by the vector  $P$ , and the price of the assets last period is denoted  $P_0$ . The liquidation value of the assets is represented by the random vector

$$\tilde{v} = \tilde{\theta} + \tilde{u}$$

where:

$$\tilde{\theta} \sim N(\bar{\theta}, \Sigma_\theta) \text{ and } \tilde{u} \sim N(0, \Sigma_u).$$

#### Market participants

There are four types of participants in the market, noise traders, derivatives dealers, informed value investors, and uninformed value investors. Noise traders objectives are uncorrelated with other participants objectives. We will denote their net trades by  $\tilde{\epsilon}$  and make the assumption that:

$$\tilde{\epsilon} \sim N(0, \Sigma_\epsilon).$$

The second type of traders are derivatives dealers. Derivatives dealers are assumed to choose their net positions  $X_{DD}$  in the underlying assets so as to remain delta-neutral. It is assumed that they were delta-neutral in the previous period at  $P = P_0$ , with net asset holdings represented by the vector  $X_{DD}(P_0)$  and with a delta of the net derivatives portfolio of  $H_P(P_0)$ . Therefore, their position in the underlying market must change as  $P$  changes in order to maintain delta-neutrality. In particular, to a linear approximation their desired position as function of  $P$  is:

$$X_{DD} = X_{DD}(P_0) - H_{P, P}(P_0)(P - P_0).$$

The third type of traders are informed value investors. Informed value investors are "informed" because they know the realisation of  $\tilde{\theta}$ . We assume that the mass of informed traders is  $\mu_I$  ("I" denotes informed), that each informed trader has CARA utility with risk tolerance parameter  $\tau$ , and initial wealth  $W_0$ . Borrowing and lending take place without constraint at the riskless rate of interest, which is normalised to 0. This allows each informed trader to choose his portfolio positions  $X_I(P, \tilde{\theta})$  to maximise:

$$E\left(-exp^{-\frac{\tilde{W}}{\tau}}|\tilde{\theta}\right)$$

such that  $\tilde{W} = W_0 + X_I'(\tilde{V} - P)$ .

It is well known that the optimal  $X_I(P, \tilde{\theta})$  satisfies:

$$\begin{aligned} X_I(P, \tilde{\theta}) &= \tau Var(\tilde{v}|\tilde{\theta})^{-1} (E(\tilde{v}|\tilde{\theta}) - P) \\ &= \tau \Sigma_u^{-1} (\tilde{\theta} - P). \end{aligned}$$

The fourth type of participant is uninformed value investors. The mass of these investors is  $\mu_{UI}$  ("UI" denotes uninformed). They have the same utility function as informed value investors but choose their positions  $X_{UI}(P)$  without knowledge of  $\tilde{\theta}$ . They do, however, know the structure of the model, and they observe  $P$  (which reveals some of the information of the informed) and condition their choices on  $P$ . In particular, the uninformed choose  $X_{UI}(P)$  to maximise:

$$E\left(-exp^{-\frac{\tilde{W}}{\tau}}|P\right)$$

such that  $\tilde{W} = W_0 + X_U'(\tilde{v} - P)$ .

It is well known that the optimal  $X_U$  satisfies:

$$X_{UI}(P) = \tau Var(\tilde{v}|P)^{-1} (E(\tilde{v}|P) - P).$$

However, the optimal  $X_{UI}(P)$  depends on  $Var(\tilde{v}|P)$  and  $E(\tilde{v}|P)$ . Both of these depend on the informativeness of  $P$  for  $\tilde{v}$ . This needs to be solved for as part of the overall general equilibrium of the model. We do this below. More specifically, the exposition that follows is designed to accomplish four goals: 1. Relate  $P$  to the private information; 2. Solve for  $E(\tilde{v}|P)$ ; 3. Solve for  $Var(\tilde{v}|P)$ ; and 4. Solve for  $P$  and do various comparative statics exercises.

### Information about $\tilde{v}$ revealed by $P$

Solving for the information revealed by  $P$  is a two-step procedure. First, we will solve for the information revealed by  $P$  as a function of  $E(\tilde{v}|P)$  and  $Var(\tilde{v}|P)$ . Second, given the information revealed by  $P$ , we will solve for  $E(\tilde{v}|P)$  and  $Var(\tilde{v}|P)$ . This section is only concerned with the first of these steps.

The market clearing condition in these markets requires that prices be set so that supply equals demand. This implies:

$$X_T = \mu_{UI} X_{UI}(P) + \mu_I X_I(\tilde{\theta}, P) + X_{DD}(P) + \varepsilon$$

$$= \mu_{UI} \tau \text{Var}(\tilde{v}|P)^{-1} (E(\tilde{v}|P) - P) + \mu_I \tau \Sigma_u^{-1} (\tilde{\theta} - P) + X_{DD}(P_0) - H_{P,P}(P_0)(P - P_0) + \varepsilon.$$

Rearranging the above equation shows that a nonlinear function of  $P$  and other known parameters is equal to  $\tilde{\theta}$  plus a function of "noise". More specifically, rearrangement produces:

$$S(P) = \tilde{\theta} + \frac{\Sigma_u \varepsilon}{\mu_I}$$

where:

$$S(P) = \frac{-I}{\mu_I} \Sigma_u [\mu_{UI} \tau \text{Var}(\tilde{v}|P)^{-1} (E(\tilde{v}|P) - P) - \mu_I \tau \Sigma_u^{-1} P + X_{DD}(P_0) - H_{P,P}(P_0)(P - P_0) - X_T]$$

Since  $S(P)$  is a function of  $P$  and publicly known variables, the above equation shows that knowledge of  $P$  is equivalent to having a noisy signal of  $\tilde{\theta}$ , the information of the informed value investors. Uninformed value investors can condition on this information. In particular, since  $\tilde{v}$ ,  $\tilde{\theta}$  and  $\tilde{\varepsilon}$  are normally distributed, we know that:

$$\begin{aligned} E(\tilde{v}|P) &= E(\tilde{v}|S(P)) \\ &= \tilde{v} + \text{Cov}[S(P), \tilde{v}] [\text{Var}(S(P))]^{-1} [S(P) - \overline{S(P)}] \\ &= \bar{\theta} + \Sigma_\theta \left[ \Sigma_\theta + \frac{\Sigma_u \Sigma_\varepsilon \Sigma_u}{\mu_I^2} \right]^{-1} [S(P) - \bar{\theta}]. \end{aligned}$$

We also know:

$$\begin{aligned}
Var(\tilde{v}|P) &= Var(\tilde{v}|S(P)) \\
&= \Sigma_v - Cov(\tilde{v}, S(P)) [Var(S(P))]^{-1} Cov(\tilde{v}, S(P))' \\
&= [\Sigma_\theta + \Sigma_u] - \Sigma_\theta \left[ \Sigma_\theta + \frac{\Sigma_u \Sigma_\varepsilon \Sigma_u}{\mu_I^2} \right]^{-1} \Sigma_\theta'.
\end{aligned}$$

At this point we have solved for  $Var(\tilde{v}|P)$ , and we have solved for  $E(\tilde{v}|P)$  as a function of  $S(P)$ ; however, we have not fully solved for  $E(\tilde{v}|P)$  since  $S(P)$  depends on  $E(\tilde{v}|P)$ . The next section presents the complete solution for  $E(\tilde{v}|P)$ .

### Complete solution for $E(\tilde{v}|P)$

The complete solution is complicated, because it is derived in a general equilibrium setting. That said, the answer is:

$$E(\tilde{v}|P) = M_0 + M_1 \bar{v} + M_2 P$$

where:

$$\begin{aligned}
M &= \left[ I + Cov(S(P), \tilde{v}) [Var(S(P))]^{-1} \frac{\tau \mu_{UI}}{\mu_I} \Sigma_u Var(\tilde{v}|P)^{-1} \right] \\
M_0 &= M^{-1} \left( Cov(S(P), \tilde{v}) [Var(S(P))]^{-1} \frac{\Sigma_u}{\mu_I} [-X_{DD}(P_0) + X_T - H_{PP}(P_0) P_0] \right) \\
M_1 &= M^{-1} \left( \left[ I - Cov(S(P), \tilde{v}) [Var(S(P))]^{-1} \right] \right) \\
M_2 &= M^{-1} \left( Cov(S(P), \tilde{v}) [Var(S(P))]^{-1} \frac{\Sigma_u}{\mu_I} \left[ \mu_{UI} \tau [Var(\tilde{v}|P)]^{-1} + \tau \mu_I \sigma_u^{-1} + H_{P,P}(P_0) \right] \right).
\end{aligned}$$

The basic form of the above answer is that  $E(\tilde{v}|P)$  is a linear combination of  $P$  and the unconditional expectation  $\bar{v}$ . This is a standard result in the rational expectations literature.

Finally, given solutions for  $E(\tilde{v}|P)$  and  $Var(\tilde{v}|P)$ , it is possible to solve for  $P$ . We do this in the next section.

### The solution for $P$

The solution for  $P$  is found by substituting the expressions for  $E(\tilde{v}|P)$  and  $Var(\tilde{v}|P)$  into the market clearing condition, and then solving for  $P$ . This yields the following expression:

$$P = \Pi^{-1} \left( X_T - \varepsilon - \tau \mu_{UI} Var(\tilde{v}|P)^{-1} (M_0 + M_1 \bar{v}) - \tau \mu_I \Sigma_u^{-1} \tilde{\theta} - X_{DD}(P_0) - H_{P,P}(P_0) \right)$$

where  $\Pi = \tau \mu_{u|} \text{Var}(y|P)^{-1} (M_2 - I) - \tau \mu_{\theta} \Sigma_u^{-1} - H_{P,P}(P_0)$ .

Similarly, we can solve for the unconditional variance of  $P$ . This yields:

$$\text{Var}(P) = \Pi^{-1} \left[ \Sigma_{\varepsilon} + \tau^2 \mu_{\theta}^2 \Sigma_u^{-1} \Sigma_{\theta} \Sigma_u^{-1} \right] \Pi^{-1}.$$