The information content of the yield curve

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Abstract

The goal of this paper is to determine empirically the information content of the nominal yield curve of riskless non-indexed bonds in Switzerland, that is, future expected inflation rates and real interest rates. Applying the three-factor term structure model proposed by Cox, Ingersoll and Ross (CIR), we estimate the model parameters by the full information maximum likelihood method for a sample of pooled time-series and cross-section data. This maximisation is subject to the condition that the theoretical yield curve fits the actual yield curve observed on the trading day under consideration as well as possible.

For a sample of 40 weeks, we obtain the puzzling result that the term structures of real spot interest rates are both upward- and downward-sloping, while the term structures of expected spot inflation rates are always upward-sloping. We attribute this result to the particular assumptions of the CIR model.

We test the model performance indirectly in two ways. First, we compare the future expected nominal spot interest rates with the nominal forward interest rates implied by the observed yield curve over a future time horizon of four years. The outcome of this test is quite satisfactory. Second, we test whether the future expected three-month nominal spot interest rate is an unbiased estimator of the future observed three-month nominal spot interest rate for future time horizons of up to 91 days. This hypothesis is accepted for future time horizons of both one day and seven days. In restricted regressions, however, we accept this hypothesis for all the future time horizons considered in this paper.

Finally, we compare the behaviour of the interest premium or inflation risk premium, respectively, between two different monetary policy regimes. We find that the interest premium has vanished since the beginning of the year 2000, when the Swiss National Bank switched from a regime with medium-term monetary targeting to a concept with inflation forecasts as a main indicator for monetary policy decisions. This reduced risk may indicate that the new concept has further increased the credibility of Swiss monetary policy.

1. Introduction

The goal of this paper is to determine empirically the information content of the nominal yield curve of riskless non-indexed bonds in Switzerland. Applying the three-factor term structure model proposed by Cox, Ingersoll and Ross (henceforth CIR), we obtain estimates of the term structure of expected spot inflation rates as well as of the term structure of real spot interest rates. We test the performance of the CIR model indirectly in two ways using three-month nominal spot interest rates. The model parameters are estimated by the full information maximum likelihood method for a sample of pooled

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time-series and cross-section data. This maximisation is subject to the condition that the theoretical yield curve fits the actual yield curve observed on the trading day under consideration as well as possible.

As regards the term structure of expected spot inflation rates, earlier attempts such as that in Frankel (1982), which relies on a macroeconomic framework, assume that the expected inflation rates are equal to the difference between nominal and real spot interest rates. This is Irving Fisher's hypothesis that the nominal interest rate moves one for one with the expected inflation rate (Fisher (1930)). Probably the first author to show that the Fisher hypothesis does not hold true in an uncertain world was his near namesake Stanley Fischer (1975) in his path-breaking paper on indexed bonds.[2] He shows that the difference between the nominal and real spot interest rate is equal to the expected spot inflation rate minus a term, which I call the "interest premium", which may have either sign.[3] Many other authors, including Bakshi and Chen (1996), Benninga and Protopapadakis (1985), Breeden (1986), Cox, Ingersoll and Ross (1981, 1985a & b), Evans and Wachtel (1992), Fama and Farber (1979) and Lucas (1982), have confirmed this result within quite different frameworks. To my knowledge, the empirical studies, however, have neglected the interest premium so far.[4] There are two exceptions to this observation. One exception is the recent paper by Evans (1998), who is able to estimate the time-varying interest premium in his investigation of index-linked bonds. However, he fails to estimate both the term structure of expected inflation rates and the interest premia endogenously within his framework. Instead, he uses an exogenous variable for the expected inflation rate, namely the Barclay's survey measure of expected inflation.[5] The other exception is the recent paper by Remolona, Wickens and Gong (1998). Using time-series data on both nominal and real discount bond prices, they are able to estimate simultaneously the expected inflation rates and the interest premia in the course of time. They find that the expected inflation rate obtained from their bond price model is an unbiased estimator of future inflation for the period 1982-97. Our approach is different in that we estimate the term structures of both expected inflation rates and interest premia entirely from *nominal* bonds by means of pooled time-series and cross-section data.

In order to calculate the nominal yield curve on the trading days under consideration, we use a non-linear optimisation to determine the nominal instantaneous forward interest rates from observed prices of coupon-bearing government bonds. The objective of the optimisation is nominal instantaneous forward rates as smooth as possible, subject to the condition that the theoretical coupon-bearing bond prices fit the observed coupon-bearing bond prices as well as possible. This optimisation procedure is a modification of the multi-objective goal attainment problem proposed by Delbaen and Lorimier (1992) and Lorimier (1995). The term structure of nominal spot interest rates or the nominal yield curve is deduced from the optimised instantaneous forward rates by numerical integration. This approach has two advantages. First, it is able to explain *any* term structure of interest rates, because no functional form of the instantaneous forward interest rates is assumed. Second, the numerical integration is more exact than the numerical differentiation. To my knowledge, the methods proposed in the literature are inferior to the one put forward by Delbaen and Lorimier. For instance, we have shown by an example that the bootstrap method is not reliable if the yield curve is sufficiently bent, or if there are no discount bonds available in the sample of bonds under consideration (Büttler (2000)). Other methods such as the regression of prices of coupon-bearing bonds on discount factors as proposed in Carleton and Cooper (1976) or various spline methods as proposed in McCulloch (1971, 1975) or in Vasicek and Fong (1982) have many drawbacks as mentioned in Shea (1984, 1985). The recent models proposed by Nelson and Siegel (1987) as well as Svensson (1995) assume an exponential function for the instantaneous forward rates. This approach has two disadvantages: it does not obey the fundamental partial differential equation to value a discount bond (Björk and Christensen (1997)) and it assumes rather than extracts the yield curve from observed data. In a

2 For a criticism of Fischer's equilibrium condition, see Fama and Farber (1979, p 643).

3 Fischer calls it just "premium". Other authors call it the "inflation risk premium" or the "purchasing power risk of the nominal bond". We prefer the neutral term "interest premium" to the term "risk premium", because the latter associates in general positive values only.

4 For a comprehensive list of empirical studies on index-linked bonds, see Evans (1998). In particular, Brown and Schaefer (1994), although applying the CIR model, do not investigate the term structure of expected inflation rates and interest premia.

5 Evans also uses the ex post realised inflation rate as a proxy for the expected inflation rate.

recent paper, we applied our optimisation procedure, the bootstrap method and the extended Nelson-Siegel method to an arbitrary yield curve. While our optimisation procedure is able to mimic the given yield curve perfectly, both the bootstrap method and the extended Nelson-Siegel method fail (Büttler (2000)).

In view of Fisher's hypothesis, the nominal yield curve contains information on real interest rates, expected inflation rates and the interest premium at least. As long as there are no indexed bonds issued in the country under consideration, we must rely on an economic model, which is able to explain simultaneously nominal and real interest rates. To my knowledge, there are two candidate models to be used, the one proposed by Cox, Ingersoll and Ross (1985a & b) on the one hand, and the one by Bakshi and Chen (1996) on the other hand. We choose the CIR model for the sake of tractability. The CIR model is a three-factor model of the nominal yield curve, the factors or state variables being the real instantaneous spot interest rate, the expected instantaneous spot inflation rate and the consumer price level. The stochastic differential equations of these three factors together with those of a set of nominal discount bond prices in *real* terms constitute the sample of pooled time-series and cross-section data to be estimated by the full information maximum likelihood method. The cross-section data refer to the various terms of the discount bonds.

The plan of the paper is as follows. In Section 2, the basic relationships to be used in the paper are explained. In particular, we define spot and forward inflation rates for a future time horizon, which correspond to the spot and forward interest rates defining the actual yield curve. Since future inflation rates are random, we derive their expected values to be used in the calculation of the term structure of expected inflation rates. In Section 3, we present the three-factor model proposed by Cox, Ingersoll and Ross. The estimation procedure is explained in Section 4 and the results are presented in Section 5, followed by conclusions.

2. Basic relationships

In the following, we will use continuously compounded rates. However, the results presented in all charts are annually compounded rates. To clarify the use of various yields, we start with the definition of the term structure. A list of variables is given in the appendix to the paper. In order to distinguish between nominal and real variables or variables associated with the inflation rate, we use two subscripts, if necessary. The first subscript of a variable, $v = \{n, r, y\}$, denotes nominal values for $v = n$, real values for $v = r$, and values associated with the inflation rate for $v = y$. The second subscript, $k = \{m, c\}$, denotes the compounding frequency with the understanding that m denotes a compounding m times a year and c denotes the continuous compounding ($m \to \infty$).

2.1 Interest rates

The spot interest rate, denoted as $R_v(t, T)$, is defined as the yield of a pure discount bond with spot price $P_v(t, T)$:

$$P_\nu(t, T) = \exp\left(-R_{\nu,c}(t, T)\left[T - t\right]\right), \quad \nu = n, r. \tag{2.1}$$

We assume that the first derivative of the pure discount bond price with respect to time exists and is bounded for any lifetime of the bond. Solving for the spot interest rate yields the following expression:

$$R_{\nu,c}(t, T) = -\frac{\ln\left(P_\nu(t, T)\right)}{T - t}, \quad \nu = n, r. \tag{2.2}$$

The term structure of spot interest rates or the yield curve is defined by equation (2.2). The instantaneous spot interest rate, denoted as $r_v(t)$, is equal to the spot interest rate with a vanishing lifetime:

$$r_{\nu,c}(t) \equiv R_{\nu,c}(t, t) = \lim_{t \leftarrow T}\left\{-\frac{\ln\left(P_\nu(t, T)\right)}{T - t}\right\}, \quad \nu = n, r. \tag{2.3}$$

The $(\tau - T)$-year forward interest rate, denoted as $F_v(t, T, \tau)$, corresponds to a forward contract on a pure discount bond with the agreement that the forward price, denoted as $\mathrm{P}_v(t, T, \tau)$, is fixed at date t

and paid at a later date T when the discount bond is delivered. The discount bond matures at a later date τ ($\tau \geqq T \geqq t$).

$$\mathcal{P}_\nu(t, T, \tau) = \exp\left(-F_{\nu,c}(t, T, \tau)\left[\tau - T\right]\right), \quad (t \leqq T \leqq \tau), \quad \nu = n, r. \tag{2.4}$$

In this case, the forward price is equal to the futures price (see Hull (1997), p 95). Again, we assume that the first derivative of the forward pure discount bond price with respect to time exists and is bounded for any lifetime of the bond. Solving for the forward interest rate yields the following expression:

$$F_{\nu,c}(t, T, \tau) = -\frac{\ln\left(\mathcal{P}_\nu(t, T, \tau)\right)}{\tau - T}, \quad \nu = n, r. \tag{2.5}$$

The price of a pure discount bond fixed at date t with maturity date τ should be equal to the price of a portfolio at date t, which consists of a pure discount bond maturing at date T plus a (τ–T)-year forward pure discount bond (see eg Hull (1997)). This leads to the following well known relationship:

$$\begin{aligned}
P_\nu(t, \tau) &= P_\nu(t, T)\, \mathcal{P}_\nu(t, T, \tau), \quad (t \leqq T \leqq \tau) \quad \Rightarrow \\
\exp\left(-R_{\nu,c}(t, \tau)\left[\tau - t\right]\right) &= \exp\left(-R_{\nu,c}(t, T)\left[T - t\right]\right) \exp\left(-F_{\nu,c}(t, T, \tau)\left[\tau - T\right]\right) \quad \Rightarrow \\
F_{\nu,c}(t, T, \tau) &= \frac{R_{\nu,c}(t, \tau)\left[\tau - t\right] - R_{\nu,c}(t, T)\left[T - t\right]}{\tau - T} \\
&= R_{\nu,c}(t, \tau) + \frac{R_{\nu,c}(t, \tau) - R_{\nu,c}(t, T)}{\tau - T}\left[T - t\right], \quad \nu = n, r.
\end{aligned} \tag{2.6}$$

It holds true that $F_\nu(t,t,T) = R_\nu(t,T)$. The instantaneous forward interest rate is obtained for a forward contract that expires in the same instant it has been initiated. Using the above equation, we obtain the following relationship:

$$\begin{aligned}
f_{\nu,c}(t, T) &= F_{\nu,c}(t, T, T) \\
&= \lim_{\tau \downarrow T} F_{\nu,c}(t, T, \tau), \quad (\tau \geqq T) \\
&= \lim_{\tau \downarrow T} \left\{R_{\nu,c}(t, \tau) + \frac{R_{\nu,c}(t, \tau) - R_{\nu,c}(t, T)}{\tau - T}\left[T - t\right]\right\} \\
&= R_{\nu,c}(t, T) + \frac{\partial R_{\nu,c}(t, T)}{\partial T}\left[T - t\right], \quad \nu = n, r.
\end{aligned} \tag{2.7}$$

It holds true that $f_\nu(t,t) = R_\nu(t,t) = r_\nu(t)$, because we have assumed that the first derivative of the pure discount bond price with respect to time exists and is bounded for any lifetime of the bond. Integration by parts of the above equation leads to the well known relationship that the spot interest rate is equal to the integral of the instantaneous forward interest rate divided by the corresponding period of time:

$$\int_{\tau=t}^{\tau=T} f_{\nu,c}(t, \tau)\, \mathrm{d}\tau = R_{\nu,c}(t, T)\left[T - t\right], \quad \nu = n, r. \tag{2.8}$$

Substituting the above equation into equation (2.1), it follows that the spot price of a pure discount bond can be written in terms of the instantaneous forward interest rate:

$$P_\nu(t, T) = \exp\left(-R_{\nu,c}(t, T)\left[T - t\right]\right) = \exp\left(-\int_{\tau=t}^{\tau=T} f_{\nu,c}(t, \tau)\, \mathrm{d}\tau\right), \quad \nu = n, r. \tag{2.9}$$

Differentiation of the logarithm of the above equation with respect to the maturity date leads to the following relationship for the instantaneous forward interest rate:

$$f_{\nu,c}(t, T) = -\frac{\partial \ln\left(P_\nu(t, T)\right)}{\partial T} = -\frac{\dfrac{\partial P_\nu(t, T)}{\partial T}}{P_\nu(t, T)}, \quad \nu = n, r. \tag{2.10}$$

If the price of a pure discount bond is given in a functional form, then the above equation can be used to determine the whole term structure of the instantaneous forward interest rate.

2.2 Inflation rates

We define the spot and forward inflation rates in an analogous way to the interest rates. Although we are dealing now with random variables, it turns out that all the previous relationships for the interest rates carry over to the various inflation rates. Let $P_y(t, T)$ denote the purchasing power of money at the future date T in nominal terms at current prices as seen from date t, and let $p(t)$ denote the consumer price level at date t, then we define the spot inflation rate, $R_{y,c}(t, T)$, and the instantaneous forward inflation rate, $f_{y,c}(t, \tau)$, as follows:

$$P_y(t, T) \equiv \frac{p(t)}{p(T)} = \exp\left(-R_{y,c}(t, T)\,[T-t]\right) = \exp\left(-\int_{\tau=t}^{\tau=T} f_{y,c}(t, \tau)\,\mathrm{d}\tau\right), \quad t \leqq T. \tag{2.11}$$

Since the future consumer price level $p(T)$ is a random variable, both the spot inflation rate, $R_{y,c}(t, T)$, and the instantaneous forward inflation rate, $f_{y,c}(t, \tau)$, are random variables, too. Taking the logarithm of the above equation, the spot inflation rate becomes

$$R_{y,c}(t, T) = -\frac{\ln\left(P_y(t, T)\right)}{T-t} = \frac{\ln\left(p(T)\right) - \ln\left(p(t)\right)}{T-t} \tag{2.12}$$

The instantaneous spot inflation rate, denoted as $r_{y,c}(t)$, is equal to the spot inflation rate with a vanishing time horizon, which turns out - as it should - to be equal to the relative change of the consumer price level.

$$r_{y,c}(t) \equiv R_{y,c}(t, t) = \lim_{t \leftarrow T}\left\{\frac{\ln\left(p(T)\right) - \ln\left(p(t)\right)}{T-t}\right\} = \frac{\left(\frac{\mathrm{d}p(t)}{\mathrm{d}t}\right)}{p(t)} \tag{2.13}$$

In the general equilibrium framework of Bakshi and Chen (1996), the instantaneous spot inflation rate, $r_{y,c}(t)$, consists of a random drift and a volatility term, both driven by macroeconomic variables. The same structure has been assumed by CIR (1985b).

The $(\tau-T)$-year forward inflation rate, denoted as $F_y(t, T, \tau)$, corresponds to a $(\tau-T)$-year forward purchasing power of money, denoted as $\mathrm{P}_y(t, T, \tau)$, at date τ at prices of the earlier date T as seen from date t ($\tau \geqq T \geqq t$). By definition, $\mathrm{P}_y(t, T, \tau) \equiv p(T) / p(\tau)$:

$$\mathcal{P}_y(t, T, \tau) \equiv \frac{p(T)}{p(\tau)} \equiv \exp\left(-F_{y,c}(t, T, \tau)\,[\tau - T]\right), \quad (t \leqq T \leqq \tau) \tag{2.14}$$

Solving for the forward interest rate yields the following expression:

$$F_{y,c}(t, T, \tau) = -\frac{\ln\left(\mathcal{P}_y(t, T, \tau)\right)}{\tau - T} = \frac{\ln\left(p(\tau)\right) - \ln\left(p(T)\right)}{\tau - T} \tag{2.15}$$

By definition, the purchasing power of money at the future date τ is equal to the purchasing power of money at an intermediate date T multiplied by the $(\tau-T)$-year forward purchasing power of money. This leads to the following relationship for the $(\tau-T)$-year forward inflation rate:

$$\begin{aligned}
P_y(t, \tau) &= P_y(t, T)\,\mathcal{P}_y(t, T, \tau), \quad (t \leqq T \leqq \tau) \quad \Rightarrow \\
\exp\left(-R_{y,c}(t, \tau)\,[\tau - t]\right) &= \exp\left(-R_{y,c}(t, T)\,[T-t]\right)\exp\left(-F_{y,c}(t, T, \tau)\,[\tau - T]\right) \quad \Rightarrow \\
F_{y,c}(t, T, \tau) &= \frac{R_{y,c}(t, \tau)\,[\tau - t] - R_{y,c}(t, T)\,[T-t]}{\tau - T} \\
&= R_{y,c}(t, \tau) + \frac{R_{y,c}(t, \tau) - R_{y,c}(t, T)}{\tau - T}\,[T-t]
\end{aligned} \tag{2.16}$$

It holds true that $F_y(t,t,T) = R_y(t,T)$. The instantaneous forward inflation rate is obtained for a vanishing time horizon. Using the above equation, we obtain the following relationship:

$$
\begin{aligned}
f_{y,c}(t,T) &= F_{y,c}(t,T,T) \\
&= \lim_{\tau \downarrow T} F_{y,c}(t,T,\tau), \quad (\tau \geqq T) \\
&= \lim_{\tau \downarrow T} \left\{ R_{y,c}(t,\tau) + \frac{R_{y,c}(t,\tau) - R_{y,c}(t,T)}{\tau - T} [T - t] \right\} \\
&= R_{y,c}(t,T) + \frac{\mathrm{d}R_{y,c}(t,T)}{\mathrm{d}T} [T - t]
\end{aligned}
\tag{2.17}
$$

It holds true that $f_y(t,t) = R_y(t,t) = r_y(t)$. Integration by parts of the above equation leads to the result that the spot inflation rate is equal to the integral of the instantaneous forward inflation rate divided by the corresponding period of time, which, in turn, is equal to the logarithm of the purchasing power of money.

$$
\int_{\tau=t}^{\tau=T} f_{y,c}(t,\tau)\,\mathrm{d}\tau = R_{y,c}(t,T)[T-t] = \ln(p(T)) - \ln(p(t)) \tag{2.18}
$$

Hence, we return to the starting definition of equation (2.11). Differentiating the integral on the left-hand side with respect to the "maturity" date T for a fixed date t, we obtain the result that the instantaneous forward inflation rate is equal to the instantaneous spot inflation rate at the future date T.

$$
f_{y,c}(t,T) = R_{y,c}(t,T) + \frac{\mathrm{d}R_{y,c}(t,T)}{\mathrm{d}T}[T-t] = \frac{\left(\frac{\mathrm{d}p(T)}{\mathrm{d}T}\right)}{p(T)} = r_{y,c}(T), \quad t \leqq T. \tag{2.19}
$$

By this result, we can extend the starting definition (2.11) by the integral of the instantaneous spot inflation rate. Again, it holds true that $f_y(t,t) = r_y(t)$.

In view of Fisher's hypothesis, we need to determine the expected value of the spot inflation rate. Let $\mathscr{E}_t$ denote the expectation operator given the information at date t, then we obtain from the above equation

$$
\mathscr{E}_t f_{y,c}(t,\tau) = \mathscr{E}_t r_{y,c}(\tau) = \mathscr{E}_t \left[\frac{\left(\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau}\right)}{p(\tau)} \right], \quad t \leqq \tau. \tag{2.20}
$$

and from equation (2.18)

$$
\mathscr{E}_t R_{y,c}(t,T)[T-t] = \int_{\tau=t}^{\tau=T} \mathscr{E}_t f_{y,c}(t,\tau)\,\mathrm{d}\tau = \mathscr{E}_t \ln\left(\frac{p(T)}{p(t)}\right) = -\mathscr{E}_t \ln(P_y(t,T)) \tag{2.21}
$$

where we have assumed that the integral of the expectation of the instantaneous forward inflation rate remains finite. Then we can reverse the order of the expectation and the time integral by Fubini's theorem (Duffie (1992)). Hence, we can calculate the expected value of the spot inflation rate, $\mathscr{E}_t R_{y,c}(t,T)$, from equations (2.21) and (2.20), given the expected values of the instantaneous spot inflation rate, $\mathscr{E}_t r_{y,c}(\tau)$, $(T \geqq \tau \geqq t)$.

2.3 Interest premium

Let the interest premium be denoted as $\eta_k(t,T)$, then the relationship between nominal and real interest rates can be written as

$$
R_{n,k}(t,T) = R_{r,k}(t,T) + \mathscr{E}_t R_{y,k}(t,T) - \eta_k(t,T), \quad k = m, c. \tag{2.22}
$$

The above equation is Fisher's hypothesis in an uncertain world (Stanley Fischer (1975)). In the CIR framework, the interest premium consists of two terms, the variance of the future consumer price level and a term which we call the wealth premium. The latter depends both on the investor's attitude towards risk, as measured by the relative risk aversion, and on the covariance between future real wealth and future inflation, which may have either sign. Hence the interest premium may have either sign, too. If investors expect to gain real wealth from future inflation, then this covariance will be positive. In this case, investors do not ask for a full compensation of the expected spot inflation rate. On the other hand, if investors expect to lose real wealth from future inflation, then this covariance will be negative. In this case, they may ask for compensation in excess of the expected spot inflation rate.

3. The CIR model

In view of Fisher's hypothesis, the nominal yield curve contains information on real interest rates, expected inflation rates and the interest premium at least. As long as there are no indexed bonds issued in the country under consideration, we must rely on an economic model which is able to explain simultaneously nominal and real interest rates. To my knowledge, there are two candidate models to be used, the one proposed by Cox, Ingersoll and Ross (1985a & b) on the one hand, and the one by Bakshi and Chen (1996) on the other hand. We choose the CIR model for the sake of tractability. The CIR model is a three-factor model of the nominal yield curve, the factors or state variables being the real instantaneous spot interest rate, the expected instantaneous spot inflation rate and the consumer price level.

To be specific, CIR propose two competitive models to explain nominal and real interest rates. Again, we choose the simpler one, which is their model 2. In equilibrium, the real instantaneous spot interest rate is given by the following square-root process:

$$\mathrm{d}r_{r,c}(t) = \kappa\left[\theta - r_{r,c}(t)\right]\mathrm{d}t + \sigma\sqrt{r_{r,c}(t)}\,\mathrm{d}z_1(t)\,, \quad 0 < \kappa,\ \theta,\ \sigma < \infty \tag{3.1}$$

The above process corresponds to a continuous time first-order autoregressive process where the randomly moving interest rate is elastically pulled towards a long-term equilibrium value, θ. The parameter κ determines the speed of adjustment, and σ denotes the constant volatility parameter, and z_1 a Gauss-Wiener process. With the square-root process, the *real* instantaneous spot interest rate remains non-negative. By means of their fundamental partial differential equation, CIR derive the price of a *real* pure discount bond in *real* terms as follows:

$$P_r\left(t, T \mid r_{r,c}(t)\right) = A(t,T)^{\psi}\exp\left(-B(t,T)\,r_{r,c}(t)\right)\text{, where}$$

$$\begin{aligned} A(t,T) &= \frac{2\,\gamma\exp\left(\dfrac{[\kappa+\lambda+\gamma][T-t]}{2}\right)}{[\kappa+\lambda+\gamma]\left[\exp(\gamma[T-t])-1\right]+2\,\gamma} \\ B(t,T) &= \frac{2\left[\exp(\gamma[T-t])-1\right]}{[\kappa+\lambda+\gamma]\left[\exp(\gamma[T-t])-1\right]+2\,\gamma} \\ \gamma &= \sqrt{[\kappa+\lambda]^2+2\,\sigma^2}\,, \quad \psi = \frac{2\,\kappa\,\theta}{\sigma^2} \end{aligned} \tag{3.2}$$

The parameter λ denotes the factor risk premium, a negative number in the CIR framework. The payoff of the real pure discount bond in real terms is equal to one unit of consumption goods, ie $P_r(T,T) = 1$.

Let $\bar{r}_{y,c}(t)$ denote the drift of the instantaneous spot inflation rate or the "expected" instantaneous spot inflation rate, respectively, at date t. CIR propose the following two random paths for the "expected" instantaneous spot inflation rate and the consumer price level to be tested empirically:

$$\begin{aligned} \mathrm{d}\bar{r}_{y,c}(t) &= \kappa_2\left[\theta_2 - \bar{r}_{y,c}(t)\right]\mathrm{d}t + \sigma_2\sqrt{\bar{r}_{y,c}(t)}\,\mathrm{d}z_2(t)\,, \quad 0 < \kappa_2,\ \theta_2,\ \sigma_2 < \infty\,, \\ r_{y,c}(t)\,\mathrm{d}t &\equiv \frac{\mathrm{d}p(t)}{p(t)} = \bar{r}_{y,c}(t)\,\mathrm{d}t + \sigma_p\sqrt{\bar{r}_{y,c}(t)}\,\mathrm{d}z_3(t)\,, \quad 0 < \sigma_p < 1\,, \\ \mathscr{C}_t&\left\{\mathrm{d}\bar{r}_{y,c}(t), \mathrm{d}p(t)\right\} = \rho\,\sigma_2\,\sigma_p\,\bar{r}_{y,c}(t)\,p(t)\,\mathrm{d}t\,. \end{aligned} \tag{3.3}$$

where $\mathscr{C}$ denotes the covariance operator, ρ denotes the correlation coefficient between the Wiener processes z_2 and z_3, and all other variables have the same meaning as above. Note that $(1/\mathrm{d}\tau)\ \mathscr{E}_t\ \{\mathrm{d}p(\tau)/p(\tau)\} = \mathscr{E}_t\ r_{y,c}(\tau) = \mathscr{E}_t\ \bar{r}_{y,c}(\tau)$ for $\tau \geqq t$, and $\mathscr{E}_t\ r_{y,c}(t) = \bar{r}_{y,c}(\tau)$. Given $\mathscr{E}_t\ \bar{r}_{y,c}(\tau)$ for $T \geqq \tau \geqq t$, we can apply equations (2.20) and (2.21) to calculate the expected spot inflation rate $\mathscr{E}_t\ R_{y,c}(t,T)$. Cox, Ingersoll and Ross derive the following price of a nominal pure discount bond in nominal terms from their fundamental partial differential equation:

$$P_n\left(t, T \mid r_{r,c}(t), \bar{r}_{y,c}(t)\right) = P_r(t, T)\ C(t, T)^{\psi_2} \exp\left(-D(t, T)\ \bar{r}_{y,c}(t)\right), \text{ where}$$

$$C(t, T) = \frac{2\,\xi \exp\left(\frac{\left[\kappa_2 + \rho\,\sigma_2\,\sigma_p + \xi\right]\left[T - t\right]}{2}\right)}{\left[\kappa_2 + \rho\,\sigma_2\,\sigma_p + \xi\right]\left[\exp(\xi\,[T - t]) - 1\right] + 2\,\xi}$$

$$D(t, T) = \frac{2\left[\exp(\xi\,[T - t]) - 1\right]\left[1 - \sigma_p^2\right]}{\left[\kappa_2 + \rho\,\sigma_2\,\sigma_p + \xi\right]\left[\exp(\xi\,[T - t]) - 1\right] + 2\,\xi} \tag{3.4}$$

$$\xi = \sqrt{\left[\kappa_2 + \rho\,\sigma_2\,\sigma_p\right]^2 + 2\,\sigma_2^2\left[1 - \sigma_p^2\right]}\,, \quad \psi_2 = \frac{2\,\kappa_2\,\theta_2}{\sigma_2^2}$$

where $P_r(t,T)$ denotes the price of a real discount bond given above. The payoff of the nominal pure discount bond in nominal terms is equal to one unit of money, ie $P_n(T,T) = 1$.

The nominal and real spot interest rates according to the CIR model 2 can be written by equation (2.2) as follows:

$$R_{r,c}(t, T) = \frac{B(t, T)\ r_{r,c}(t) - \psi \ln\left(A(t, T)\right)}{T - t}$$
$$R_{n,c}(t, T) = \frac{B(t, T)\ r_{r,c}(t) - \psi \ln\left(A(t, T)\right) - \psi_2 \ln\left(C(t, T)\right) + D(t, T)\ \bar{r}_{y,c}(t)}{T - t} \tag{3.5}$$

Taking the limit as $T \to t$, we find for the nominal and real instantaneous spot interest rates by means of L'Hopital's rule:

$$R_{r,c}(t, t) = r_{r,c}(t)$$
$$R_{n,c}(t, t) = r_{n,c}(t) = r_{r,c}(t) + \left[1 - \sigma_p^2\right] \bar{r}_{y,c}(t) \tag{3.6}$$

The last equation is Fisher's hypothesis (2.22) for instantaneous interest rates, where the instantaneous interest premium is given by $\eta_k(t,t) = \sigma_p^2\,\bar{r}_{y,c}(t) = (1/\mathrm{d}t)\ \mathscr{V}_t\ \{\mathrm{d}p(t)\ /\ p(t)\} = \mathrm{d}t\ \mathscr{V}_t\ r_{y,c}(t)$. If there are no indexed bonds issued in the country under consideration, this equation allows you to calculate the real instantaneous spot interest rate; otherwise it determines the volatility parameter σ_p.

Before moving to the estimation of the CIR model, it may be appropriate to add a few remarks. The price of a nominal discount bond in *real* terms, $P_n(t,T)\ /\ p(t)$, at date t depends on the *observed* values of three factors or state variables, namely the real instantaneous spot interest rate, $r_{r,c}(t)$, the drift of the instantaneous spot inflation rate, $\bar{r}_{y,c}(\tau)$, and the consumer price level, $p(t)$. However, the price of a nominal discount bond in *nominal* terms, $P_n(t,T)$, as given in equation (3.4) does *not* depend on the consumer price level. In view of the empirical estimation, it is a great advantage of the CIR model that it provides a closed-form solution, which comes at the expense of some unrealistic features of the CIR model. First, there is no correlation between the real interest rate and the inflation rate, that is, monetary impulses are artificially superimposed on real shocks. Second, the inflation process of equation (3.3) does not allow for negative inflation rates (ie deflation rates). It is an empirical fact that moderate deflation rates could be observed for several industrial countries, for instance in the 1950s and 1990s. Third, the adjustment processes for the real instantaneous spot interest rate and the "expected" instantaneous spot inflation rate as given in equations (3.1) and (3.3) do not allow for the phenomenon of overshooting the long-run equilibrium value, nor for the phenomenon of oscillating around the long-run equilibrium value. Again, it is an empirical fact that both phenomena could be observed in the past. Despite these disadvantages, the nominal yield curve according to the CIR model as given in equation (3.5) may exhibit a wide variety of possible shapes, including the well known normal, inverse and humped shapes.

4. Estimation: pooled time-series and cross-section data

In view of the empirical estimation of the model parameters described in the previous section, we wish to use all the information contained in the CIR model. This may best be accomplished by considering pooled time-series and cross-section data. The pooled time-series and cross-section data will consist of a sample of the three stochastic processes for the real instantaneous spot interest rate, the expected instantaneous spot inflation rate and the consumer price level on the one hand as well as of a sample of the stochastic processes for a selected number of prices of nominal pure discount bonds in *real* terms on the other hand. The cross-section data refer to the term structure of the yield curve. (The first three processes mentioned above have, of course, a term of zero years.) The estimation procedure maximises the full information maximum likelihood function of the pooled time-series and cross-section sample subject to the constraint that the sample of theoretical discount bond prices fits the sample of observed discount bond prices as well as possible on the trading day under consideration.

Let us consider the stochastic processes for the discount bond prices first. Let $\Pi(\cdot)$ denote the price of a nominal pure discount bond in *real* terms, ie $\Pi(t, T) = P_n(t, T) / p(t)$. Applying Ito's lemma to equation (3.4), we derive the following stochastic differential equation for the price of a nominal pure discount bond in real terms:

$$\begin{aligned} d\Pi(\cdot) = {} & [1 - \lambda\, B(\cdot)]\, r_{r,c}(t)\, \Pi(\cdot)\, dt - B(\cdot)\, \Pi(\cdot)\, \sigma \sqrt{r_{r,c}(t)}\, dz_1(t) \\ & - D(\cdot)\, \Pi(\cdot)\, \sigma_2 \sqrt{\bar{r}_{y,c}(t)}\, dz_2(t) - \Pi(\cdot)\, \sigma_p \sqrt{\bar{r}_{y,c}(t)}\, dz_3(t) \end{aligned} \tag{4.1}$$

By this equation, the bond price change is driven by the three Wiener processes associated with the real instantaneous spot interest rate, the expected instantaneous spot inflation rate and the consumer price level as given in equations (3.1) and (3.3). The expressions for $\Pi(\cdot)$, $B(\cdot)$ and $D(\cdot)$ are given in equations (3.2) and (3.4).

Next, we consider discrete time steps Δt (which may be variable) and select H discount bonds with remaining life periods (terms) $\tau_j = T_j - t$, $j = 1, 2, \ldots, H$. In the estimation to follow, the terms of the bonds will vary between three and 26 years. Let $\Pi_{j,t}$ denote the bond price for term τ_j at date t, and similarly for B and D. Since we cannot expect that the CIR model perfectly fits the data, we introduce for each bond a new volatility as follows:

$$\begin{aligned} \Delta\Pi_{j,t} = {} & [1 - \lambda\, B_j]\, r_{r,c,t-1}\, \Pi_{j,t-1}\, \Delta t - B_j\, \Pi_{j,t-1}\, \sigma \sqrt{r_{r,c,t-1}}\, \Delta z_{1,t} - D_j\, \Pi_{j,t-1}\, \sigma_2 \sqrt{\bar{r}_{y,c,t-1}}\, \Delta z_{2,t} \\ & - \Pi_{j,t-1}\, \sigma_p \sqrt{\bar{r}_{y,c,t-1}}\, \Delta z_{3,t} + \sigma_0\, \Pi_{j,t-1}\, \Delta z_{3+j,t}, \quad j = 1, \ldots, H. \end{aligned} \tag{4.2}$$

where σ_0 is a new volatility parameter which is common to each bond selected. Note that we introduced H new Wiener processes, one for each bond selected, and that the expressions B and D depend on the various terms $\tau_j = T_j - t$, $j = 1, 2, \ldots, H$, but not on a particular date t.

Let $\Delta\mathbf{z}(t)$ denote the column vector of the $(3 + H)$ discrete Wiener processes. This vector has mean zero and the following $((3 + H) \times (3 + H))$ variance-covariance matrix $\mathbf{\Sigma}(t) = \mathcal{C}(\Delta\mathbf{z}(t), \Delta\mathbf{z}(t)) = \mathcal{E}\{\Delta\mathbf{z}(t)\, \Delta\mathbf{z}(t)'\} - \mathcal{E}\Delta\mathbf{z}(t)\, \{\mathcal{E}\Delta\mathbf{z}(t)\}'$.

$$\underset{((3+H)\times 1)}{\Delta\mathbf{z}(t)} = \left[\begin{array}{ccc:ccc} \Delta z_{1,t} & \Delta z_{2,t} & \Delta z_{3,t} & \Delta z_{4,t} & \cdots & \Delta z_{3+H,t} \end{array}\right]'$$

$$\underset{((3+H)\times(3+H))}{\boldsymbol{\Sigma}(t)} = \left[\begin{array}{ccc:cccc} \Delta t & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \Delta t & \rho\,\Delta t & 0 & \cdots & \cdots & 0 \\ 0 & \rho\,\Delta t & \Delta t & 0 & \cdots & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \Delta t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \Delta t \end{array}\right] \tag{4.3}$$

By the properties of the Wiener process, the vector $\Delta\mathbf{z}(t)$ is normally distributed with mean zero and variance-covariance matrix $\boldsymbol{\Sigma}(t)$, that is,

$$\underset{((3+H)\times 1)}{\Delta\mathbf{z}(t)} \sim \mathcal{N}\left(\underset{((3+H)\times 1)}{\mathbf{0}}, \underset{((3+H)\times(3+H))}{\boldsymbol{\Sigma}(t)} \right) \tag{4.4}$$

where $\mathcal{N}(\cdot)$ denotes the Gaussian (normal) distribution.

Next, let $\Delta\mathbf{y}(t)$ denote the column vector of the $(3 + H)$ discrete, trend-adjusted increments of the variables considered, that is,

$$\underset{((3+H)\times 1)}{\Delta\mathbf{y}(t)} = \left[\begin{array}{c} \Delta r_{r,c,t} - \kappa\left[\theta - r_{r,c,t-1}\right]\Delta t \\ \Delta \bar{r}_{y,c,t} - \kappa_2\left[\theta_2 - \bar{r}_{y,c,t-1}\right]\Delta t \\ \Delta p_t - \bar{r}_{y,c,t-1}\, p_{t-1}\,\Delta t \\ \hdashline \Delta \Pi_{1,t} - \left[1 - \lambda\, B_1\right] r_{r,c,t-1}\, \Pi_{1,t-1}\,\Delta t \\ \Delta \Pi_{2,t} - \left[1 - \lambda\, B_2\right] r_{r,c,t-1}\, \Pi_{2,t-1}\,\Delta t \\ \vdots \\ \Delta \Pi_{H,t} - \left[1 - \lambda\, B_H\right] r_{r,c,t-1}\, \Pi_{H,t-1}\,\Delta t \end{array}\right] \tag{4.5}$$

Furthermore, let $\mathbf{G}(t)$ denote the $((3 + H) \times (3 + H))$ volatility matrix as follows:

$$\underset{((3+H)\times(3+H))}{\mathbf{G}(t)} = \left[\begin{array}{c:c} \underset{(3\times 3)}{\mathbf{G}_{11}(t)} & \underset{(3\times H)}{\mathbf{0}} \\ \hdashline \underset{(H\times 3)}{\mathbf{G}_{21}(t)} & \underset{(H\times H)}{\mathbf{G}_{22}(t)} \end{array}\right] \tag{4.6}$$

where the first (3 x 3) submatrix is defined as

$$\underset{(3\times 3)}{\mathbf{G}_{11}(t)} = \begin{bmatrix} \sigma\sqrt{r_{r,c,t-1}} & 0 & 0 \\ 0 & \sigma_2\sqrt{\bar{r}_{y,c,t-1}} & 0 \\ 0 & 0 & \sigma_p\, p_{t-1}\sqrt{\bar{r}_{y,c,t-1}} \end{bmatrix} \tag{4.6a}$$

the second (H x 3) submatrix as

$$\underset{(H\times 3)}{\mathbf{G}_{21}(t)} = \begin{bmatrix} -B_1\,\Pi_{1,t-1}\,\sigma\sqrt{r_{r,c,t-1}} & -D_1\,\Pi_{1,t-1}\,\sigma_2\sqrt{\bar{r}_{y,c,t-1}} & -\Pi_{1,t-1}\,\sigma_p\sqrt{\bar{r}_{y,c,t-1}} \\ -B_2\,\Pi_{2,t-1}\,\sigma\sqrt{r_{r,c,t-1}} & -D_2\,\Pi_{2,t-1}\,\sigma_2\sqrt{\bar{r}_{y,c,t-1}} & -\Pi_{2,t-1}\,\sigma_p\sqrt{\bar{r}_{y,c,t-1}} \\ \vdots & \vdots & \vdots \\ -B_H\,\Pi_{H,t-1}\,\sigma\sqrt{r_{r,c,t-1}} & -D_H\,\Pi_{H,t-1}\,\sigma_2\sqrt{\bar{r}_{y,c,t-1}} & -\Pi_{H,t-1}\,\sigma_p\sqrt{\bar{r}_{y,c,t-1}} \end{bmatrix} \tag{4.6b}$$

and the last (H x H) submatrix as

$$\underset{(H \times H)}{\mathbf{G}_{22}(t)} = \begin{bmatrix} \sigma_0 \,\Pi_{1,t-1} & 0 & \cdots & 0 \\ 0 & \sigma_0 \,\Pi_{2,t-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_0 \,\Pi_{H,t-1} \end{bmatrix} \tag{4.6c}$$

Taking for the pre-period variables their realised (observed) values, the trend-adjusted increments are then normally distributed as follows (Goldberger (1964)):

$$\underset{((3+H)\times 1)}{\Delta \mathbf{y}(t)} = \underset{((3+H)\times(3+H))}{\mathbf{G}(t)} \underset{((3+H)\times 1)}{\Delta \mathbf{z}(t)} \sim \mathcal{N}\Big(\underset{((3+H)\times 1)}{\mathbf{0}}, \underset{((3+H)\times(3+H))}{\mathbf{S}(t)}\Big), \quad \mathbf{S}(t) = \mathbf{G}(t)\,\boldsymbol{\Sigma}(t)\,\mathbf{G}(t)' \tag{4.7}$$

If we had not introduced new volatility terms for the selected bonds, then the rank of the variance-covariance matrices $\boldsymbol{\Sigma}(t)$ and $\mathbf{S}(t)$ would be three at most, hence we could only estimate the processes for the real instantaneous spot interest rate, the drift of the instantaneous spot inflation rate and the consumer price level.

Our goal is to maximise the likelihood of a given sample of the vector of trend-adjusted increments $\Delta\mathbf{y}(t)$ for each date t. For computational ease, we wish to reduce the normal distribution as given in equation (4.7) to a standard multivariate normal distribution. This can be accomplished by the following transformation (Goldberger (1964)):

$$\underset{((3+H)\times 1)}{\Delta \mathbf{x}(t)} = \underset{((3+H)\times(3+H))}{\left(\mathbf{Q}(t)^{-1}\right)'} \underset{((3+H)\times 1)}{\Delta \mathbf{y}(t)} \sim \mathcal{N}\Big(\underset{((3+H)\times 1)}{\mathbf{0}}, \underset{((3+H)\times(3+H))}{\mathbf{I}}\Big), \quad \mathbf{S}(t) = \mathbf{Q}(t)'\,\mathbf{Q}(t) \tag{4.8}$$

where the ((3 + H) x (3 + H)) upper-triangle matrix $\mathbf{Q}(t)$ is the Cholesky decomposition of the variance-covariance matrix $\mathbf{S}(t)$ as shown in the equation above, and $\mathbf{I}$ denotes the identity matrix. The upper-triangle matrix, $\mathbf{Q}(t)$, and its inverse, $\mathbf{Q}(t)^{-1}$, can easily be computed recursively. The probability density function of the vector $\Delta\mathbf{y}(t)$ is then equal to the standard multivariate normal distribution for the vector $\Delta\mathbf{x}(t)$ divided by the absolute value of the determinant of the Jacobian of the above transformation (4.8), which is equal to the upper-triangle matrix $\mathbf{Q}(t)$. If the volatility matrix $\mathbf{G}(t)$ has full rank, then the variance-covariance matrix $\mathbf{S}(t)$ has full rank, too, and it is positive definite. Since the determinant of a positive definite matrix is positive, it follows that the determinant of the upper-triangle matrix $\mathbf{Q}(t)$ is also positive, that is,

$$|\mathbf{S}(t)| = |\mathbf{Q}(t)'\,\mathbf{Q}(t)| = |\mathbf{Q}(t)'|\,|\mathbf{Q}(t)| = |\mathbf{Q}(t)|^2 > 0 \quad \Rightarrow \quad |\mathbf{Q}(t)| = \sqrt{|\mathbf{S}(t)|} > 0 \tag{4.9}$$

where $|\mathbf{Q}(t)|$ denotes the determinant of $\mathbf{Q}(t)$. Note that, by the transformation (4.8), all the elements of the vector $\Delta\mathbf{x}(t)$ are independent of each other, that is, each element has a univariate standard normal distribution.

Denote the (10 x 1) column vector of the CIR model parameters as $\boldsymbol{\beta} = [\kappa, \theta, \sigma, \lambda, \kappa_2, \theta_2, \sigma_2, \sigma_p, \rho, \sigma_0]'$. Let $L(\Delta\mathbf{y}(t) | \boldsymbol{\beta})$ denote the logarithm of the likelihood of a given sample $\Delta\mathbf{y}(t)$ in terms of parameters $\boldsymbol{\beta}$. From equations (4.7) to (4.9), the logarithm of the likelihood function can be written as follows:

$$\begin{aligned} \mathcal{L}\{\Delta\mathbf{y}(t) \,|\, \boldsymbol{\beta}\} &= \ln\left(\frac{\prod_{j=1}^{3+H} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\Delta x_j^2\{\Delta\mathbf{y}(t) \,|\, \boldsymbol{\beta}\}}{2} \right)}{|\mathbf{Q}(t)|} \right) \\ &= -\left[\frac{3+H}{2} \ln(2\pi) + \frac{1}{2} \sum_{j=1}^{3+H} \Delta x_j^2\{\Delta\mathbf{y}(t) \,|\, \boldsymbol{\beta}\} + \ln\{|\mathbf{Q}(t)|\} \right] \end{aligned} \tag{4.10}$$

Due to the transformation (4.8), the likelihood function is just the logarithm of the algebraic product of the univariate probability masses of the (3 + H) independent increments Δx_j divided by the determinant of the matrix $\mathbf{Q}(t)$.

Consider next the correlation between the trend-adjusted increments $\Delta\mathbf{y}$ at various dates. By the properties of the Wiener processes, these covariances become zero for any time period s greater than zero, that is,

$$\mathscr{C}(\Delta\mathbf{y}(t), \Delta\mathbf{y}(t+s)) = \mathscr{C}(\mathbf{G}(t)\,\Delta\mathbf{z}(t), \mathbf{G}(t+s)\,\Delta\mathbf{z}(t+s)) = \underset{((3+H)\times(3+H))}{\mathbf{0}}, \quad s > 0. \tag{4.11}$$

By the equation above, all the likelihood functions $\mathscr{L}(\Delta\mathbf{y}(t)\,|\,\boldsymbol{\beta})$ at different dates are independent of each other. Hence, the overall likelihood function is equal to the sum of the single logarithmic likelihood functions.

Finally, let $P_n^{\text{obs}}(s, T_k)$ denote the observed prices of the nominal pure discount bonds in nominal terms on the trading day under consideration, s, for various term dates T_k, $k = 1, \ldots, K$. These terms need not be the same as the terms considered in the selection of the various bond price processes of equation (4.2). The goal of the estimation is to maximise the overall likelihood function subject to the condition that the theoretical bond prices do not deviate from the bond prices observed on the trading day under consideration by more than a given tolerance ϵ, that is,

$$\begin{aligned} &\max_{\{\boldsymbol{\beta}\}} \sum_{t \leq s} \mathscr{L}(\Delta\mathbf{y}(t)\,|\,\boldsymbol{\beta}), \quad s \in \mathscr{S}, \quad \text{s. t.} \\ &-\epsilon_k \leqq \left(\frac{P_n(s, T_k\,|\,\boldsymbol{\beta})}{P_n^{\text{obs}}(s, T_k)} - 1\right) 100 \leqq \epsilon_k, \quad k = 1, \ldots, K. \end{aligned} \tag{4.12}$$

where s denotes the actual trading day and $\mathscr{S}$ the set of trading days considered in the paper. Note that a different percentage tolerance may be given for each bond selected on the trading day for which we investigate the actual yield curve.

In this paper, we run the optimisation as given in equation (4.12) above for a sample of 40 trading days between 14 August 2000 and 14 May 2001. These 40 trading days are approximately weekly spaced. Hence, we obtain a set of 40 parameter vectors $\boldsymbol{\beta}$. The observed nominal discount bond prices are obtained from a set of observed coupon-bearing government bonds by means of the constrained optimisation described in the introduction above (Büttler (2000)). For each of the 40 trading days, we consider a sample of pooled time-series and cross-section data which starts at the beginning of February 1998, the earliest date for which bond data with the required information are available from our database. The number of nominal pure discount bonds selected for the cross-section data has been chosen to be five, that is, $H = 5$. Their terms vary between three and 26 years. The number of bonds selected for the constraints in equation (4.12) has been chosen to be 20, that is, $K = 20$. Their terms are equally spaced between zero years and the maximum term of the set of observed discount bond prices which define the actual yield curve on the trading day under consideration. Since indexed bonds are not traded in Switzerland, the time-series data of the real instantaneous spot interest rate have been calculated from equation (3.6); they depend on the parameter set, $\boldsymbol{\beta}$, however.

5. Results

The estimates of the parameters for the 40 trading days are depicted in Figure 1. Except for the speed of adjustment of the real instantaneous spot interest rate process, κ, the parameter estimates are quite stable over time. Note that the long-run equilibrium values, θ and θ_2, are related to their corresponding speeds of adjustment, κ and κ_2, respectively. For a speed of adjustment of zero, say, any long-run equilibrium value is compatible.

Given the estimated parameters, $\boldsymbol{\beta}$, for each trading day considered, the real yield curve, $R_r(t, T)$, has been calculated from equation (3.5). The 40 real yield curves and the 40 observed nominal yield curves are depicted in Figure 2. Although the nominal yield curves are upward-sloping (ie normal) or U-shaped, the real yield curves turn out to be upward-sloping during the first half of the sample period, but downward-sloping (ie inverse) during the second half of the sample period. Since downward-sloping real yield curves are puzzling in view of the falling trend of recent inflation rates, we checked whether we had encountered a local rather than a global maximum of the logarithmic likelihood function. Indeed, we found many local maxima of the logarithmic likelihood function, which result mostly in an *upward*-sloping real yield curve, and consequently in a *downward*-sloping term structure of expected spot inflation rates, but we could not find other maxima with greater function values than the ones reported here. We conclude, therefore, that the downward-sloping real yield curves result, most likely, from the assumptions of the CIR model, namely from the impossibility of the future expected inflation rate overshooting its long-run equilibrium value. Furthermore, the real instantaneous spot interest rate is not correlated with the drift of the instantaneous spot inflation rate in the CIR model.

In order to calculate the term structure of the expected spot inflation rates, $\mathscr{E}_t\, R_{y,c}(t,T)$, for a given trading date t, we need to know the expected instantaneous spot inflation rate, $\mathscr{E}_t\, r_{y,c}(\tau) = \mathscr{E}_t\, \bar{r}_{y,c}(\tau)$ for $T \geqq \tau \geqq t$, by equations (2.20) and (2.21). The future expected instantaneous spot inflation rate can be derived in three different ways at least. First, since the transition probability density function of the drift of the instantaneous spot inflation rate is given by the non-central chi square probability density function, the expected values of the drift of the instantaneous spot inflation rate can be calculated straightforwardly from this distribution (Cox et al (1985b)). Second, the drift of the instantaneous spot inflation rate at date t can be written as an integral over time of the respective process as given in equation (3.3). Taking the expectation on the one hand and differentiating the integral equation on the other hand yields a first-order ordinary differential equation for the expected instantaneous spot inflation rate (Duffie (1992)). Third, the discrete-time version of the stochastic differential equation of the drift of the instantaneous spot inflation rate as given in equation (3.3) leads to a stochastic finite difference equation. Taking the expectation on the one hand and the limit as the time step goes to zero on the other hand, the same ordinary differential equation as mentioned in the second way can be derived. Similarly, we derive the first-order ordinary differential equations for the variance and the covariance of the drift of the instantaneous spot inflation rate. The solutions of these three differential equations are given below.

$$
\begin{gathered}
\mathscr{E}_s \bar{r}_{y,c}(t) = \theta_2 + \left[\mathscr{E}_s \bar{r}_{y,c}(s) - \theta_2\right] \mathrm{e}^{-\kappa_2 [t-s]}, \quad t \geqq s, \\
\mathscr{V}_s \bar{r}_{y,c}(t) = \frac{\sigma_2^2\, \mathscr{E}_s \bar{r}_{y,c}(s)}{\kappa_2}\left[\mathrm{e}^{-\kappa_2[t-s]} - \mathrm{e}^{-2\kappa_2[t-s]}\right] + \frac{\sigma_2^2\, \theta_2}{2\,\kappa_2}\left[1 - \mathrm{e}^{-\kappa_2[t-s]}\right]^2, \quad t \geqq s, \\
\mathscr{C}_s\left(\bar{r}_{y,c}(t), \bar{r}_{y,c}(t+\tau)\right) = \mathrm{e}^{-\kappa_2 \tau}\, \mathscr{V}_s \bar{r}_{y,c}(t), \quad \tau \geqq 0, \quad t \geqq s
\end{gathered} \tag{5.1}
$$

where $\mathscr{V}$ denotes the variance operator. Note that t and s denote two arbitrary dates, whereas τ denotes a non-negative time period. As seen from date s, the initial value is equal to the realised value, that is, $\mathscr{E}_s\, \bar{r}_{y,c}(s) = \bar{r}_{y,c}(s)$. By the third equation above, the auto-covariances decline exponentially with the speed of adjustment and time. Using equations (5.1), (2.20) and (2.21), the expected value of the spot inflation rate becomes as follows:

$$
\begin{gathered}
\mathscr{E}_t R_{y,c}(t,T) = \theta_2 + \frac{\left[\mathscr{E}_t \bar{r}_{y,c}(t) - \theta_2\right]}{\kappa_2}\, \frac{\left[1 - \mathrm{e}^{-\kappa_2[T-t]}\right]}{T-t}, \quad T \geqq t, \\
\lim_{T \to \infty} \mathscr{E}_t R_{y,c}(t,T) = \theta_2, \quad \lim_{t \leftarrow T} \mathscr{E}_t R_{y,c}(t,T) = \mathscr{E}_t \bar{r}_{y,c}(t) = \bar{r}_{y,c}(t).
\end{gathered} \tag{5.2}
$$

The term structures of the expected spot inflation rates, $\mathscr{E}_t\, R_{y,c}(t,T)$, for the 40 trading days considered are depicted in Figure 3. Note that the expected spot inflation rates with terms zero are equal to the realised drifts of the instantaneous spot inflation rates on the trading day considered. The overall picture suggests that the term structure of expected spot inflation rates is quite stable over time.

Next, we calculate the future expected three-month nominal spot interest rate over a future time horizon of four years from the estimated parameters. (We choose the three-month interest rate because three-month Libor is the operational target rate of the Swiss National Bank.) From equation (3.5), the expected three-month nominal spot interest rate is given by

$$
\begin{gathered}
\mathscr{E}_s R_{n,c}(t,T) = \frac{B(t,T)\, \mathscr{E}_s r_{r,c}(t) - \psi \ln\left(A(t,T)\right) - \psi_2 \ln\left(C(t,T)\right) + D(t,T)\, \mathscr{E}_s \bar{r}_{y,c}(t)}{T-t} \\
t \geqq s, \quad T = t + \tfrac{1}{4}.
\end{gathered} \tag{5.3}
$$

where the expected instantaneous spot inflation rate is given in equation (5.1). The expected real instantaneous spot interest rate is given below.

$$
\begin{gathered}
\mathscr{E}_s r_{r,c}(t) = \theta + \left[\mathscr{E}_s r_{r,c}(s) - \theta\right] \mathrm{e}^{-\kappa[t-s]}, \quad t \geqq s, \quad \mathscr{E}_s r_{r,c}(s) = r_{r,c}(s), \\
\mathscr{V}_s r_{r,c}(t) = \frac{\sigma^2\, \mathscr{E}_s r_{r,c}(s)}{\kappa}\left[\mathrm{e}^{-\kappa[t-s]} - \mathrm{e}^{-2\kappa[t-s]}\right] + \frac{\sigma^2\, \theta}{2\,\kappa}\left[1 - \mathrm{e}^{-\kappa[t-s]}\right]^2, \quad t \geqq s, \\
\mathscr{C}_s\left(r_{r,c}(t), r_{r,c}(t+\tau)\right) = \mathrm{e}^{-\kappa \tau}\, \mathscr{V}_s r_{r,c}(t), \quad \tau \geqq 0, \quad t \geqq s.
\end{gathered} \tag{5.4}
$$

We compare the future expected three-month nominal spot interest rate with the three-month nominal forward interest rate which we calculate from the *observed* nominal yield curve by equation (2.6). Apart from a risk premium, these two interest rates should exhibit the same forecasting profile over the future time horizon. This may be taken as a plausibility test of the CIR model. The sample of the 40 trading days considered is depicted in Figures 4a-4h. As you can see in these figures, the forecasting profiles are quite similar in most cases.

As another indirect test of the performance of the CIR model, we address the question of whether future expected three-month Libor, calculated from the estimated model parameters according to equation (5.3), is an unbiased estimator of future three-month Libor. Let $R_{n,c}^{\text{obs}}(t,T)$ denote the nominal spot interest rate which is observed at date t and which has a term of three months ($T-t$ = 1 / 4 years), then the ordinary least-squares regressions are written as follows:

$$R_{n,c}^{\text{obs}}(t,T) = \alpha_0 + \alpha_1 \mathscr{E}_s R_{n,c}(t,T) + u(s), \quad s \in \mathscr{S}, \quad t > s, \quad T = t + \tfrac{1}{4}, \\ t - s = 1, 7, 14, \ldots, 91 \text{ days.} \tag{5.5}$$

where again $\mathscr{S}$ denotes the set of the 40 trading dates considered. It is assumed that the disturbances, $u(s)$, are identically and independently distributed normal variates with a zero mean value and a constant variance. We consider 14 different time horizons of up to 91 days into the future. If expected three-month Libor is an unbiased estimator of future three-month Libor, then α_0 = 0 and α_1 = 1. The regression results are shown in Table 1 and depicted in Figures 5a-5d. For future time horizons of both one day and seven days, we accept the hypothesis that expected three-month Libor is an unbiased estimator of future three-month Libor. As one can see in Figures 5a-5d, the observations are clustered in a rather small range which is due to the small sample period. If we could consider a sample period of 10 years, say, then the observations would vary between 0 and 10%. Hence, it might be reasonable to argue that the 14 regressions would look different for a larger sample period. To account for this phenomenon, we rerun the 14 regressions subject to the condition that the coefficient of the constant term is equal to zero, that is, α_0 = 0. The results of these restricted least-squares regressions are shown in Table 2 and depicted in Figures 5a-5d. For all 14 future time horizons considered, we now accept the hypothesis that expected three-month Libor is an unbiased estimator of future three-month Libor.

Finally, we compare the behaviour of the interest premium or inflation risk premium, respectively, between two monetary regimes. At the beginning of the year 2000, the Swiss National Bank switched from a concept of medium-term monetary targeting to a concept with inflation forecasts as a main indicator for guiding monetary policy decisions. The old monetary policy mentioned above was operated mainly by foreign exchange swaps, whereas the new concept relies on repurchase agreements (repos) with commercial banks for short terms of one day up to several weeks. By these operations, the Swiss National Bank keeps three-month Libor - its operational target rate - within a particular band. Due to the limited database, we cannot estimate the CIR model by the full information maximum likelihood (FIML) method described in this paper before 2000. For the period 1999-2001, however, we applied the CIR model to observed yield curves by means of a multi-objective goal attainment (MOGA) method described elsewhere (Büttler (2000)). The MOGA method does not use time-series data at all, because it only requires that the theoretical yield curve fits the actual yield curve observed on a particular trading day as well as possible, given two other objective functions. The parameters estimated by the MOGA method are less stable than those estimated by the FIML method. Furthermore, some estimates are associated with a local rather than a global maximum in terms of the likelihood function. Although the term structures of expected inflation rates estimated from the MOGA method may deviate considerably from those obtained from the FIML method, the interest premia are close to each other during the period from January to May 2001, given weekly spaced data. The difference in the interest premium between these two methods is depicted in Figure 6. The maximum difference in absolute value is 31 basis points. The interest premia obtained from the MOGA method for the period between 1999 and 2001 are depicted in Figure 7. As you can see, the interest premia for all the terms considered between zero and 25 years are declining over this period. Since the middle of 2001, they have been almost zero. This reduced risk may indicate that the new concept has further increased the credibility of Swiss monetary policy.

6. Conclusions

Applying the CIR model, we determine empirically the term structure of expected spot inflation rates and the term structure of real spot interest rates from the nominal yield curves of 40 consecutive weeks. The smooth evolution of these curves over the course of time suggests that the empirical estimation is quite stable. We find the puzzling result that half the real yield curves are upward-sloping, while the other half are downward-sloping, but all the expected inflation rate curves are upward-sloping. We attribute this phenomenon to the fact that the future expected inflation rate cannot overshoot its long-run equilibrium value in the CIR model. We test the performance of the CIR model indirectly in two ways. First, we compare the time profile of the future expected nominal three-month spot interest rate with that of the three-month nominal forward interest rate implied by the observed nominal yield curve on the trading day under consideration. This test is quite satisfactory. Second, we test whether expected three-month Libor, calculated from the estimated model parameters, is an unbiased estimator of future three-month Libor for 14 different time horizons of up to 91 days into the future. We accept this hypothesis for future time horizons of both one day and seven days. With a restriction on the coefficient of the constant term, however, we accept this hypothesis for all 14 future time horizons considered. Finally, we compare the behaviour of the interest premium or inflation risk premium, respectively, between two different monetary policy regimes. We find that the interest premium has vanished since the beginning of the year 2000, when the Swiss National Bank switched from a regime with medium-term monetary targeting to a concept with inflation forecasts as a main indicator for monetary policy decisions. This reduced risk may indicate that the new concept has further increased the credibility of Swiss monetary policy.

Appendix: list of variables, functions and symbols

Subscripts:

$x_{v,k}(\cdot)$ — The first subscript of the variable x, $v = \{n, r, y\}$, denotes nominal values for $v = n$, real values for $v = r$, and values associated with the inflation rate for $v = y$. If necessary, we use a second subscript, $k = \{m, c\}$, which denotes the compounding frequency with the understanding that m denotes a compounding m times a year and c denotes the continuous compounding ($m \to \infty$).

Variables in Roman letters:

$A(t,T)$ — See equation (3.2).

$B(t,T)$ — See equation (3.2).

$C(t,T)$ — See equation (3.4).

$D(t,T)$ — See equation (3.4).

$F_v(t,T,\tau)$ — $v = \{n, r\}$. The $(\tau - T)$-year forward interest rate corresponding to a forward contract on a pure discount bond with the agreement that the forward price is fixed at date t and paid at a later date T when the discount bond will be delivered. The discount bond matures at a later date $\tau\,(\tau \geqq T \geqq t)$. It holds true that $F_v(t,t,T) = R_v(t,T)$.

$F_y(t,T,\tau)$ — The $(\tau - T)$-year forward inflation rate corresponding to future consumer price levels at future dates τ and T as seen from date $t\,(\tau \geqq T \geqq t)$. It holds true that $F_y(t,t,T) = R_y(t,T)$.

$f_v(t,T)$ — $\equiv F_v(t,T,T)$; $v = \{n, r\}$. The instantaneous forward interest rate corresponding to a forward contract on a pure discount bond with the agreement that the forward price is fixed at date t and paid at a later date T when the discount bond will be delivered. The discount bond matures at the same instant it is delivered. It holds true that $f_v(t,t) = r_v(t)$.

$f_y(t,T)$ — $\equiv F_y(t,T,T)$. The instantaneous forward inflation rate corresponding to a consumer price level at the future date T as seen from date $t\,(T \geqq t)$. It holds true that $f_y(t,t) = r_y(t)$.

$\mathbf{G}(t)$ — Volatility matrix. See equation (4.6).

H — Number of bonds selected for the cross-section data. See equation (4.2).

$\mathbf{I}$ — Identity matrix. See equation (4.8).

K — Number of bonds selected for the constraints. See equation (4.12).

$p(t)$ — The price level of consumer goods or the cost of living index, respectively, at date t.

$P_v(t,T)$ — $v = \{n, r\}$. The spot price of a pure discount bond, which is fixed and paid at the settlement date t. The debtor of the pure discount bond redeems one monetary unit when the bond matures at date T, but does not pay out any coupons during the bond's life.

$P_y(t,T)$ — $\equiv p(t) \,/\, p(T)$, $(T \geqq t)$. The purchasing power of money at the future date T in nominal terms at current prices as seen from date t.

$P_n^{\text{obs}}(s,T_k)$ — Observed prices of the nominal pure discount bonds in nominal terms on the trading day under consideration, s, for various term dates T_k, $k = 1, \ldots, K$.

$\mathrm{P}_v(t,T,\tau)$ — $v = \{n, r\}$. The $(\tau - T)$-year forward price of a forward contract on a pure discount bond with the agreement that the forward price is fixed at date t and paid at a later date T when the pure discount bond will be delivered. The pure discount bond matures at a later date $\tau\,(\tau \geqq T \geqq t)$. In this case, the forward price is equal to the futures price of a discount bond (see Hull (1997), p 95). It holds true that $\mathrm{P}_v(t,t,T) = P_v(t,T)$.

$\mathrm{P}_y(t,T,\tau)$ — $\equiv p(T) \,/\, p(\tau)$. The $(\tau - T)$-year forward purchasing power of money at date τ at prices of the earlier date T as seen from date $t\,(\tau \geqq T \geqq t)$. It holds true that $\mathrm{P}_y(t,t,T) = P_y(t,T)$.

$\mathbf{Q}(t)$ Upper-triangle matrix. See equation (4.8).

$R_v(t, T)$ $v = \{n, r\}$. The spot interest rate of a pure discount bond with its price fixed at date t and which matures at date T ($T \geqq t$). The spot interest rate is also denoted as the yield of the discount bond.

$R^{\text{obs}}_{n,c}(t,T)$ The nominal spot interest rate which is observed at date t and whose contract matures at date T.

$R_y(t, T)$ The spot inflation rate corresponding to a consumer price level at the future date T as seen from date t ($T \geqq t$).

$r_v(t)$ $\equiv R_v(t, t)$; $v = \{n, r\}$. The instantaneous spot interest rate of a pure discount bond with its price fixed at date t and which matures at the same instant.

$r_y(t)$ $\equiv R_y(t, t)$. The instantaneous spot inflation rate at date t.

$\bar{r}_y(t)$ The drift of the instantaneous spot inflation rate or the "expected" instantaneous spot inflation rate, respectively, at date t. $\mathcal{E}_t\, r_y(t) = \mathcal{E}_t\, \bar{r}_y(t) = \bar{r}_y(t)$.

$\mathbf{S}(t)$ Variance-covariance matrix. See equation (4.7).

$x_j(t)$ Transformed, trend-adjusted variables considered in the sample of pooled time-series and cross-section data, $j = 1, \ldots, 3 + H$. See equation (4.8).

$y_j(t)$ Trend-adjusted variables considered in the sample of pooled time-series and cross-section data, $j = 1, \ldots, 3 + H$. See equation (4.5).

$z_j(t)$ Wiener processes, $j = 1, \ldots, 3 + H$. A Wiener process has normally distributed increments with mean zero and variance dt. Any two increments at two different dates are independent. See equations (3.1), (3.3), (4.2) and (4.3).

$u(s)$ Disturbance on the trading day s in linear regression. See equation (5.5).

Variables in Greek letters:

α_0, α_1 Regression coefficients. See equation (5.5).

$\boldsymbol{\beta}$ $\equiv [\kappa, \theta, \sigma, \lambda, \kappa_2, \theta_2, \sigma_2, \sigma_p, \rho, \sigma_0]'$. Parameter vector to be estimated. See equation (4.10).

ψ See equation (3.2).

ψ_2 See equation (3.4).

$\in$ Error tolerance. See equation (4.12).

γ See equation (3.2).

η_k $k = \{m, c\}$. The interest premium.

ξ See equation (3.4).

κ Speed of adjustment of the increments of the real instantaneous spot interest rate. See equation (3.1).

κ_2 Speed of adjustment of the increments of the instantaneous spot inflation rate. See equation (3.3).

λ Factor risk premium. See equation (3.2).

$\Pi(t, T)$ $\equiv P_n(t, T)\ /\ p(t)$. The spot price of a pure nominal discount bond in real terms which is fixed and paid at the settlement date t.

ρ Correlation coefficient. See equation (3.3).

σ Volatility parameter of the increments of the real instantaneous spot interest rate. See equation (3.1).

σ_2 Volatility parameter of the increments of the instantaneous spot inflation rate. See equation (3.3).

σ_p	Volatility parameter of the increments of the consumer price level. See equation (3.3).
σ_0	Volatility parameter of the increments of the nominal discount bond prices in real terms. See equation (4.2).
$\mathbf{\Sigma}(t)$	Variance-covariance matrix. See equation (4.3).
θ	Long-run equilibrium value of the real instantaneous spot interest rate. See equation (3.1).
θ_2	Long-run equilibrium value of the instantaneous spot inflation rate. See equation (3.3).

Functions:

$\mathcal{N}(\mu, \mathbf{\Sigma})$	Gaussian or normal distribution with mean vector μ and variance-covariance matrix $\mathbf{\Sigma}$. For the (n x 1) column vector $\mathbf{x}$, the notation $\mathbf{x} \sim \mathcal{N}(\mu, \mathbf{\Sigma})$ means that $\mathbf{x}$ has normal distribution. The normal probability density function is given by $f(\mathbf{x}) = [2\pi]^{-n/2} \mid \mathbf{\Sigma} \mid^{-1/2} \exp\left(-\frac{1}{2}[\mathbf{x}-\boldsymbol{\mu}]' \mathbf{\Sigma}^{-1} [\mathbf{x}-\boldsymbol{\mu}]\right)$.
$\mathrm{L}(\mathbf{x} \mid \boldsymbol{\beta})$	Logarithm of the likelihood function for a given sample $\mathbf{x}$ in terms of parameters $\boldsymbol{\beta}$.
exp	$\equiv$ e. The exponential function.
ln	The natural logarithm.

Symbols:

$\mathcal{E}$	The expectation operator. For a variable x with probability density function $f(x)$, the expectation of x is defined as $\mathcal{E}x \equiv \int f(x)\,\mathrm{d}x$.
$\mathcal{C}$	The covariance operator. For variables x and y, the covariance between x and y is defined as $\mathcal{C}(x,y) \equiv \mathcal{E}\{(x-\mathcal{E}x)\,(y-\mathcal{E}y)\}$.
$\mathcal{V}$	The variance operator. For a variable x, the variance is defined as $\mathcal{V}x \equiv \mathcal{E}\{(x-\mathcal{E}x)^2\}$.
$\lvert\cdot\rvert$	Determinant of a matrix.
$'$	Transposition mark of a vector or of a matrix.
Δ	Backward difference operator in discrete time, ie $\Delta x(t) = x(t) - x(t-1)$.
$\mathcal{S}$	The set of trading days or settlement days considered in the paper. The 40 trading days are approximately weekly spaced between 14 August 2000 and 14 May 2001.

Figure 1a: Parameter estimates

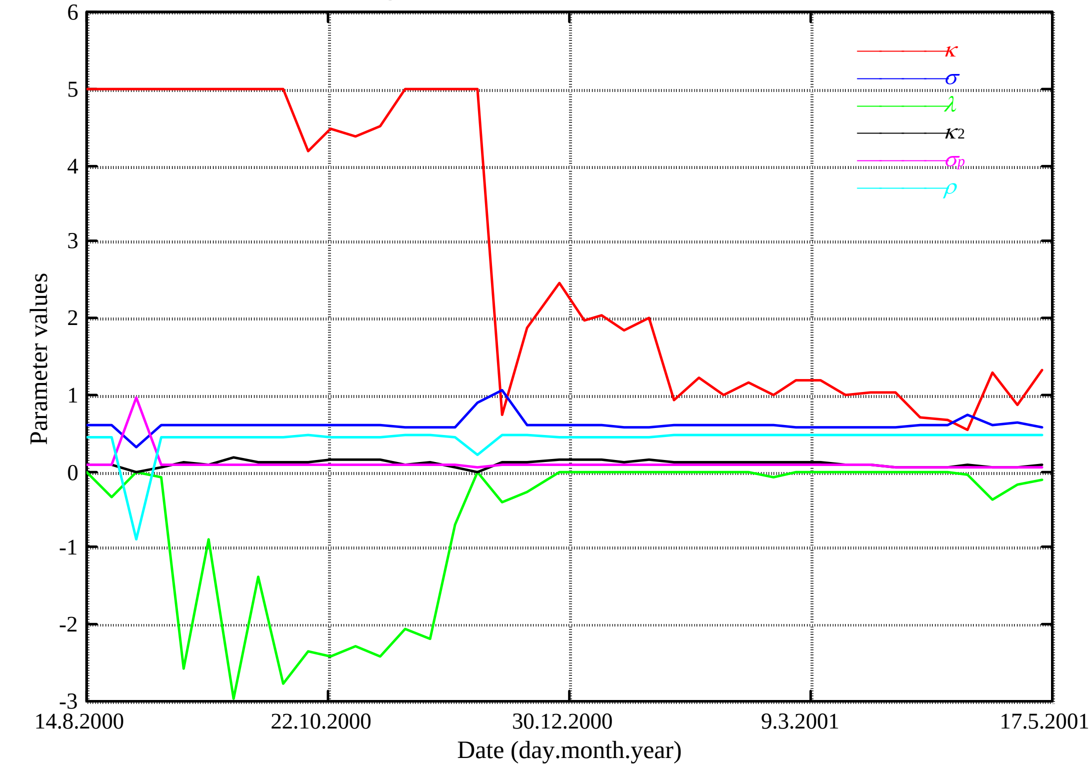


Figure 1b: Parameter estimates continued

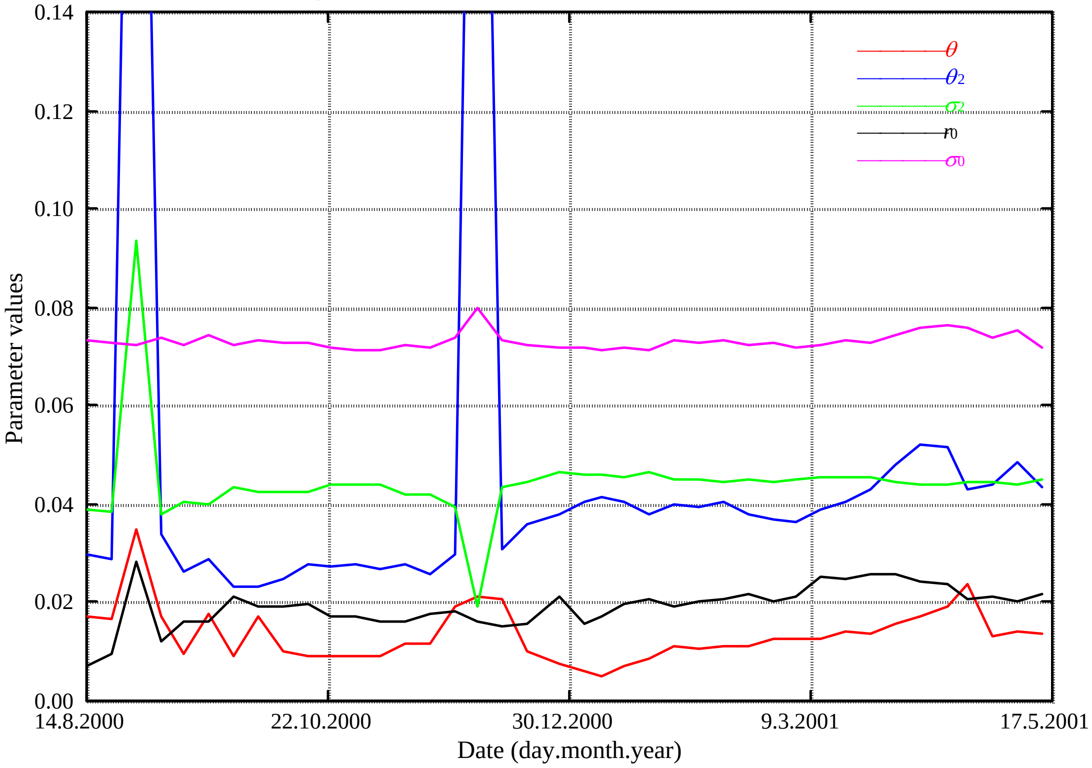


Figure 2: Nominal and real yield curves in percent per annum

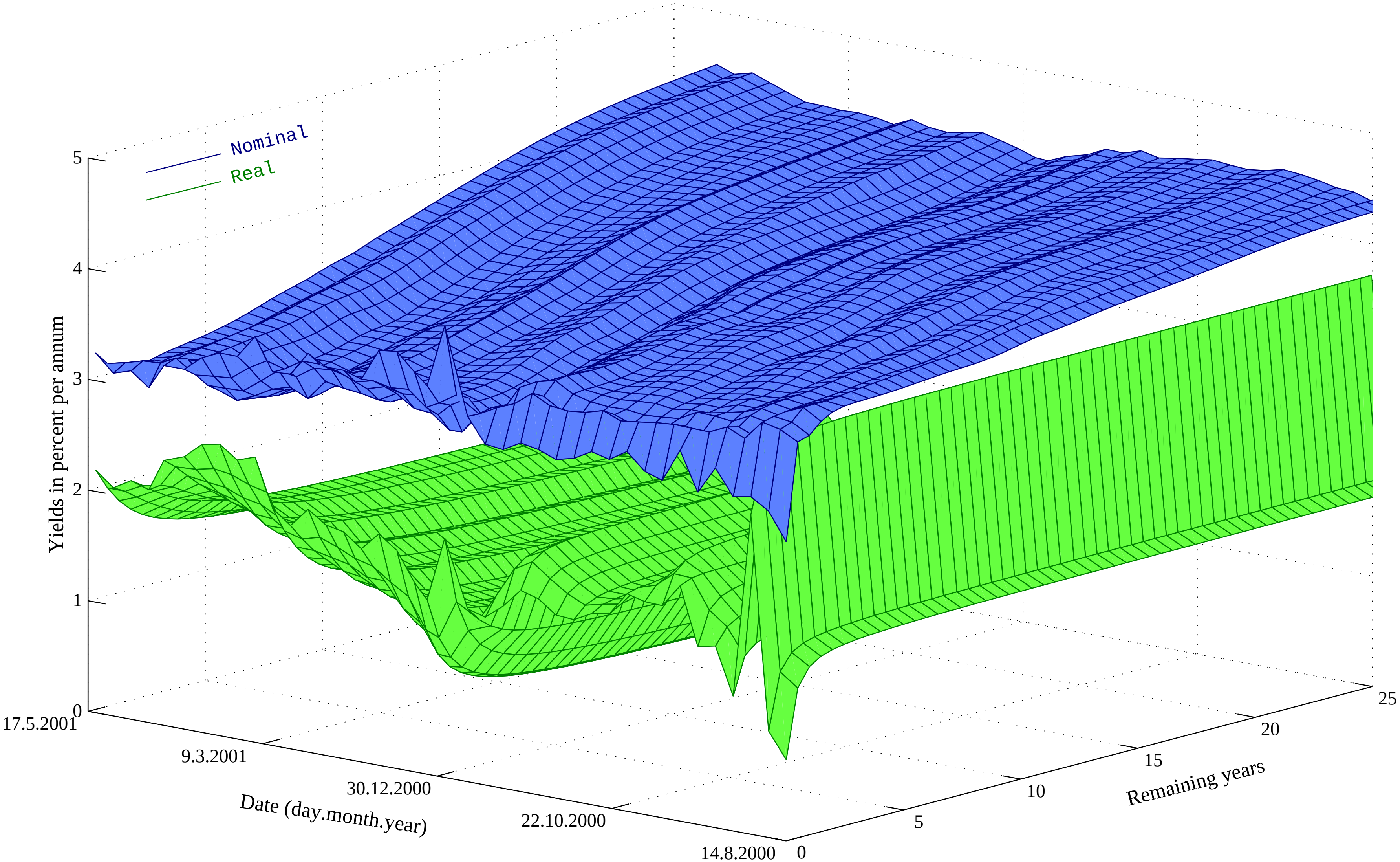


Figure 3: Expected spot inflation rates in percent per annum

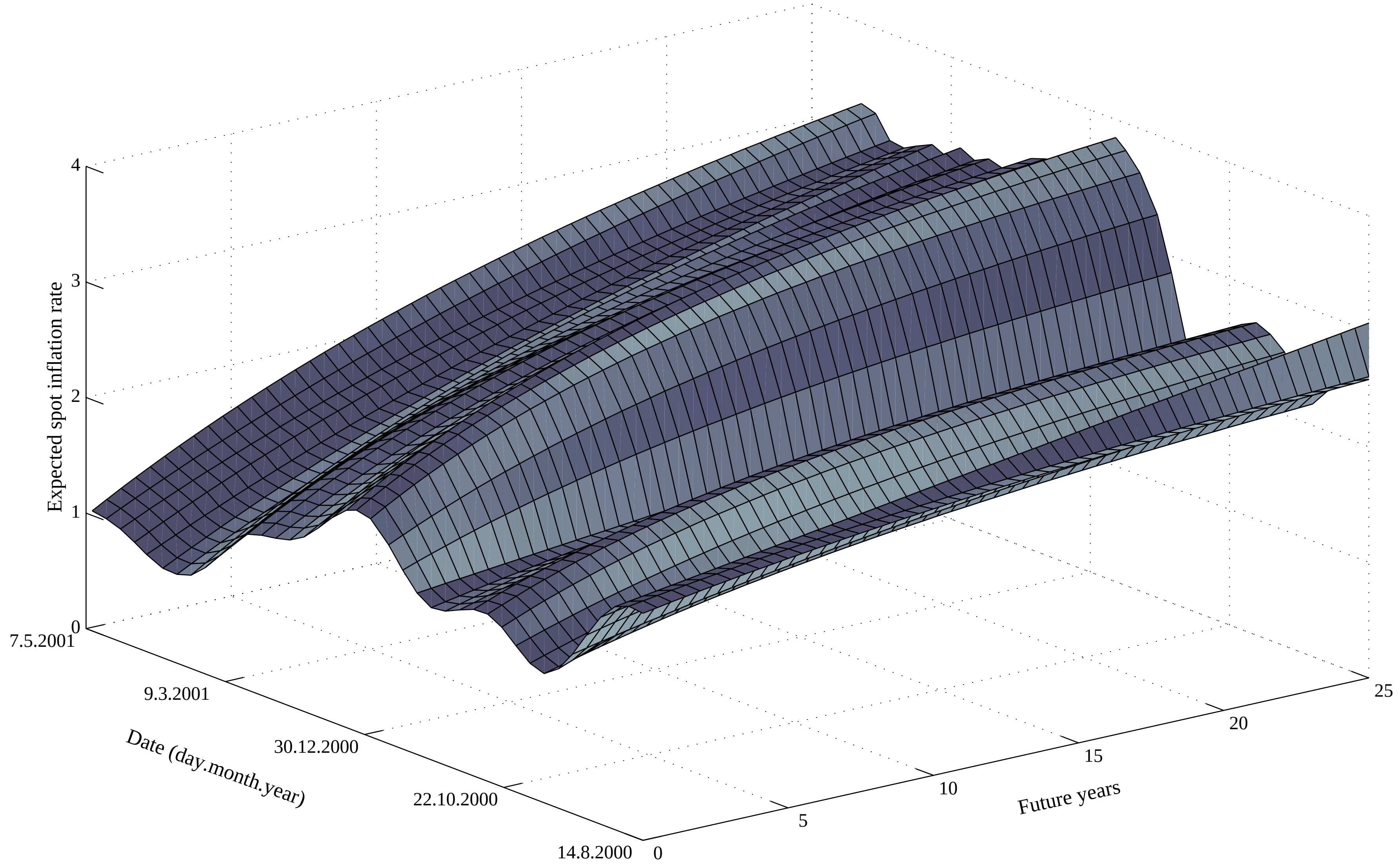


Figure 4a

Expected three-month spot interest rates in percentages per annum

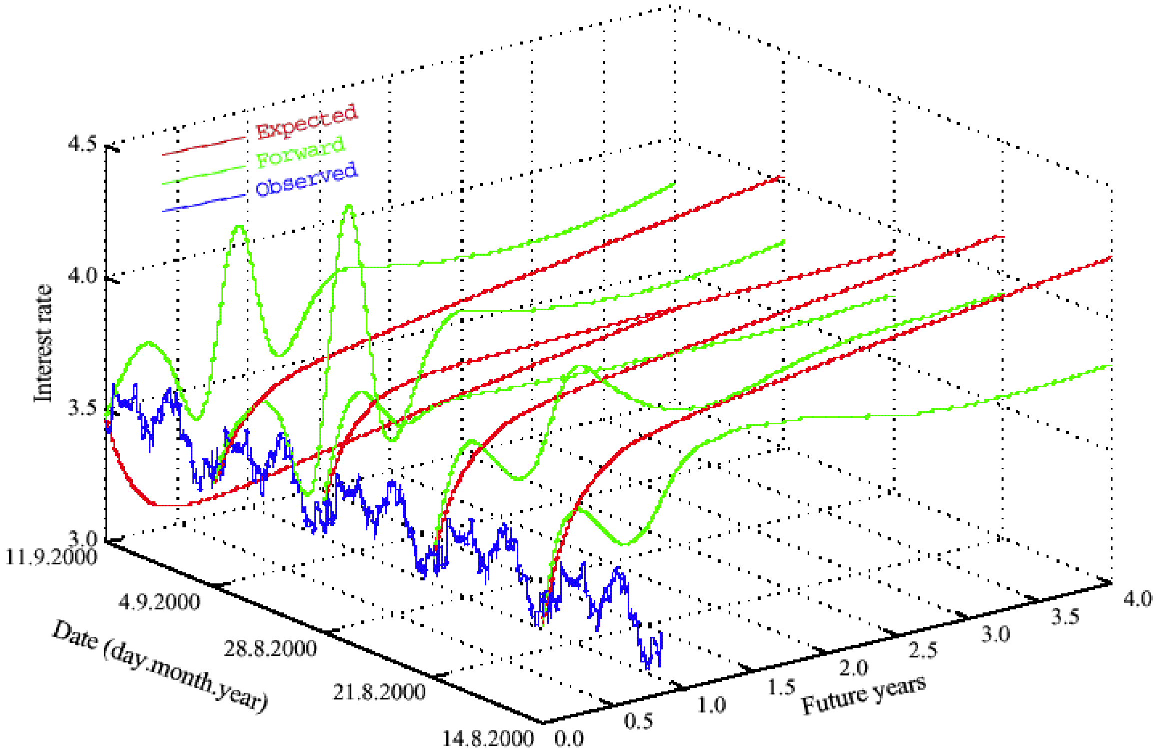


Figure 4b

Expected three-month spot interest rates in percentages per annum

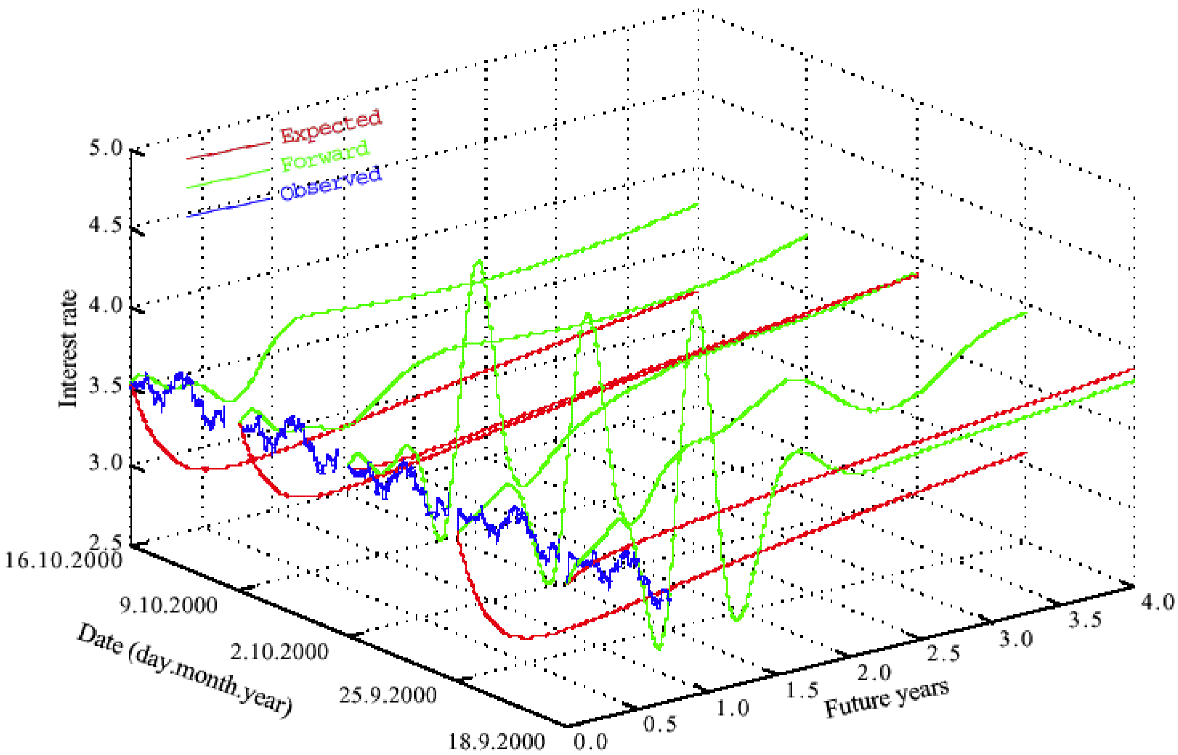


Figure 4c

Expected three-month spot interest rates in percentages per annum

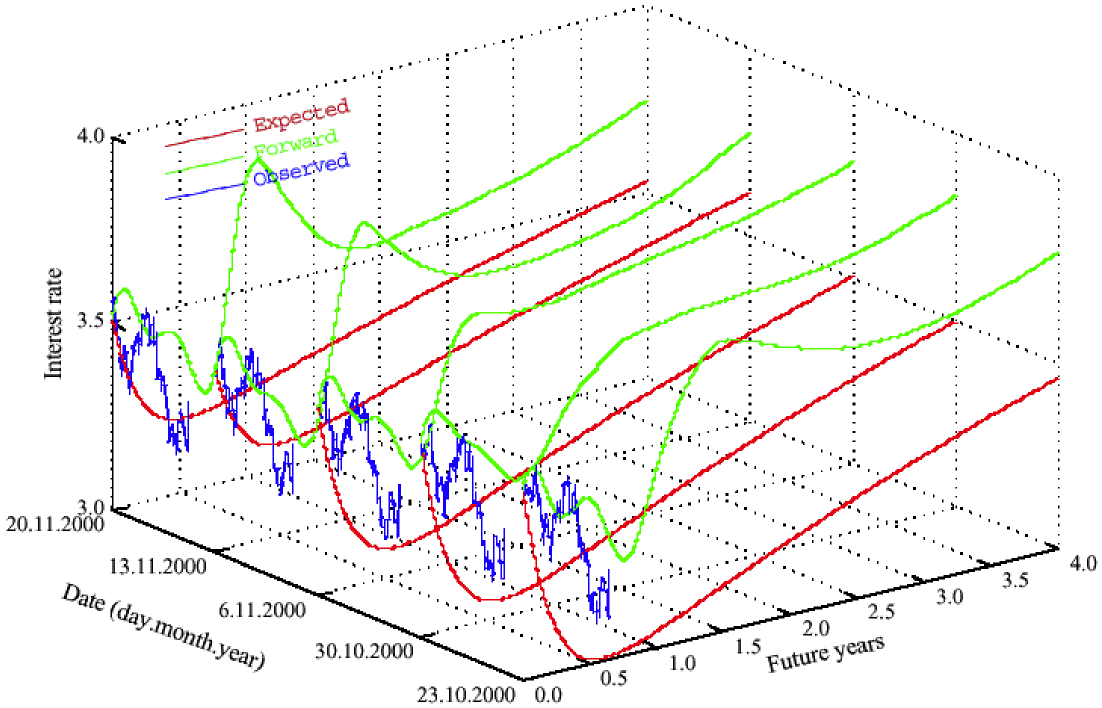


Figure 4d

Expected three-month spot interest rates in percentages per annum

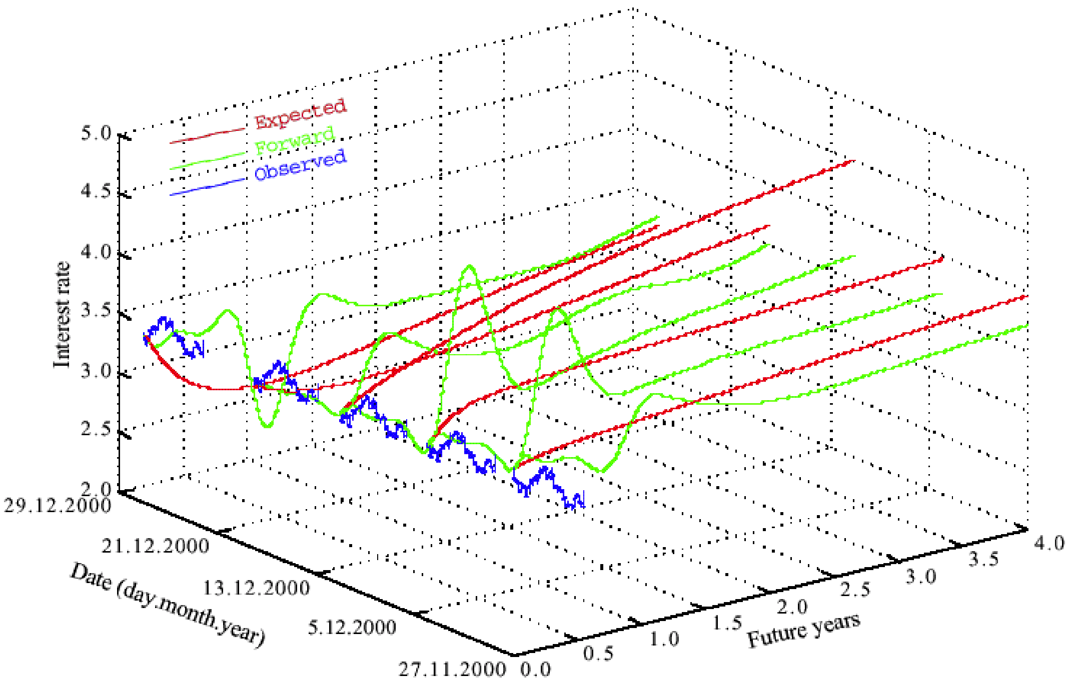


Figure 4e

Expected three-month spot interest rates in percentages per annum

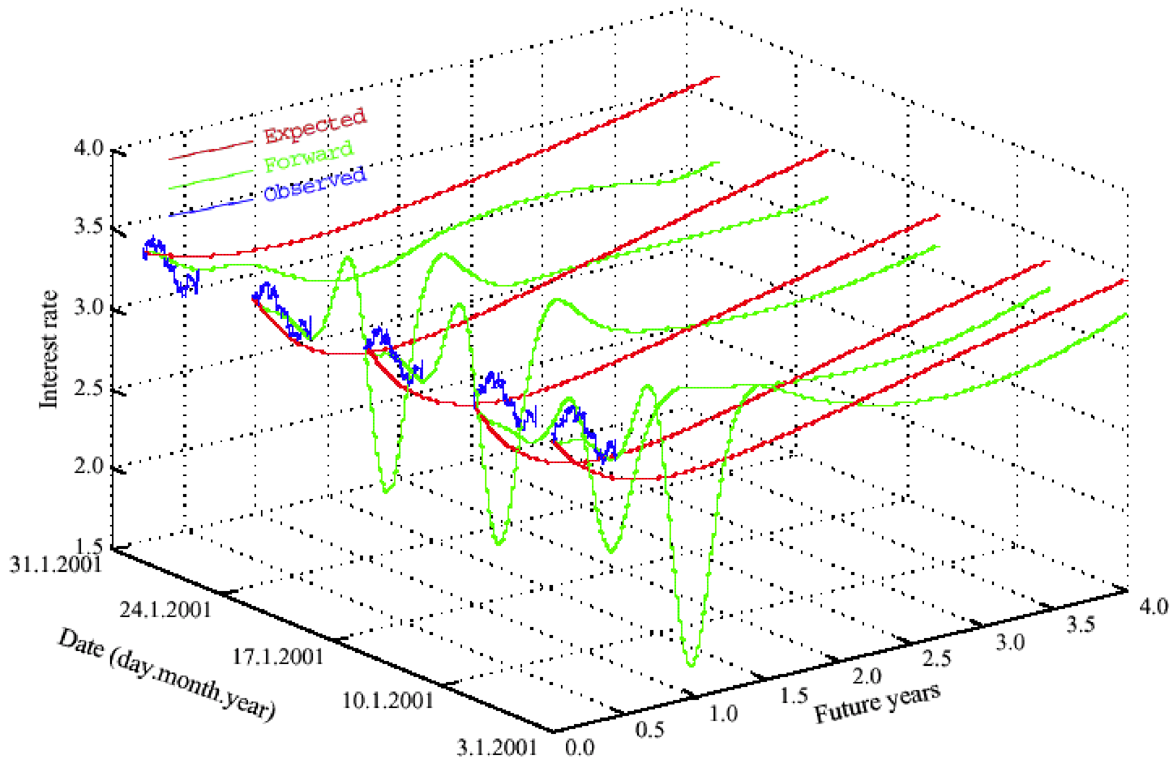


Figure 4f

Expected three-month spot interest rates in percentages per annum

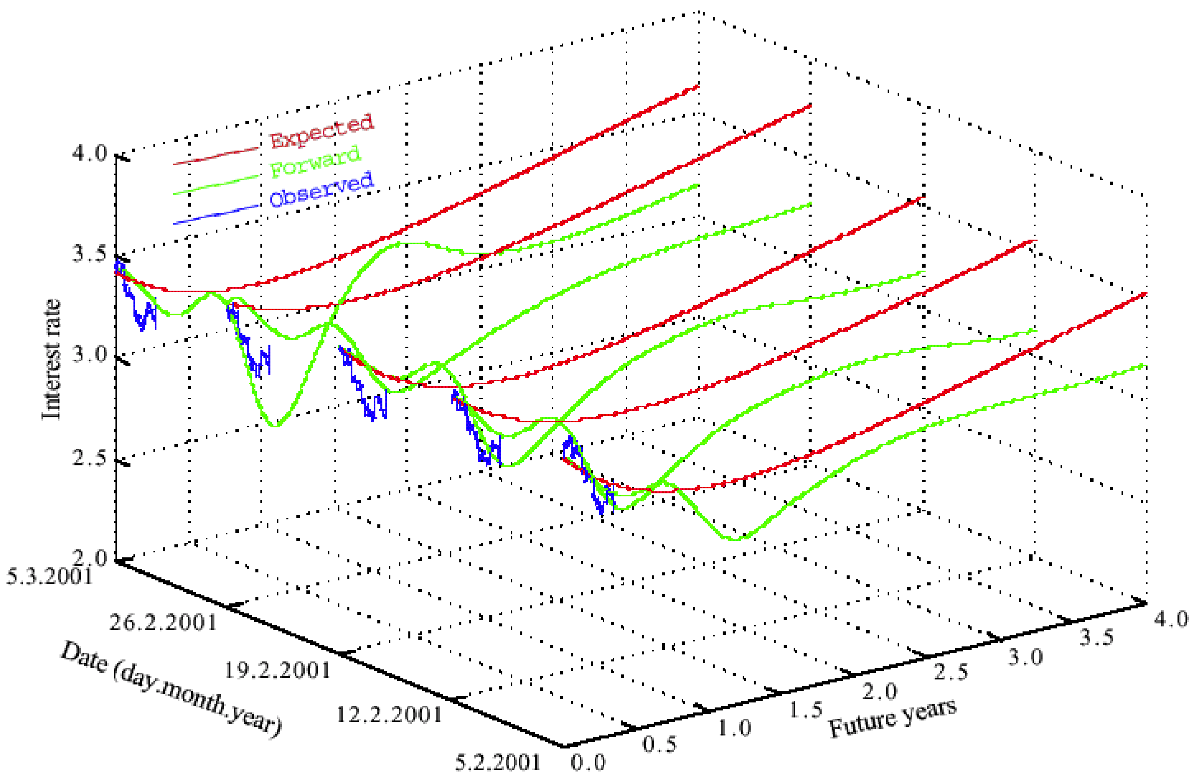


Figure 4g

Expected three-month spot interest rates in percentages per annum

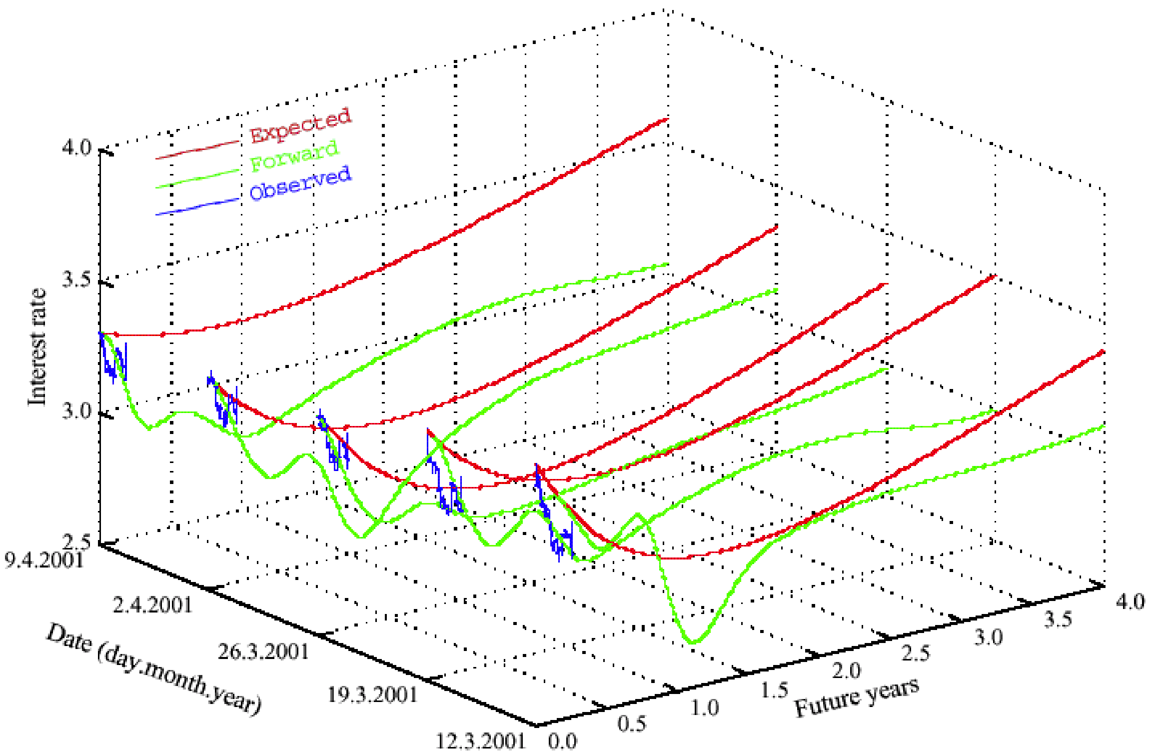


Figure 4h

Expected three-month spot interest rates in percentages per annum

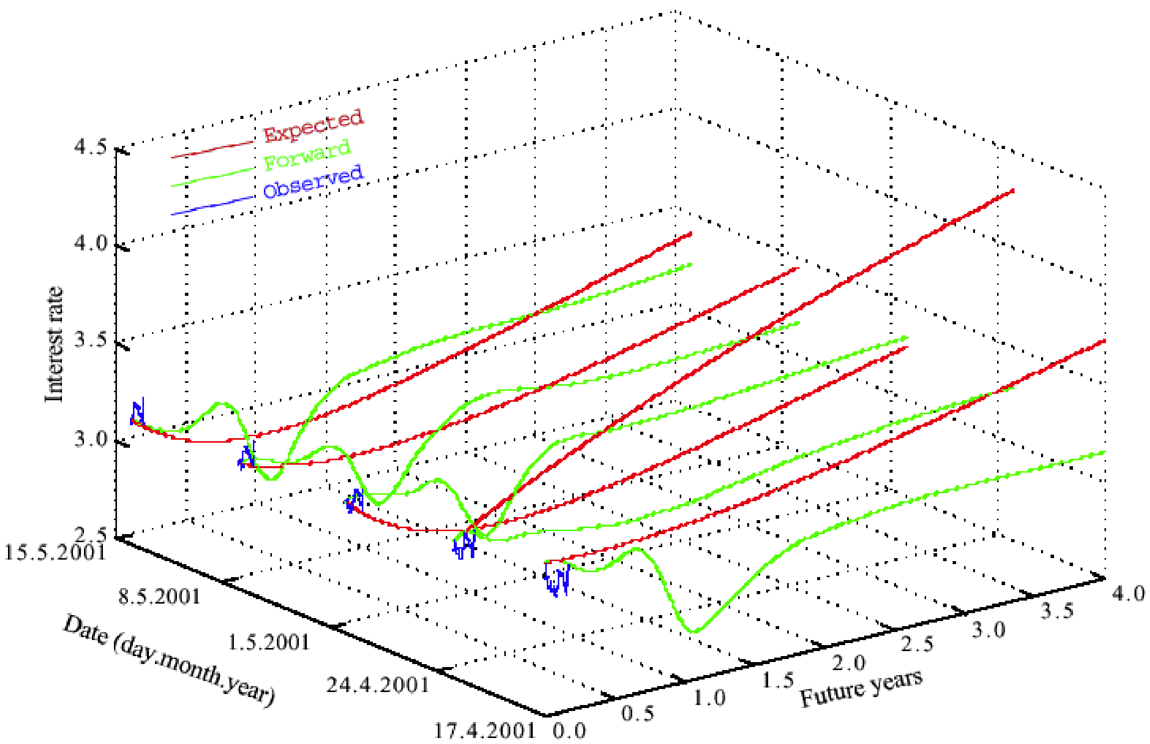


Table 1

Ordinary least-squares regressions

Independent variables	Future time horizon in days						
	1	**7**	**14**	**21**	**28**	**35**	**42**
Constant term	0.0496 (0.1151)	0.4750 (0.3438)	1.0890 (0.4315)	1.8123 (0.4846)	2.0643 (0.4853)	2.5239 (0.4723)	2.5909 (0.4406)
Expected 3M Libor	0.9862 (0.0337)	0.8636 (0.1011)	0.6843 (0.1273)	0.4697 (0.1435)	0.3943 (0.1442)	0.2570 (0.1407	0.2364 (0.1317)
Coeff. of determ.	0.9575	0.6577	0.4319	0.2199	0.1644	0.0807	0.0782
F-ratio	0.2847	1.4230	4.0122	7.6567	9.8006	15.2113	18.4928
Accept hypothesis	yes	yes	no	no	no	no	no
Independent variables	**Future time horizon in days**						
	49	**56**	**63**	**70**	**77**	**84**	**91**
Constant term	2.6048 (0.4097)	2.7086 (0.3728)	2.7893 (0.3464)	2.6624 (0.3171)	2.6536 (0.2982)	2.5929 (0.2782)	2.6242 (0.2684)
Expected 3M Libor	0.2309 (0.1227)	0.1975 (0.1120)	0.1715 (0.1042)	0.2080 (0.0956)	0.2084 (0.0901)	0.2255 (0.0842)	0.2138 (0.0814)
Coeff. of determ.	0.0852	0.0757	0.0665	0.1108	0.1234	0.1588	0.1536
F-ratio	21.5131	27.6284	33.6554	36.5186	40.6396	44.5484	48.6205
Accept hypothesis	no	no	no	no	no	no	no

Comments: The dependent variable is observed three-month Libor. Values in parentheses are standard deviations. The size of the sample of settlement days is 40. We test the joint hypothesis that the coefficient of the constant term is equal to zero and that the coefficient of expected three-month Libor is equal to one. The hypothesis is “accepted” if the *F*-ratio is less than the corresponding critical *F*-value. The critical one-tailed *F*-value is equal to 3.2448 for a probability of 95%.

Table 2

Restricted least-squares regressions

Independent variables	Future time horizon in days						
	1	**7**	**14**	**21**	**28**	**35**	**42**
Constant term	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
Expected 3M Libor	1.0007 (0.0011)	1.0031 (0.0032)	1.0055 (0.0042)	1.0060 (0.0052)	1.0070 (0.0057)	1.0084 (0.0061)	1.0097 (0.0063)
Partial coeff. of det.	0.9575	0.6577	0.4319	0.2199	0.1644	0.0807	0.0782
t-ratio	0.6197	0.9685	1.2863	1.1529	1.2285	1.3680	1.5519
Accept hypothesis	yes	yes	yes	yes	yes	yes	yes
Independent variables	**Future time horizon in days**						
	49	**56**	**63**	**70**	**77**	**84**	**91**
Constant term	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
Expected 3M Libor	1.0102 (0.0063)	1.0097 (0.0062)	1.0096 (0.0061)	1.0095 (0.0060)	1.0086 (0.0059)	1.0086 (0.0058)	1.0075 (0.0058)
Partial coeff. of det.	0.0852	0.0757	0.0665	0.1108	0.1234	0.1588	0.1536
t-ratio	1.6137	1.5694	1.5668	1.5892	1.4522	1.4916	1.2812
Accept hypothesis	yes	yes	yes	yes	yes	yes	yes

Comments: The dependent variable is observed three-month Libor. Values in parentheses are standard deviations. The size of the sample of settlement days is 40. We test the hypothesis that the coefficient of expected three-month Libor is equal to one, given the restriction that the coefficient of the constant term is equal to zero. The hypothesis is “accepted” if the *t*-ratio is less than the corresponding critical *t*-value. The critical two-tailed *t*-value is equal to 2.0244 for a probability of 95%.

Figure 5a: 1-day ahead forecast

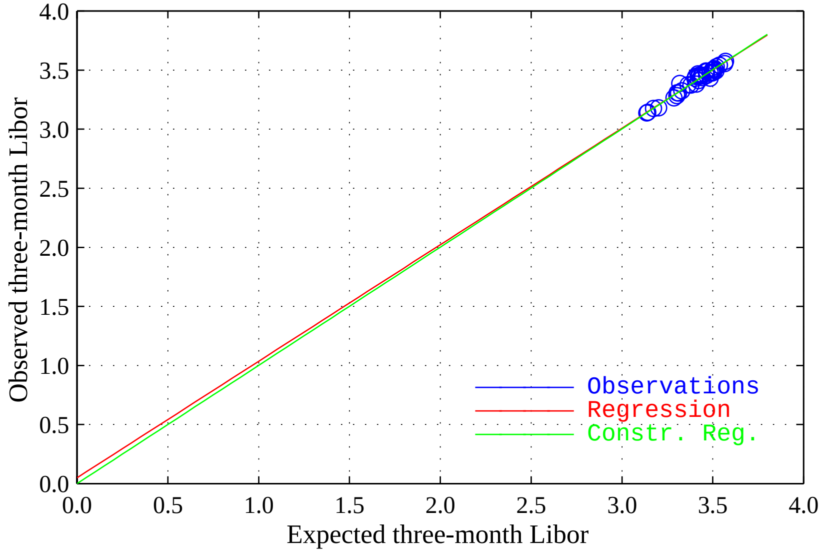


Figure 5b: 7-day ahead forecast

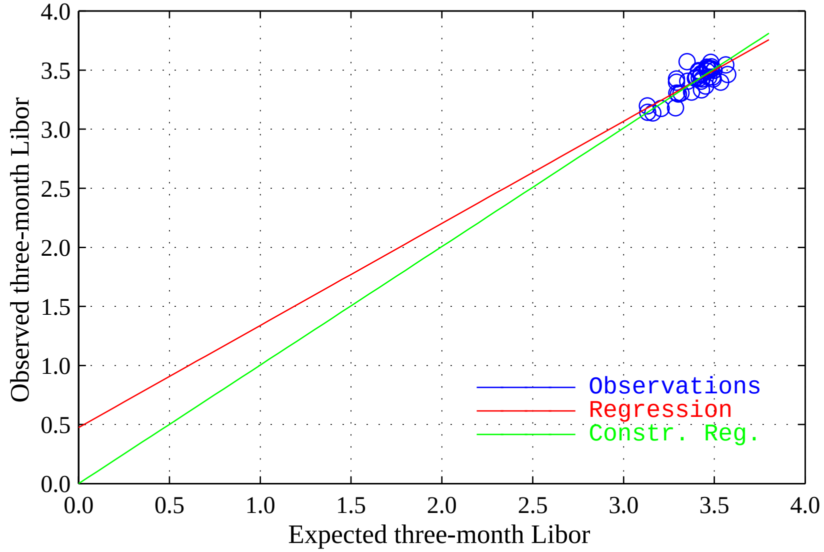


Figure 5c: 84-day ahead forecast

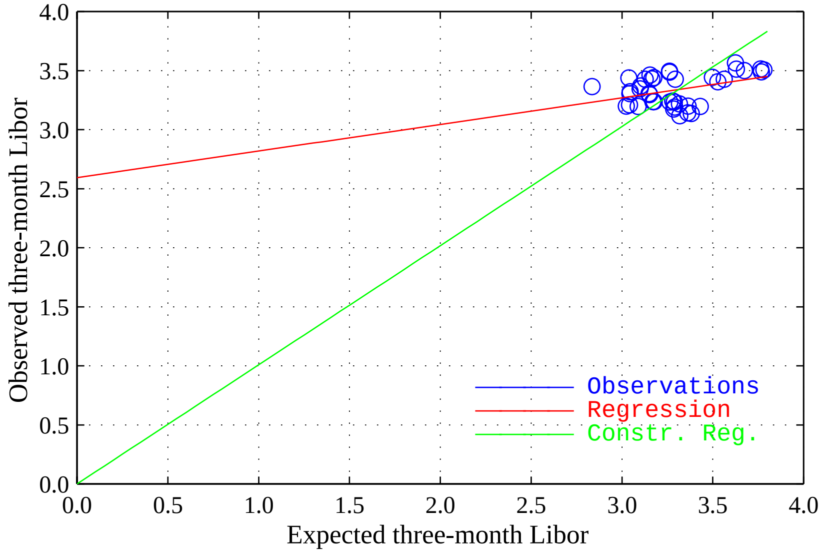


Figure 5d: 91-day ahead forecast

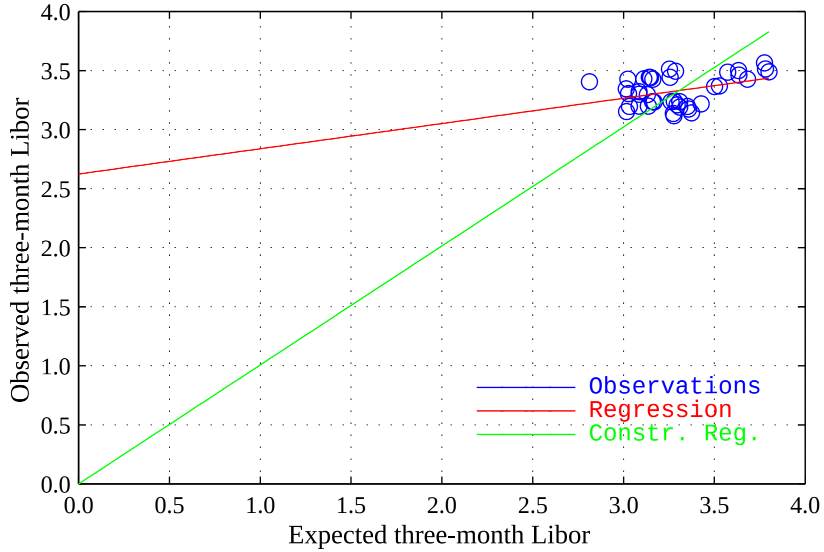


Figure 6: Premium difference between FIML and MOGA methods in percentage points

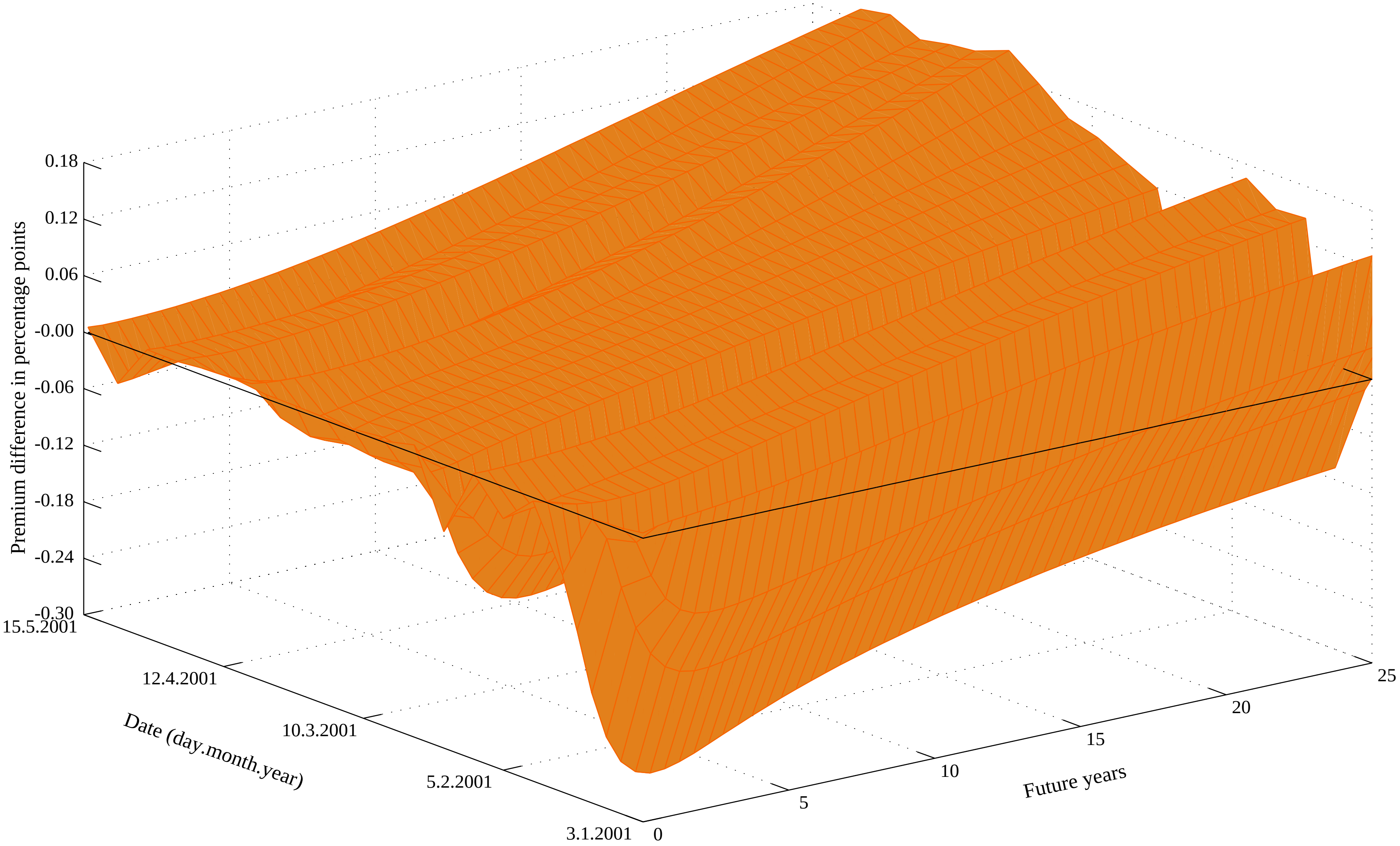
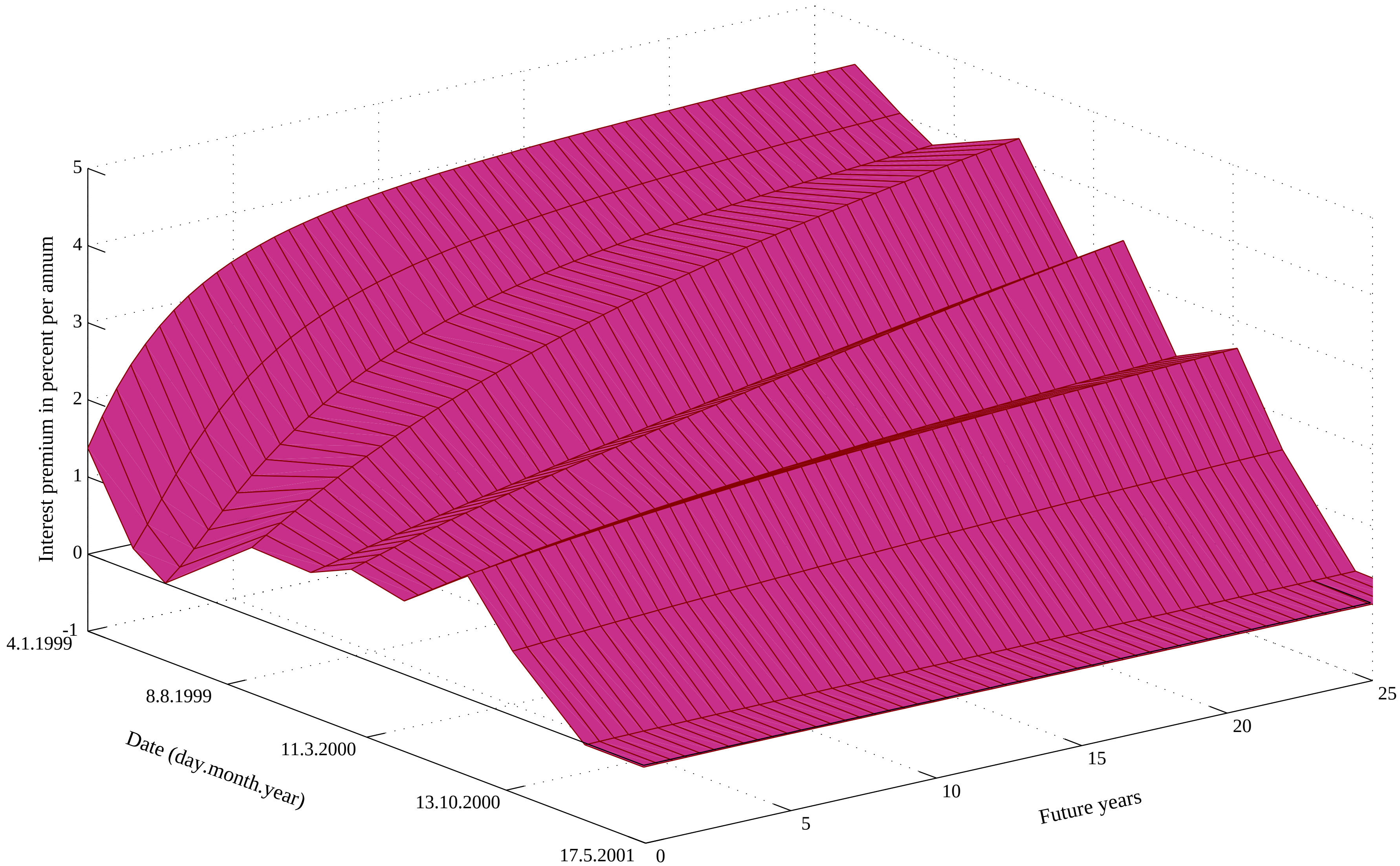


Figure 7: Interest premium in percent per annum



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