

# The effect of VaR-based risk management on asset prices and the volatility smile<sup>1</sup>

Arjan Berkelaar, World Bank,<sup>2</sup> Phornchanok Cumperayot, Erasmus University, Rotterdam, and Roy Kouwenberg, Aegon Asset Management, The Hague

## Abstract

Value-at-risk (VaR) has become the standard criterion for assessing risk in the financial industry. Given the widespread usage of VaR, it becomes increasingly important to study the effects of VaR-based risk management on the prices of stocks and options. We solve a continuous-time asset pricing model, based on Lucas (1978) and Basak and Shapiro (2001), to investigate these effects. We find that the presence of risk managers tends to reduce market volatility, as intended. However, in some cases VaR risk management undesirably raises the probability of extreme losses. Finally, we demonstrate that option prices in an economy with VaR risk managers display a volatility smile.

## 1. Introduction

Many financial institutions and non-financial firms nowadays publicly report value-at-risk (VaR), a risk measure for potential losses. Internal uses of VaR and other sophisticated risk measures are on the rise in many financial institutions, where, for example, a bank's risk committee may set VaR limits, both amounts and probabilities, for trading operations and fund management. At the industrial level, supervisors use VaR as a standard summary of market risk exposure.<sup>3</sup> An advantage of the VaR measure, following from extreme value theory, is that it can be computed without full knowledge of the return distribution. Semi-parametric or fully non-parametric estimation methods are available for downside risk estimation. Furthermore, at a sufficiently low confidence level the VaR measure explicitly focuses risk managers' and regulators' attention on infrequent but potentially catastrophic extreme losses.

Given the widespread use of VaR-based risk management, it becomes increasingly important to study the effects on the stock market and the option market of these constraints. For example, institutions with a VaR constraint might be willing to buy out-of-the-money put options on the market portfolio in order to limit their downside risk. If multiple institutions follow the same risk management strategy, then this will clearly lift the equilibrium prices of these options. Also the shape of the stock return distribution in equilibrium will be affected by the collective risk management efforts. As a result, it might even be the case that the distribution of stock returns will become more heavy-tailed. This would imply that the attempt to handle market risk, and thus to reduce default risk, has adversely raised the probability of such events.

Recently, Basak and Shapiro (2001) have derived the optimal investment policies for investors who maximise utility, subject to a VaR constraint, and found some surprising features of VaR usage. They show, in a partial equilibrium framework, that a VaR risk manager often has a higher loss in extremely

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<sup>2</sup> Corresponding author: World Bank, Investment Management Department (MC7-300), 1818 H Street NW, Washington DC 20433, USA, tel: +1 202 473 7941, fax: +1 202 477 9015, e-mail: [aberkelaar@worldbank.org](mailto:aberkelaar@worldbank.org). This paper reflects the personal views of the authors and not those of the World Bank. We would like to thank Suleyman Basak and Alex Shapiro for their helpful comments.

<sup>3</sup> The Bank for International Settlements (BIS) mandates internationally active financial institutions in the G10 countries to report VaR estimates and to maintain regulatory capital to cover market risk.

bad states than a non-risk manager. The risk manager reduces his losses in states that occur with  $(100 - \alpha)\%$  probability, but seems to ignore the  $\alpha\%$  of states that are not included in the computation of VaR. Starting from this equilibrium framework based on the Lucas pure exchange economy, in this paper we aim to further investigate Basak and Shapiro's (2001) very interesting and relevant question regarding the usefulness of VaR-based risk management.

In our economic setup, agents maximise the expected utility of intermediate consumption up to a finite planning horizon  $T$  and the expected utility of terminal wealth at the horizon. A portion of the investors in the economy are subject to a VaR risk management constraint, which restricts the probability of losses at the planning horizon  $T$ . As a result of our setup, asset prices do not drop to zero at the planning horizon and, moreover, we can ignore the unrealistic jump in asset prices that occurs just after the horizon of the VaR constraint, as in Basak and Shapiro (2001). We find that the VaR agents' investment strategies, depending on the state of nature, directly determine market volatility, the equilibrium stock price and the implied volatilities of options. In general VaR-based risk management tends to reduce the volatility of the stock returns in equilibrium and hence the regulation has the desired effect. In most cases the stock return distribution has a relatively thin left tail and positive skewness, which reduces the probability of severe losses relative to a benchmark economy without risk managers.

However, we also find that in some cases VaR-based risk management adversely amplifies default risk through a relatively heavier left tail of the return distribution. In very bad states the VaR risk managers switch to a gambling strategy that pushes up market risk. The adverse effects of this gambling strategy are typically strong when the investors consume a large share of their wealth, or when the VaR constraint has a relatively high maximum loss probability  $\alpha$ . Additionally, we study option prices in the VaR economy. We find that the presence of VaR risk managers tends to reduce European option prices, and hence the implied volatilities of these options. Moreover, we find that the implied volatilities display a smile, as often observed in practice, unlike the benchmark economy, where implied volatility is constant.

We conclude that VaR regulation performs well most of the time, as it reduces the volatility of the stock returns and it limits the probability of losses. However, in some special cases, the VaR constraint can also adversely increase the likelihood of extremely negative returns. This negative side effect typically occurs if the investors in the economy have a strong preference for consumption instead of terminal wealth, or when the VaR constraint is rather loose (ie with high  $\alpha$ ). Note that the negative consequences of VaR-based risk management are mainly due to the "all or nothing" gambling attitude of the optimal investment strategy in case of losses, which might seem rather unnatural. In this paper we argue that the gambling strategy of a VaR risk manager might not be that unnatural for many investors, as it is closely related to the optimal strategy of loss-averse agents with the utility function of prospect theory.

Prospect theory is a framework for decision-making under uncertainty developed by the psychologists Kahneman and Tversky (1979), based on behaviour observed in experiments. The utility function of prospect theory is defined over gains and losses, relative to a reference point. The function is much steeper over losses than over gains and also has a kink in the reference point. Loss-averse agents dislike losses, even if they are very small, and therefore their optimal investment strategy tries to keep wealth above the reference point.<sup>4</sup> Once a loss-averse investor's wealth drops below the reference point, he tries to make up his previous losses by following a risky investment strategy. Hence, similar to a VaR agent, a loss-averse agent tries to limit losses most of the time, but starts taking risky bets once his wealth drops below the reference point. The optimal investment strategy under a VaR constraint might therefore seem rather natural for loss-averse investors. Or, conversely, one could argue that a VaR constraint imposes a minimum level of "loss aversion" on all investors affected by the regulation.

This paper is organised as follows: in Section 2, we define our dynamic economy and the market-clearing conditions required in order to solve for the equilibrium prices. Individual optimal investment decisions are also discussed. The general equilibrium solutions and analysis are presented in Section 3. We focus on the total return distribution of stocks and the prices of European options in the presence of VaR risk managers. Section 4 investigates the similarity between risk management

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<sup>4</sup> This behaviour is induced by the kink in the utility function, ie first-order risk aversion; see Berkelaar and Kouwenberg (2001a).

policies based on VaR and the optimal investment strategy of loss-averse investors. Section 4 finally summarises the paper and presents our conclusions.

## 2. Economic setting

### 2.1 A dynamic economy

In this section, the pure exchange economy of Lucas (1978) is formulated in a continuous-time stochastic framework. Suppose in a finite horizon,  $[0, T]$ , economy, there are heterogeneous economic agents with constant relative risk aversion (CRRA). The agents are assumed to trade one riskless bond and one risky stock continuously in a market without transaction costs.<sup>5</sup> There is one consumption good, which serves as the numeraire for other quantities, ie prices and dividends are measured in units of this good. The bond is in zero net supply, while the stock is in constant net supply of 1 and pays out dividends at the rate  $\delta(t)$ , for  $t \in [0, T]$ . The dividend rate is presumed to follow a Geometric Brownian motion:<sup>6</sup>

$$d\delta(t) = \mu_\delta \delta(t) dt + \sigma_\delta \delta(t) dB(t) \quad (1)$$

with  $\mu_\delta > 0$  and  $\sigma_\delta > 0$  constant.

The equilibrium processes of the riskless money market account  $S_0(t)$  and the stock price  $S_1(t)$  are the following diffusions, as will be shown in Section 3.1:

$$dS_0(t) = r(t)S_0(t)dt, \quad (2)$$

$$dS_1(t) + \delta(t) = \mu(t)S_1(t)dt + \sigma(t)S_1(t)dB(t),$$

where the interest rate  $r(t)$ , the drift rate  $\mu(t)$  and the volatility  $\sigma(t)$  are adapted processes and possibly path-dependent.

As we assume a dynamically complete market, these price processes ensure the existence of a unique state price density (or pricing kernel)  $\xi(t)$ , following the process

$$\frac{d\xi(t)}{\xi(t)} = -r(t)dt - \kappa(t)dB(t), \quad \xi(0) = 1, \quad (3)$$

where  $\kappa(t) = (\mu(t) - r(t))/\sigma(t)$  denotes the process for the market price of risk (Sharpe ratio).

Following from the law of one price, the pricing kernel  $\xi(t)$  relates future dividend payments  $\delta(s)$ ,  $s \in (t, T]$  to today's stock price  $S_1(t)$ :

$$S_1(t) = \frac{1}{\xi(t)} E_t \left[ \int_t^T \xi(s) \delta(s) ds \right]. \quad (4)$$

Intuitively the stock price is the price you pay to achieve a certain dividend in each state at each time  $t$ . Equation (4) is simply an over-time summation of the Arrow-Debreu security prices, discounting the future dividend payouts to today's value. The state price density process will therefore play an important role in deriving the equilibrium prices.

<sup>5</sup> Basak and Shapiro (2001) assume  $N$  risky assets. However, our results are robust to the number of assets.

<sup>6</sup> All mentioned processes are assumed to be well defined and satisfy the appropriate regularity conditions. For technical details, see Karatzas and Shreve (1998).

## 2.2 Preferences, endowments and risk management

Suppose there are two groups of agents in the economy: non-risk-managing and risk-managing agents. Agents belonging to the former group freely optimise their investment strategy, ie without risk management constraints, whereas the latter group is obligated to take a VaR restriction as a side constraint when structuring portfolios.<sup>7</sup> We assume that a proportion  $\lambda$  of the agents is not regulated, while the remaining proportion  $(1 - \lambda)$  is. Each agent is endowed at time zero with initial wealth  $W_i(0)$ . We use subscript  $i = 1$  for the unregulated agents and  $i = 2$  for the risk managers. For both groups of agents we define a non-negative consumption process  $c_i(t)$  and a process for the amount invested in stock  $\pi_i(t)$ . The wealth  $W_i(t)$  of the agents then follows the process below:

$$dW_i(t) = r(t)W_i(t)dt + (\mu(t) - r(t))\pi_i(t)dt - c_i(t)dt + \sigma(t)\pi_i(t)dB(t), \quad (5)$$

for  $i = 1, 2; \forall t \in [0, T]$ .

As in the case of asset prices, today's wealth can be related to future consumption and terminal wealth through the state price density process  $\xi(t)$ :

$$W_i(t) = \frac{1}{\xi(t)} E_t \left[ \int_t^T \xi(s)c_i(s)ds + \xi(T)W_i(T) \right]. \quad (6)$$

The agents maximise their utility from intertemporal consumption in  $[0, T]$  and terminal wealth at the planning horizon  $T$ , which are represented by  $U_i(c_i(t))$  and  $H_i(W_i(T))$  respectively. The parameter  $\rho_i > 0$  determines the relative importance of utility from terminal wealth compared to utility from consumption. The planning problem for an unregulated agent then is:

$$\begin{aligned} \max_{c_1, \pi_1} \quad & E \left[ \int_0^T U_1(c_1(s))ds + \rho_1 H_1(W_1(T)) \right]. \\ \text{s.t.} \quad & dW_1(t) = r(t)W_1(t)dt + (\mu(t) - r(t))\pi_1(t)dt - c_1(t)dt + \sigma(t)\pi_1(t)dB(t), \\ & W_1(t) \geq 0, \text{ for } \forall t \in [0, T]. \end{aligned} \quad (7)$$

Additionally, in order to limit the likelihood of large losses, the risk managers have to take a VaR constraint into account. Based on the practical implementation of VaR and its interpretation by Basak and Shapiro (2001), at the horizon  $T$  the maximum likely loss with probability  $(1 - \alpha)\%$  over a given period, namely  $\text{VaR}(\alpha)$ , is mandated to be equal to or below a prespecified level. More precisely, the agents are allowed to consume continuously but make sure that, only with probability  $\alpha\%$  or less, their wealth  $W_2(T)$  falls below the critical floor level  $\underline{W}$ . Therefore, the second group of agents faces the following optimisation problem with the additional VaR constraint:

$$\begin{aligned} \max_{c_2, \pi_2} \quad & E \left[ \int_0^T U_2(c_2(s))ds + \rho_2 H_2(W_2(T)) \right] \\ \text{s.t.} \quad & dW_2(t) = r(t)W_2(t)dt + (\mu(t) - r(t))\pi_2(t)dt - c_2(t)dt + \sigma(t)\pi_2(t)dB(t), \\ & W_2(t) \geq 0, \text{ for } \forall t \in [0, T], \\ & P[W_2(T) \geq \underline{W}] \geq 1 - \alpha. \end{aligned} \quad (8)$$

We assume that all agents have constant relative risk aversion over intertemporal consumption  $U_i(c_i(t)) = V_{CRRRA}(c_i(t))$  and over terminal wealth  $H_i(W_i(T)) = V_{CRRRA}(W_i(T))$  for  $i = 1, 2$ , where  $V_{CRRRA}(\cdot)$  is a power utility function:

$$V_{CRRRA}(x) = \frac{1}{1 - \gamma} x^{1 - \gamma}, \text{ for } \gamma > 0; x > 0. \quad (9)$$

<sup>7</sup> It should be noted that the superfluous risk management critique (see Modigliani and Miller (1958), Stiglitz (1969a,b and 1974), DeMarzo (1988), Grossman and Vila (1989) and Leland (1998)), does not hold at the individual level. The critique states that risk management is irrelevant for institutions and firms since individuals can undo any financial restructuring by trading in the market. This paper considers individual agents, and hence this line of reasoning is invalid here.

Note that the power utility function (9) is increasing and strictly concave and hence agents are assumed to be risk-averse. By assuming a common power utility function for intertemporal consumption and terminal wealth, we can isolate the effect of VaR-based risk management on asset prices. Our main purpose is to study the influence of risk management on the equilibrium price of the risky asset and on option prices.

In this paper, the VaR horizon coincides with the investment horizon, which is different from Basak and Shapiro's (2001) work. In their work, agents are concerned with the optimal consumption path over their lifetime, while obligated to a one-time-only risk evaluation. The VaR condition is supposed to be satisfied at some intermediate time, before the end of the agent's life. As a consequence of this setup, a severe jump in the price level occurs when the VaR condition is lifted. In addition, since agents consume everything at the planning horizon  $T$  and wealth drops to zero at that time, the corresponding asset prices go to zero.

In this paper, the horizon is just a subperiod of the lifetime in Basak and Shapiro (2001). Thus, agents can evaluate their VaR performance at the end of each period, eg a 10-day or an annual report, along the way maximising the utility from their intertemporal consumption as well as their terminal wealth. In our perspective, this adjustment makes the model more realistic. At the horizon agents may end up with claims on the assets and prices do not necessarily drop to zero. Moreover, within our setup we can ignore the jump in asset prices that occurs directly after the VaR horizon in Basak and Shapiro (2001). Note that this jump in asset prices occurs because all regulated investors drop the VaR constraint collectively, at a prespecified point in time. We think that such a coordinated abandonment of risk management policies is rather hypothetical and therefore we do not analyse the consequences of a jump in asset prices in this paper.

### 2.3 Equilibrium conditions and optimal decisions

In order to investigate the equilibrium asset prices in an economy with VaR risk managers, in this section we discuss the conditions that should be satisfied in any general equilibrium. In equilibrium, each agent optimises his individual consumption-investment problem. Moreover, as the consumption good cannot be stored, it follows that aggregate consumption in the economy has to equal aggregate dividends at each time  $t \in [0, T]$ . Additionally, from Walras law it follows that all markets have to clear, eg the good and the riskless securities markets, given that the stock market is in equilibrium at each time  $t \in [0, T]$ . Combined, this gives the following set of equilibrium conditions:

$$\lambda c_1^*(t) + (1 - \lambda)c_2^*(t) = \delta(t), \quad (10)$$

$$\lambda \pi_1^*(t) + (1 - \lambda)\pi_2^*(t) = S_1(t),$$

$$\lambda W_1^*(t) + (1 - \lambda)W_2^*(t) = S_1(t),$$

where  $c_i^*(t)$  and  $\pi_i^*(t)$  are the optimal consumption and investment decisions for each type of agent  $i = 1, 2$  and  $W_i^*(t)$  is the corresponding optimal wealth process.

Typically the optimal policies for agents with power utility can be derived with dynamic programming, as in Merton (1969), as well as with the martingale methodology of Cox and Huang (1989), Karatzas et al (1987) and Pliska (1986). However, for the risk managers, the binding VaR restriction induces non-concavity into the optimisation problem (through the wealth function). Following Basak and Shapiro (2001), we derive the optimal policies for the regulated agents with power utility by applying the martingale methodology. The dynamic optimisation problem melts down to the following problem:

$$\begin{aligned} \max_{c_i, \pi_i} \quad & E \left[ \int_0^T U_i(c_i(s)) ds + \rho_i H_i(W_i(T)) \right] \\ \text{s.t.} \quad & E \left[ \int_0^T \xi(s) c_i(s) ds + \xi(T) W_i(T) \right] \leq \xi(0) W_i(0), \\ & W_i(T) \geq 0, \end{aligned} \quad (11)$$

for  $i = 1, 2$ , with an additional constraint for the risk managers:

$$P[W_2(T) \geq \underline{W}] \geq 1 - \alpha. \quad (12)$$

In the next proposition we characterise the optimal consumption paths and terminal wealth profiles for both groups of agents:

**Proposition 1**

For any state price density following the process (3), the optimal intertemporal consumption policies  $c_i^*(t)$  and terminal wealth profiles  $W_i^*(T)$  of both groups of agents  $i = 1, 2$  are

$$\begin{aligned}
 c_1^*(t) &= I_1(y_1 \xi(t)) = (y_1 \xi(t))^{-1/\gamma}, \\
 c_2^*(t) &= I_2(y_2 \xi(t)) = (y_2 \xi(t))^{-1/\gamma}, \\
 W_1^*(T) &= (y_1 \xi(T)/\rho_1)^{-1/\gamma}, \\
 W_2^*(T) &= \begin{cases} I_2(y_2 \xi(T)/\rho_2) & \text{for } \xi(T) < \underline{\xi} \\ \underline{W} & \text{for } \underline{\xi} \leq \xi(T) \leq \bar{\xi} \\ I_2(y_2 \xi(T)/\rho_2) & \text{for } \bar{\xi} \leq \xi(T) \end{cases}
 \end{aligned} \tag{13}$$

where  $I_i(z)$  denotes the inverse of the marginal utility function  $z = U'_i(x)$ ,  $\underline{\xi} = \rho_2 U'_2(\underline{W})/y_2$ ,  $\bar{\xi}$  is such that  $P[\xi(T) > \bar{\xi}] = \alpha$  and  $y_i \geq 0$  is a Lagrange multiplier that satisfies

$$E\left[\int_0^T \xi(s) c_i^*(s) ds + \xi(T) W_i^*(T)\right] = \xi(0) W_i(0), \text{ for } i = 1, 2. \tag{14}$$

In order to facilitate the derivation of equilibrium prices in the following sections, we additionally use the following proposition proved by Karatzas et al (1990) and Basak (1995). It provides a different representation of the equilibrium conditions by applying the martingale methodology.

**Proposition 2**

If there exists a state price density process  $\xi(t)$  satisfying

$$\delta(t) = \lambda I_1(y_1 \xi(t)) + (1 - \lambda) I_2(y_2 \xi(t)), \tag{15}$$

where  $y_1$  and  $y_2$  are Lagrange multipliers defined in (14), then the equilibrium conditions (10) are satisfied by the corresponding optimal consumption and investment policies.

Before we actually derive the equilibrium prices, we will first discuss the optimal investment strategies in partial equilibrium of the unrestricted agents (benchmark agents), the regulated agents (the VaR risk managers) and the portfolio insurers. Portfolio insurers are VaR risk managers that do not tolerate any losses below  $\underline{W}$ , ie they represent the extreme case  $\alpha = 0$ .<sup>8</sup> It is noteworthy that portfolio insurers fully insure against all states of nature at the minimal wealth level  $\underline{W}$ , whereas VaR risk managers with  $\alpha > 0$  only partially insure. VaR risk managers with  $\alpha > 0$  do not necessarily insure in expensive states that occur with very low probability.

In Figure 1, we display the optimal terminal wealth profiles for the three types of agents. All agents have initial endowment of  $W(0) = 1$  and power utility over consumption and over terminal wealth with risk aversion parameter  $\gamma = 1$ , which is in limit equivalent to log utility. We set the trade-off between consumption and terminal wealth at  $\rho_1 = \rho_2 = 10$ , to avoid excessive consumption. The maximum loss probability  $\alpha$  of the risk managers and the portfolio insurers is equal to 1% and 0 respectively. The critical wealth  $\underline{W}$  at time  $T = 1$  is 0.95. The interest rate and the Sharpe ratio are given by 5% and 0.4. Endogenous thresholds,  $\underline{\xi}$  and  $\bar{\xi}$ , are then calculated.

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<sup>8</sup> See Basak (1995), Basak and Shapiro (1999), Grossman and Vila (1989) and Grossman and Zhou (1996).

To be more specific, throughout the paper, states of the world are roughly classified into three ranges. First, good states are the states in which the VaR constraint is not binding since the optimal (terminal) wealth, as a consequence of individual optimisation, is likely to be above or equal to the minimal wealth threshold. Second, intermediate states ( $\underline{\xi}, \bar{\xi}$ ] are the states in which the unrestricted individual's wealth is likely to end up below the threshold. Thus, VaR risk management becomes binding in these states. Last, extremely bad states are the small area of worst cases with a total probability mass of  $\alpha$ . In these states the VaR managers will no longer try to keep wealth above  $\underline{W}$ . Note that sizes of these ranges depend on the minimal wealth threshold  $\underline{W}$  and the probability of loss  $\alpha$  for the VaR managers.

Figure 1 shows that the optimal terminal wealth function of the benchmark agents is decreasing smoothly from good states of the world (low  $\xi(T)$ ) to bad states (high  $\xi(T)$ ), whereas the partial insurers and full insurers behave differently. In the good states region (low  $\xi(T)$ ), they all have smoothly decreasing curves. However, for the portfolio insurers to guarantee terminal wealth in intermediate and extreme states, from  $\underline{\xi}$  onwards, they have to pay based on the pricing kernel, ie the worse the state implies the higher the price to obtain a fixed amount of wealth in that particular state. Therefore, the portfolio insurers consume relatively less in good states in order to pay for the insurance in bad states.

Unlike the portfolio insurers, the VaR-based risk managers are not obligated to insure in the most expensive states (high  $\xi(T)$ ). In very bad states, from  $\bar{\xi}$  onwards, terminal wealth dramatically drops and remains lower than the benchmark level. In terms of probability, the VaR managers have high probability of achieving the critical wealth threshold that they insure in intermediate states. However, lower payoff in bad states leads to a heavier tail in the lower quantiles of the wealth distribution, compared to other types of investors. Note that the area on the right of  $\bar{\xi}$  contains only  $\alpha\%$  of probability mass. Hence, the area where the VaR risk manager tolerates losses is less probable than it might seem in the figure.

Next we analyse the dynamic (state-varying) optimal investment policies of the VaR managers relative to the benchmark agents and the portfolio insurers. In order to do so, we assume for now Geometric Brownian motions for the dividend rate and the stock price with constant interest rate  $r = 5\%$  and constant market price of risk  $\kappa = 0.4$ . Later on we will see that in the general equilibrium framework the interest rate and the market price of risk are indeed constant. As in the previous example, initial wealth is  $W(0) = 1$  and  $T = 1$ , while the current time is  $t = 0.75$ .

In Figure 2, the optimal wealth of three kinds of investors is shown at an intermediate point in time ( $t = 0.75$ ). The figure demonstrates the common feature of a steeply decreasing wealth function, with a slower descent as the states are worsening. As in Figure 1, the benchmark agent's wealth is a decreasing convex function, while the portfolio insurers always keep their wealth above the level  $\exp(-r(T-t))\underline{W}$ . The time- $t$  wealth profile of the VaR risk manager with  $\alpha = 1\%$  lies in between these two extremes. It looks like a smoothed version of the terminal wealth profile at the horizon  $T$ , without the abrupt jump at  $\bar{\xi}$ .

Figure 3 displays the optimal portfolio weight of stocks relative to the benchmark. It reveals how VaR-based investors change their investment strategy to insure their terminal wealth. In good states the VaR risk-managing agents follow the benchmark behaviour. In the middle range, when the state price is relatively low they replicate the portfolio insurers' strategy, which is to invest in riskless assets. In bad states the VaR agents greatly increase their exposure to the risky asset, as they try to maximise the probability of their wealth staying above the critical threshold  $\underline{W}$ . However, to avoid bankruptcy, the risk managers eventually reduce their risk exposure again.<sup>9</sup>

Concluding: the VaR restriction reduces the exposure to risky assets in good states relative to the benchmark, while in extremely bad states it stimulates risky-asset holding in a desperate attempt to make up for low wealth (gambling). Thus, outside the area where the VaR constraint is binding, logically the VaR agents' wealth is always lower than in the benchmark case. In the following section,

<sup>9</sup> Please note that states with a pricing kernel higher than three in Figure 3 have almost no probability of occurring and are only shown for illustrative purposes.

we will investigate the impact of this investment strategy on market risk, the equilibrium asset prices and the total return distribution. The usefulness of VaR regulation, implicitly or explicitly, will be discussed in Section 5.

### 3. General equilibrium with VaR risk managers

#### 3.1 Closed-form solutions

In this section, we expand the individual optimisation problem of the previous section to a Lucas general equilibrium model, based on the market clearing conditions (10). In the economy, there is a fraction  $\lambda$  of agents, who are unrestricted, and the remaining fraction  $1 - \lambda$ , who apply VaR-based risk management. Our economy is formulated such that, in some subperiod, agents maximise their utility from intermediate consumption,  $t \in [0, T]$ , and from terminal wealth at the horizon  $T$ . The VaR restriction is imposed on the risk managers at time  $T$ , coinciding with their planning horizon.

As a first step in solving the equilibrium model, it is convenient to derive the equilibrium state price density as a function of aggregate dividends by inverting equation (15). Given the stochastic process for the state price density in (3), as a second step we can infer the equilibrium interest rate and market price of risk. The following proposition summarises these general results:

#### **Proposition 3**

In any economy with  $0 \leq \lambda \leq 1$  equilibrium exists and the state price density is given by

$$\xi(t) = (v(y_1, y_2) \delta(t))^{-\gamma}, \quad (16)$$

where

$$v(y_1, y_2) = (\lambda y_1^{-1/\gamma} + (1-\lambda) y_2^{-1/\gamma})^{-1}. \quad (17)$$

The equilibrium interest rate and market price of risk processes are constant:

$$r(t) = \gamma(\mu_\delta - 1/2(1+\gamma)\sigma_\delta^2), \quad (18)$$

$$k(t) = \gamma\sigma_\delta.$$

An important conclusion from proposition (3) is that the interest rate  $r$  and the market price of risk  $k$  are constant in equilibrium. Furthermore, the fraction of risk managers in the economy does not affect the interest rate and the market price of risk. Hence, in our setup the presence of risk managers will only have an impact on the price level of the stock, on its drift rate  $\mu$  and volatility  $\sigma$ . Note, however, that the mean and volatility always have to move in lockstep due to the constant market price of risk. In addition, based on the assumption of a Geometric Brownian for the dividend process, equation (16) implies a log-normal state price density.

The constant interest rate and market price of risk arise due to the assumption that both groups of agents share an identical power utility function over intertemporal consumption. Although this economic setup leads to some inflexibility, a major advantage is that we can derive closed-form solutions for the equilibrium prices and hence fully analyse the economic problem at hand. Basak (1995) and Basak and Shapiro (2001) also impose equivalent assumptions in order to study equilibrium with portfolio insurers and value-at-risk regulation respectively. Without an identical utility function over consumption for both groups of agents, we would have to resort to numerical techniques as in Grossman and Zhou (1996).<sup>10</sup>

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<sup>10</sup> The same holds if we leave out consumption and only assume utility over terminal wealth.

Given the state price density of proposition (3), we can derive the equilibrium stock price from the equilibrium conditions (10). Once a closed-form expression has been obtained, the drift rate  $\mu$  and volatility  $\sigma$  of the stock price process follow straightforwardly from Ito's lemma. Below we first present the equilibrium price in a benchmark economy with unrestricted investors only ( $\lambda = 1$ ), before we consider our general results in the presence of risk managers:

**Proposition 4**

The equilibrium price of the risky asset in an economy with unregulated agents only ( $\lambda = 1$ ) is

$$S_1(t) = \delta(t) \left( a(t) + \rho_1^{1/\gamma} e^{\eta(T-t)} \right), \tag{19}$$

with

$$a(t) = \frac{1}{\eta} \left( e^{\eta(T-t)} - 1 \right), \quad \eta = (1-\gamma)\mu_\delta - 1/2\gamma(1-\gamma)\sigma_\delta^2. \tag{20}$$

The stock price follows a Geometric Brownian motion with constant drift rate and volatility given by

$$\mu(t) = \gamma \left( \mu_\delta + 1/2(1-\gamma)\sigma_\delta^2 \right), \quad \sigma(t) = \sigma_\delta. \tag{21}$$

In the case of unregulated agents with power utility, the interest rate is constant and the stock price follows a Geometric Brownian motion, resembling the familiar Black-Scholes assumptions for option pricing. As a result, it has a log-normal (Gaussian-class) distribution. If we additionally introduce VaR-based risk managers ( $\lambda < 1$ ), then the equilibrium stock price process changes quite drastically:

**Proposition 5**

The equilibrium price of the risky asset in an economy with both unregulated agents and VaR risk managers is

$$\begin{aligned} S_1(t) = & a(t)\delta(t) + \lambda(y_1/\rho_1)^{-1/\gamma} v(y_1, y_2) e^{\Gamma(t)} \delta(t) \\ & + (1-\lambda)(y_2/\rho_2)^{-1/\gamma} v(y_1, y_2) e^{\Gamma(t)} \delta(t) \left( 1 - N(-d(\delta(t), t, \underline{\xi})) + N(-d(\delta(t), t, \bar{\xi})) \right) \\ & + (1-\lambda) W e^{-r(T-t)} \left( N(-f(\delta(t), t, \underline{\xi})) - N(-f(\delta(t), t, \bar{\xi})) \right) \end{aligned} \tag{22}$$

with

$$f(\delta, t, x) = \frac{\log(x) + \gamma(\log(\delta(t)) + \log(v(y_1, y_2))) + \left( r - \frac{1}{2}\kappa^2 \right) (T-t)}{\kappa\sqrt{(T-t)}}, \tag{23}$$

$$d(\delta, t, x) = f(\delta, t, x) + \frac{1}{\gamma} \kappa\sqrt{(T-t)} \tag{24}$$

$$\Gamma(t) = \frac{1-\gamma}{\gamma} \left( r + \frac{1}{2}\kappa^2 \right) (T-t) + \frac{1}{2} \left( \frac{1-\gamma}{\gamma} \right)^2 \kappa^2 (T-t) \tag{25}$$

and  $N(\cdot)$  is the standard-normal cumulative distribution function.

In an economy with VaR-regulated investors, the drift rate  $\mu_t$  and the volatility  $\sigma_t$  of the stock price process are no longer constant, which can be easily verified by applying Ito's lemma to the stock price formula (22). Figures 4 and 5 show that at a point in time the expected drift rate and the volatility can be either high or low relative to the benchmark case, depending on the state of the world. In the next section we will analyse these drift rates and the volatility processes. Moreover, the impact of VaR-based risk management on the equilibrium stock price, its total return distribution and the equilibrium option prices will be examined respectively.

### 3.2 Drift rate, volatility and prices in equilibrium

For the numerical examples, we set the parameters of the dividend process as  $\mu_\delta = 0.056$  and  $\sigma_\delta = 0.115$ , based on monthly S&P 500 index data from 1980 up to 1999. All agents have initial wealth  $W(0) = 1$  and maximise a power utility over intertemporal consumption and terminal wealth with risk aversion  $\gamma = 1$ . We set the trade-off between consumption and terminal wealth at  $\rho_1 = \rho_2 = 10$ . The maximum loss probability  $\alpha$  is 100%, 1% and 0 respectively for the unregulated agents, the risk managers and the portfolio insurers. The critical wealth  $\underline{W}$  threshold at the planning horizon  $T = 1$  is 0.95, while the current time is  $t = 0.75$ . The insurer economy and the risk manager economy both contain 50% of the benchmark agents and 50% of their own population (ie  $\lambda = 1/2$ ).

In order to take a closer look at the stock price process, first we present the expected drift rate and the volatility of the stock returns as functions of the pricing kernel. In Figures 4 and 5, one can see that in all three types of economies the market volatility and the drift rate move in lockstep as a consequence of the constant market price of risk. In the benchmark economy, the mean and the volatility of the stock price are constant in every state at all times, corresponding to the Black-Scholes assumptions for option pricing. In the portfolio insurance economy, the expected returns and the volatility are lower in most states, compared to the benchmark. Intuitively, the portfolio insurers want to hold less equity. In order to clear the market, the attractiveness of stocks has to be increased and, hence, the equilibrium volatility is reduced. The portfolio insurance strategy therefore reduces volatility and stabilises the economy.

For exactly the same reasons, the volatility is reduced in intermediate states in the partially insured VaR economy. However, in bad states the VaR agents abandon the portfolio insurance strategy and instead they start to increase their demand for risky assets, as can be seen in Figure 3.<sup>11</sup> The overwhelming demand caused by this gambling policy leads to a relatively high volatility and drift rate, in order to clear the market. Eventually, in very bad states, the portfolio weight of the VaR agents returns to normal levels in order to avoid bankruptcy. As a result, the equilibrium volatility and drift rate are then pulled back to the benchmark level.

We will now analyse the consequences of the presence of VaR risk managers on the equilibrium stock price. Figure 6 displays the equilibrium price of the risky asset in different states of nature, ranging from good to bad, for the benchmark economy with  $\lambda = 1$  and the mixed economies with  $\lambda = 1/2$ , at the intermediate time  $t = 0.75$ . For all economies, the price of the risky asset is generally high in good states of the world (low  $\xi(t)$ ) and low in bad states (high  $\xi(t)$ ). This follows from the inverse relation between the state price density and aggregate dividends in proposition 3, namely a higher pricing kernel level implies lower aggregate dividends and consequently a lower stock price.

We start by explaining the price function in the portfolio insurance economy (the dotted line). In good states the price function in the economy with portfolio insurers has a similar shape as in the benchmark case; however, it lies at a lower level. This follows from the fact that the constrained agents have less wealth and therefore they demand less stocks: the equilibrium price has to be lower in order to clear the market. In intermediate and bad states, the prices in the portfolio insurance economy are higher than in the benchmark economy. In these states the equilibrium volatility in the portfolio insurance economy is considerably lower than in the benchmark case. The low volatility increases the demand for stocks and hence also increase the equilibrium price.

The price function in the economy with VaR risk managers is similar to the pattern in the portfolio insurance economy in good and intermediate states. However, in bad states, the VaR risk managers switch from the portfolio insurance strategy to a risky gambling strategy and this alters the shape of the price function. Due to the increased demand for stocks in bad states, the volatility increases in order to clear the market. While volatility increases and makes the stock less attractive, the price function drops rather fast and ends up below the benchmark case. Finally, in extremely bad states, the VaR agents reduce their risk exposure to avoid bankruptcy, volatility returns to regular levels and the shape of the price function becomes similar to the benchmark case again.

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<sup>11</sup> Note that Figure 3 is not used for a quantitative analysis but for qualitative interpretation only, since the drift rate and volatility do not coincide with the equilibrium case in Figures 4 and 5.

### 3.3 The return distribution and option prices in equilibrium

Figures 7, 8 and 9 show the annualised total return distribution of the stock, including the estimated dividend yield. As the dividend payout rate changes stochastically through time, an exact calculation of the total dividend payments would require extensive simulation. Instead we approximate the stochastic dividend payout  $\int_0^t \delta(z) dz$  between time zero and  $t$  by  $(\delta(t) - \delta(0))t$ . As the dividend process is the same in each economy, we do not think that the error inherent in this approximation will have a serious impact on our analysis or conclusions.

In the benchmark economy, the returns are log-normally distributed with slight positive skewness and kurtosis close to three, as the stock price follows a Geometric Brownian motion. Figure 7 shows the effect on the stock return distribution of a VaR constraint applied to 50% of the investors (ie  $\lambda = 50\%$ ), with critical wealth level  $\underline{W} = 0.95$  and maximum loss probability  $\alpha = 0\%$ , 1% and 5%. In an economy with VaR risk managers or portfolio insurers, the volatility of the stock return distribution is clearly lower. Moreover, the probability of negative returns up to  $-25\%$  has also decreased substantially. Hence, the VaR restriction seems to stabilise the economy. Note, however, that the probability of negative returns in excess of  $-25\%$  has increased in the case of a loose VaR constraint with  $\alpha = 5\%$ , due to increased risk-taking at lower levels of wealth.

Figure 8 shows the return distribution in economies with a VaR constraint with  $\alpha = 1\%$ , at different critical wealth levels  $\underline{W} = 0.90, 0.95$  and  $0.99$ . The volatility of the stock returns tends to decrease as the critical wealth level becomes higher and hence the VaR constraint becomes tighter. Note that in the case of a very tight constraint, with  $\underline{W} = 0.99$ , the left tail of the distribution becomes thicker and extreme returns below  $-30\%$  are more likely than in the unregulated economy. In general, though, the economies with a VaR constraint are less volatile and the regulation has the intended effect.

Another important parameter affecting the economy is  $\rho_i$ , which determines the trade-off between the utility of intertemporal consumption and the utility of terminal wealth. The previous computations were made with  $\rho_1 = \rho_2 = 10$ , putting emphasis on terminal wealth. Figure 9 shows the return distribution in an economy where intertemporal consumption is more important, with  $\rho_1 = \rho_2 = 1$ . In this case the left tail of the distribution in the VaR economy becomes quite thick. The investors consume a large share of their wealth and therefore have more difficulties meeting the VaR constraint in case of losses: this leads to a more risky investment strategy and hence a heavy left tail.

So far, we have found that the presence of risk managers has a profound impact on the equilibrium stock price and its return distribution. Now we will concentrate on the option market for European call and put contracts. The prices of these options can be computed easily by discounting the payoffs at maturity with the pricing kernel, as in no-arbitrage equation (4). The option payoffs are a function of the equilibrium stock price, which we have derived in closed form. We only have to compute the expectation in equation (4), which can be implemented straightforwardly as the pricing kernel follows a Geometric Brownian motion. Once we have calculated the option prices for a wide range of strike prices, we transform them into implied volatilities with the Black-Scholes formula, in order to facilitate interpretation and comparisons.

Figure 10 shows the implied volatility in economies with a VaR constraint ( $\alpha = 0\%$ , 1% and 5%) and in the benchmark economy. In the benchmark case we observe a constant implied volatility, following from the Geometric Brownian motion of the stock price and the constant interest rate. In the economies with a VaR constraint a remarkable option smile can be recognised, as often observed in practice. Note that the implied volatility in economies with a VaR constraint is lower than in the benchmark case most of the time. Moreover, as the VaR constraint becomes more strict (ie from  $\alpha = 5\%$  to  $\alpha = 0\%$ ), the implied volatility decreases. Similar volatility smiles can be observed if we change other parameters of the VaR economy such as  $\underline{W}$  and  $\rho_2$ . Option prices are generally lower in VaR economies, as a result of the reduced volatility of the stock return distribution.

## 4. VaR risk management and loss aversion

The equilibrium analysis in the previous section shows that, in general, VaR-based risk management reduces the volatility of stock returns. However, in some cases the VaR constraint adversely increases the probability of extreme losses. The negative consequences of VaR-based risk management are mainly due to the “all or nothing” gambling attitude of the optimal investment strategy in case of losses,

which might seem rather unnatural. In this section our aim is to demonstrate that the gambling strategy of a VaR risk manager might not be that unnatural for many investors, as it is closely related to the optimal strategy of loss-agents with the utility function of prospect theory.

Prospect theory was proposed by Kahneman and Tversky (1979) as a descriptive model for decision-making under uncertainty, given the strong violations of the traditional utility paradigm observed in practice. In experiments, Kahneman and Tversky (1979) found that people are concerned about changes in wealth, rather than the level of wealth itself. Moreover, individuals treat gains and losses relative to their reference point differently: the pain of a loss is felt much more strongly than the payoff of an equivalent gain. Furthermore, in the domain of gains people are risk-averse, while they become *risk-seeking* in the domain of losses.

Kahneman and Tversky (1979) quantified these empirical findings in prospect theory: individuals maximise an S-shaped value function (26), which is convex for losses and concave for gains relative to the reference point  $\theta$ :

$$U_{LA}(x) = \begin{cases} -A(\theta - x)^{\gamma_L}, & \text{for } x \leq \theta \\ +B(x - \theta)^{\gamma_G}, & \text{for } x > \theta, \end{cases} \quad (26)$$

where  $A > 0$  and  $B > 0$  to ensure that  $U_{LA}(\cdot)$  is an increasing function and  $0 < \gamma_L \leq 1$ ,  $0 < \gamma_G \leq 1$ . Moreover,  $A > B$  holds in the case of loss aversion. The parameters of the loss-averse utility function were estimated by Tversky and Kahneman (1992) as  $\gamma_L = \gamma_G = 0.88$  and  $A/B = 2.25$ , based on experiments. An illustration of the value function can be found in Figure 11. We denote agents who maximise this value function with  $A > B$  as “loss-averse”.

Berkelaar and Kouwenberg (2001a) derive the optimal wealth profile of a loss-averse investor, in a similar dynamic economy as in Section 2 of this paper:

**Proposition 6**

For any state price density following the process (3), the optimal intertemporal consumption policy  $c_{LA}^*(t)$  and terminal wealth profiles  $W_{LA}^*(T)$  of a loss-averse agent are

$$c_{LA}^*(t) = I(y_{LA}\xi(t)) = (y_{LA}\xi(t))^{\gamma_L - 1}, \quad (27)$$

$$W_{LA}^*(T) = \begin{cases} \theta + \left( \frac{y_{LA}\xi(T)}{\rho_{LA}B\gamma_G} \right)^{1/(\gamma_G - 1)} & \text{for } \xi(T) < \bar{\xi}/y_{LA} \\ 0 & \text{for } \xi(T) \geq \bar{\xi}/y_{LA}, \end{cases} \quad (28)$$

where  $I(z)$  denotes the inverse of the marginal utility function over consumption  $z = U'(x)$ , and  $\bar{\xi}$  solves  $f(\bar{\xi}) = 0$  with

$$f(x) = \frac{1 - \gamma_G}{\gamma_G} \left( \frac{1}{x} \right)^{\gamma_G/(1 - \gamma_G)} (\rho_{LA}B\gamma_G)^{1/(1 - \gamma_G)} - \theta x + \rho_{LA}A\theta^{\gamma_L} \quad (29)$$

and  $y_{LA} \geq 0$  is a Lagrange multiplier, satisfying

$$E \left[ \int_0^T \xi(s) c_{LA}^*(s) ds + \xi(T) W_{LA}^*(T) \right] = \xi(0) W_{LA}(0). \quad (30)$$

Figure 12 shows the portfolio weights corresponding to this optimal wealth profile for a loss-averse investor that puts more emphasis on terminal wealth than on consumption ( $\rho_{LA} = 100$ ), assuming a constant interest rate  $r = 5\%$  and a constant Sharpe ratio  $\kappa = 0.4$ . Note that the investment strategy of the loss-averse agent is qualitatively similar to the optimal policy of a VaR constrained agent. The strategy is cautious in good states of the world with a low pricing kernel, and more risky in bad states of

the world with a high pricing kernel. Loss-averse agents dislike losses, even if they are very small, and therefore their optimal investment strategy tries to keep wealth above the reference point in good states of the world by investing more wealth in the riskless asset.<sup>12</sup>

Once a loss-averse investor's wealth drops below the reference point, he tries to make up his losses by following a risky investment strategy. The risk-seeking behaviour stems from the convex shape of the utility function of prospect theory below the reference point. Hence, similar to a VaR agent, a loss-averse agent tries to limit losses most of the time, but starts taking risky bets once his wealth drops below the reference point. The optimal investment strategy under a VaR constraint might therefore seem rather natural for loss-averse investors. Or, conversely, one could argue that a VaR constraint imposes a minimum level of "loss aversion" on all investors affected by the regulation.

Not surprisingly, loss-averse agents have a similar impact on asset prices in equilibrium as VaR-constrained agents. We refer to Berkelaar and Kouwenberg (2001b) for a closed-form solution of the asset price in a dynamic economy with loss-averse agents. Figure 13 compares the stock return distribution in an economy with 50% loss-averse agents to an economy with 50% VaR-constrained agents, while Figure 14 shows the volatility smile in the economies. Both figures demonstrate a clear resemblance in prices. We conclude that asset prices in an economy with a VaR constraint correspond qualitatively to prices in an economy with loss-averse investors who put emphasis on terminal wealth.

## 5. Conclusions

In order to investigate the effect of VaR-based risk management on asset prices in equilibrium, we adopted the continuous-time equilibrium model of Basak and Shapiro (2001). The asset pricing model is modified such that agents maximise their expected utility from intermediate consumption and terminal wealth at the planning horizon. While dynamically optimising their consumption-investment plan, the VaR restriction is imposed on the investors at the planning horizon. The VaR horizon thus coincides with the investment horizon and can be interpreted as a subperiod of the investor's lifetime (eg an official reporting period of 10 days or one year). Within this setup we can ignore two unrealistic price movements that occur in the model of Basak and Shapiro (2001); (1) a jump in prices just after the VaR horizon, and (2) prices dropping to zero at the planning horizon.

We derived the closed-form equilibrium solutions for the asset prices in the model. Our main findings are as follows: the presence of VaR risk managers generally reduces market volatility. However, in very bad states the optimal investment strategy of VaR risk managers is to take a large exposure to stocks, pushing up market risk and creating a hump in the equilibrium price function. In some special cases this strategy can adversely increase the probability of extreme negative returns. This effect is amplified in economies where investors consume a large share of their wealth, in economies where the VaR constraint has a high maximum loss probability  $\alpha$  and in economies with a high VaR threshold  $\underline{W}$ .

We also derived the prices of European options in our economy and found that implied volatility is typically lower in the presence of VaR risk managers. Moreover, VaR risk management creates an implied volatility smile in our economy, as often observed in practice. As a final conclusion, we would like to stress that VaR-based risk management has a stabilising effect on the economy as a whole for most parameter settings in our model. However, it is important to note that the VaR restriction in some cases might worsen catastrophic states that occur with a very small chance, due to the gambling strategy of the VaR risk managers in bad states of the world.

The gambling strategy of VaR-based risk managers in bad states of the world is optimal within a standard dynamic investment model, but might seem rather unnatural for investors in the real world. In the last section of the paper we demonstrated that this gambling strategy in bad states is shared by loss-averse investors, who maximise the utility function of prospect theory (Kahneman and Tversky (1979)). As the optimal investment strategies of loss-averse agents and VaR risk managers are quite similar, it might be relatively easy for the group of loss-averse investors to adopt VaR-based risk management.

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<sup>12</sup> See footnote 4.

## Appendix

### Proof of proposition 1

We refer to Cox and Huang (1989), Karatzas et al (1987) and Pliska (1986) for the optimal consumption and investment policies for agents with power utility. Basak and Shapiro (2001) derive the optimal policies for the portfolio choice problem with an additional VaR constraint.

### Proof of proposition 2

This proof can be found in Karatzas et al (1990) and Basak (1995).

### Proof of proposition 3

If we substitute the optimal consumption policies (13) into equilibrium relationship (15), then we find:

$$\delta(t) = \lambda(y_1 \xi(t))^{-1/\gamma} + (1-\lambda)(y_2 \xi(t))^{-1/\gamma} = (\lambda y_1^{-1/\gamma} + (1-\lambda)y_2^{-1/\gamma}) \xi(t)^{-1/\gamma}, \quad (31)$$

and hence the state price density in equilibrium is

$$\xi(t) = (\lambda y_1^{-1/\gamma} + (1-\lambda)y_2^{-1/\gamma})^\gamma \delta(t)^{-\gamma} = (v(y_1, y_2) \delta(t))^{-\gamma}. \quad (32)$$

By applying Ito's lemma, we can derive that  $\xi(t)$  follows the stochastic process below:

$$\begin{aligned} d\xi(t) &= -(\gamma\mu_\delta - 1/2\gamma(\gamma+1)\sigma_\delta^2)(v(y_1, y_2)\delta(t))^{-\gamma} dt - \gamma\sigma_\delta(v(y_1, y_2)\delta(t))^{-\gamma} dB(t) \\ &= -(\gamma\mu_\delta - 1/2\gamma(\gamma+1)\sigma_\delta^2)\xi(t)dt - \gamma\sigma_\delta\xi(t)dB(t). \end{aligned} \quad (33)$$

Equating the processes (3) and (33), we can determine the constant interest rate  $r$  and the constant market price of risk  $\kappa$ .

### Proof of proposition 4

The equilibrium stock price in an economy with unregulated agents is a special case of proposition 5 with  $\lambda = 1$  (see proof below). The drift rate  $\mu(t)$  and volatility  $\sigma(t)$  of the process can be derived by applying Ito's lemma to the stock price.

### Proof of proposition 5

The price of the risky asset can be derived from the third equilibrium condition in (10):

$$S_1(t) = \lambda W_1^*(t) + (1-\lambda)W_2^*(t). \quad (34)$$

Given the optimal policies of a normal agent and the process for  $\delta(t)$ , we can derive:

$$\begin{aligned} W_1^*(t) &= \frac{1}{\xi(t)} E_t \left[ \int_t^T \xi(s) c_1^*(s) ds + \xi(T) W_1^*(T) \right] \\ &= \frac{1}{\xi(t)} \left( E_t \left[ \int_t^T \xi(s) (y_1 \xi(s))^{-1/\gamma} ds \right] + E_t \left[ \xi(T) (y_1 \xi(T) / \rho_1)^{-1/\gamma} \right] \right) \\ &= \frac{1}{\xi(t)} y_1^{-1/\gamma} v(y_1, y_2)^{1-\gamma} \left( E_t \left[ \int_t^T \delta(t)^{1-\gamma} ds \right] + \rho_1^{1/\gamma} E_t \left[ \delta(T)^{1-\gamma} \right] \right) \\ &= y_1^{-1/\gamma} v(y_1, y_2) \delta(t)^\gamma \left( a(t) \delta(t)^{1-\gamma} + \rho_1^{1/\gamma} e^{n(T-t)} \delta(t)^{1-\gamma} \right) \\ &= y_1^{-1/\gamma} v(y_1, y_2) \left( a(t) + \rho_1^{1/\gamma} e^{n(T-t)} \right) \delta(t). \end{aligned} \quad (35)$$

Similarly, we find for the VaR constrained agents:

$$W_2^*(t) = \frac{1}{\xi(t)} E_t \left[ \int_t^T \xi(s) c_2^*(s) ds + \xi(T) W_2^*(T) \right] \quad (36)$$

$$= y_2^{-1/\gamma} v(y_1, y_2) a(t) \delta(t) + \frac{1}{\xi(t)} E_t \left[ \xi(T) W_2^*(T) \right] \quad (37)$$

$$= y_2^{-1/\gamma} v(y_1, y_2) a(t) \delta(t) + \frac{1}{\xi(t)} E_t \left[ \xi(T) (y_2 \xi(T) / \rho_2)^{-1/\gamma} 1_{\{\xi(T) < \underline{\xi} \cup \xi(T) \geq \bar{\xi}\}} + \xi(T) W 1_{\{\xi \leq \xi(T) < \bar{\xi}\}} \right]$$

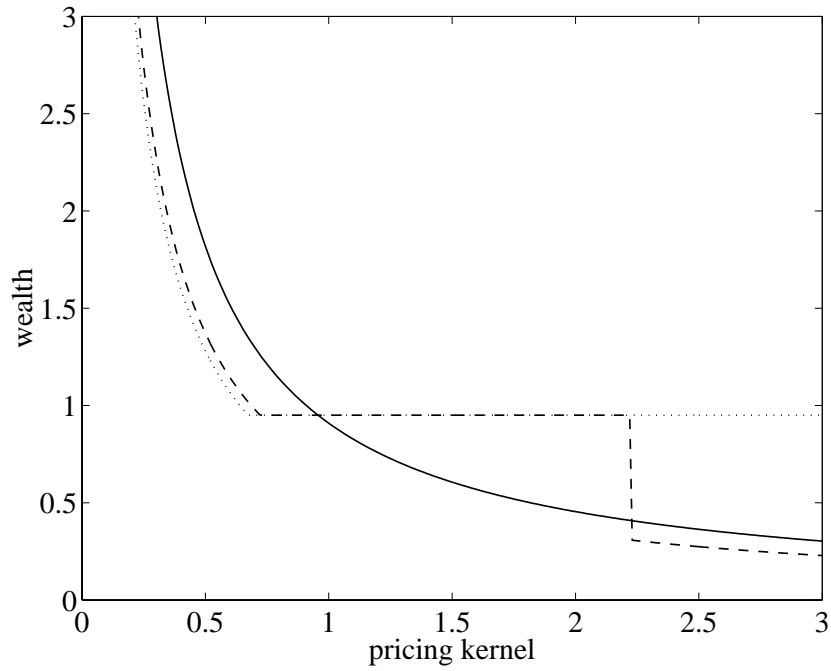
$$= y_2^{-1/\gamma} v(y_1, y_2) a(t) \delta(t) + (y_2, \rho_2)^{-1/\gamma} v(y_1, y_2) e^{\Gamma(t)} \delta(t) \left( 1 - N(-d(\delta(t), t, \underline{\xi})) + N(-d(\delta(t), t, \bar{\xi})) \right) + W e^{-r(T-t)} \left( N(-f(\delta(t), t, \underline{\xi})) - N(-f(\delta(t), t, \bar{\xi})) \right), \quad (38)$$

where  $1_{\{\xi(T) \geq \bar{\xi}\}}$  denotes the indicator function. Finally, by substituting (35) and (36) into (34), we obtain the equilibrium price.

### Proof of proposition 6

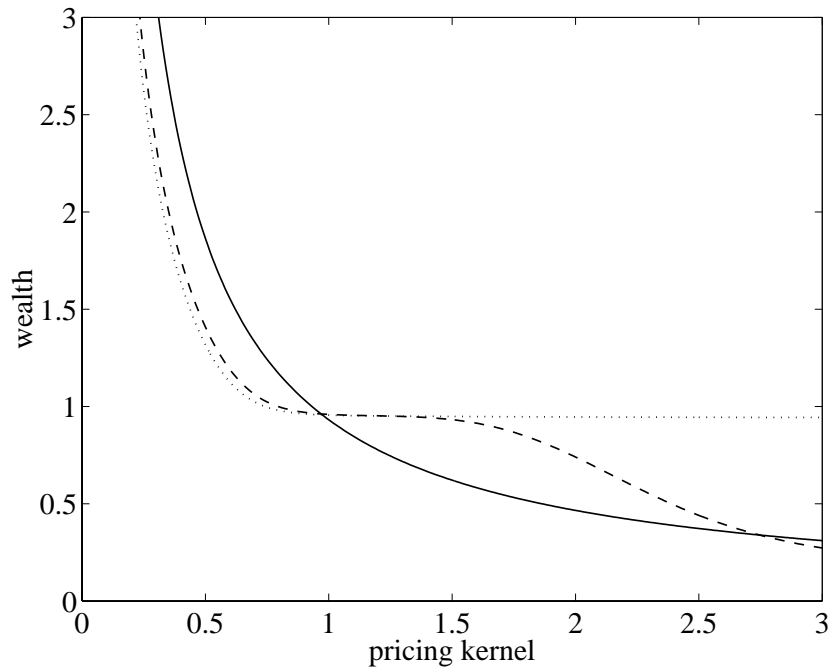
This proof can be found in Berkelaar and Kouwenberg (2001b).

Figure 1  
Wealth at time  $T = 1$



This figure shows the optimal wealth at time  $T = 1$ , for the unregulated agent (solid line), the VaR risk manager (dashed line) and the portfolio insurer (dotted line).

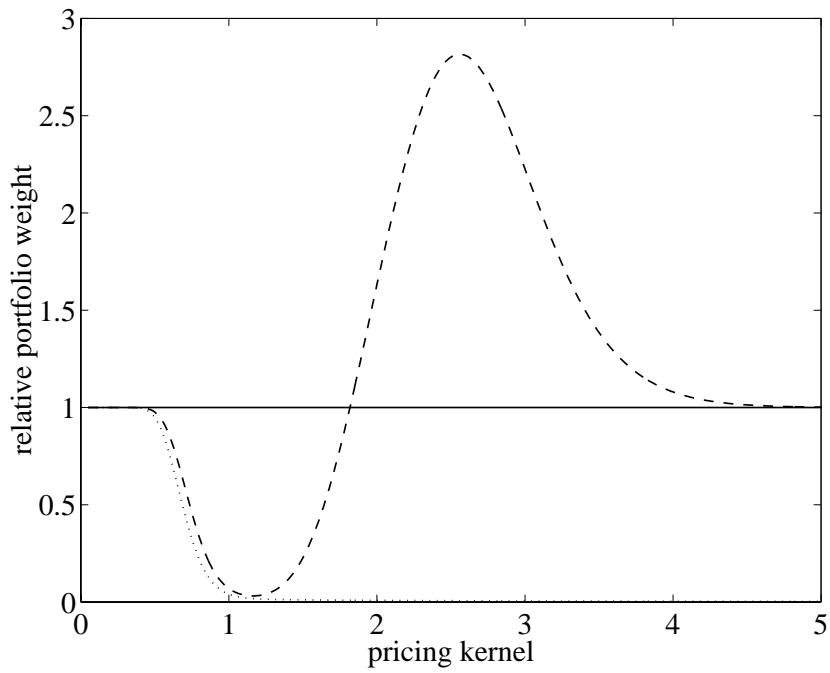
Figure 2  
Wealth at time  $t = 0.75$



This figure shows the optimal wealth at time  $t = 0.75$ , for the unregulated agent (solid line), the VaR risk manager (dashed line) and the portfolio insurer (dotted line).

Figure 3

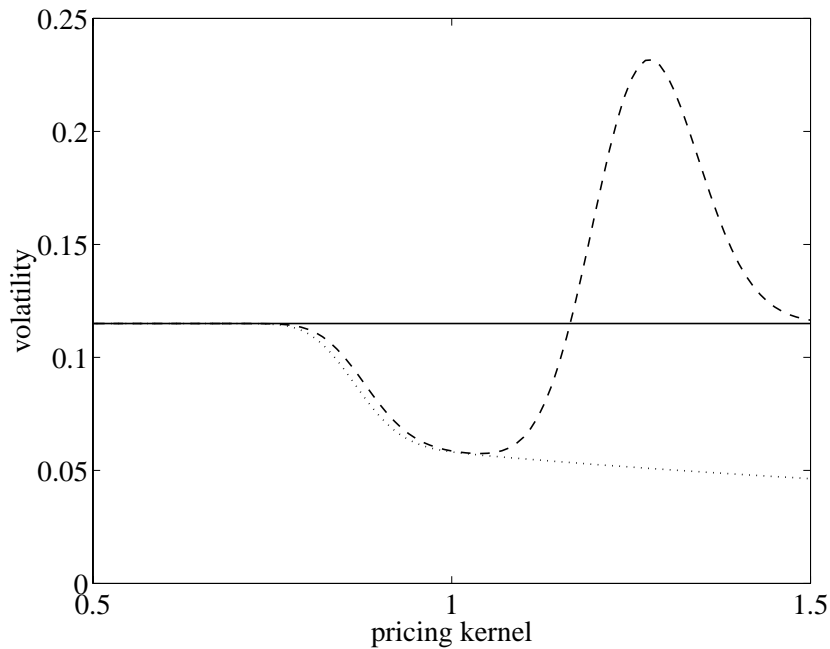
**Relative portfolio weight of stocks**



This figure shows the portfolio weight of stocks relative to the unregulated agent at time  $t = 0.75$ , for the unregulated agent (solid line), the VaR risk manager (dashed line) and the portfolio insurer (dotted line).

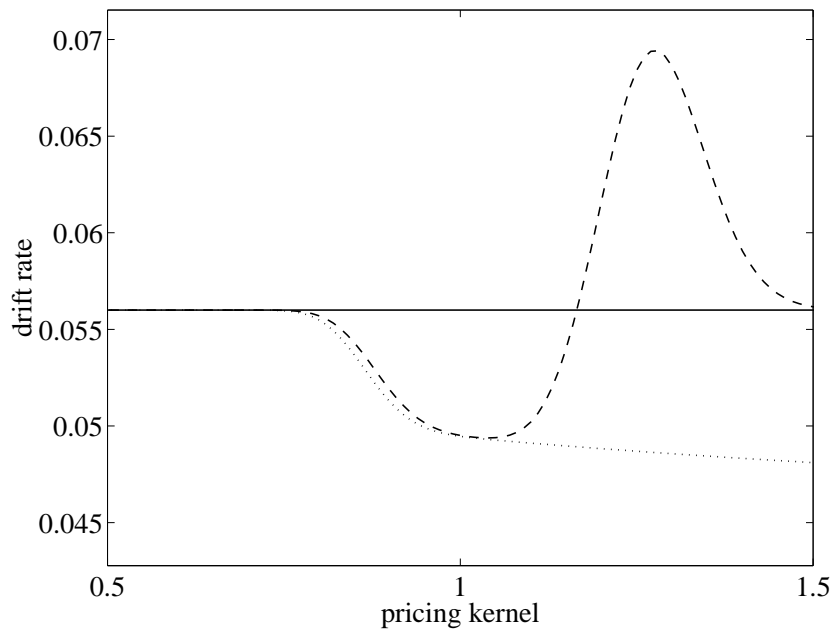
Figure 4

**Volatility in equilibrium**



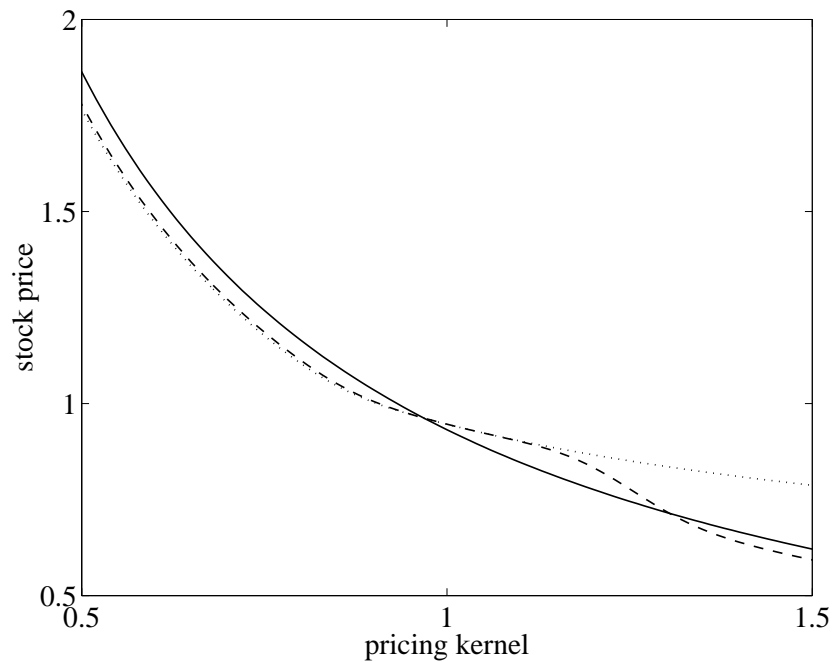
This figure shows the volatility of the stock price process in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers (dashed line) and the economy with 50% portfolio insurers (dotted line).

Figure 5  
Drift rate in equilibrium



This figure shows the drift rate of the stock price process in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers (dashed line) and the economy with 50% portfolio insurers (dotted line).

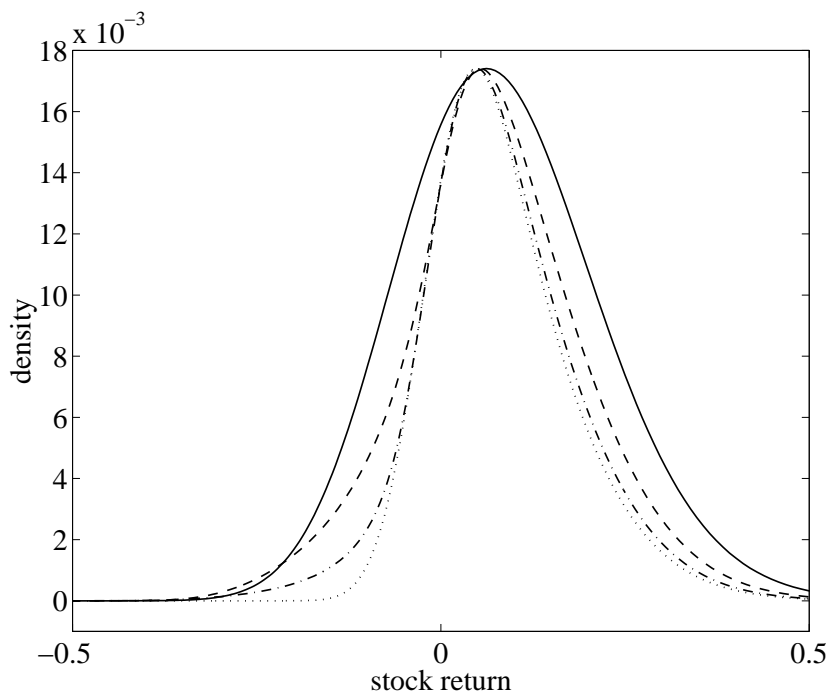
Figure 6  
Stock price in equilibrium



This figure shows the stock price in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers (dashed line) and the economy with 50% portfolio insurers (dotted line).

Figure 7

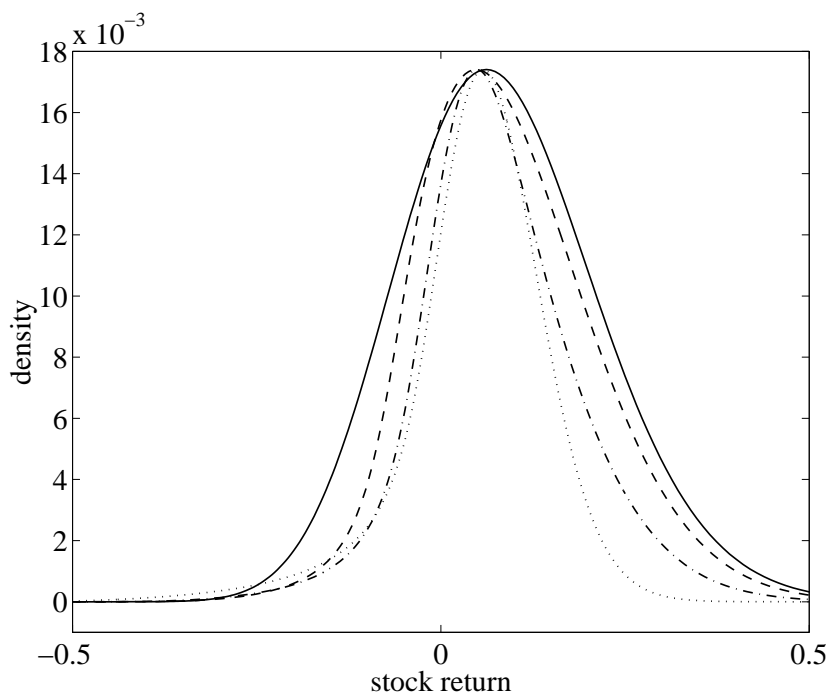
**Stock return distribution in equilibrium**



This figure shows the return distribution of stocks in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers with  $\alpha = 5\%$  (dashed line), the economy with 50% VaR risk managers with  $\alpha = 1\%$  (dashed-dotted line), and the economy with 50% portfolio insurers (dotted line).

Figure 8

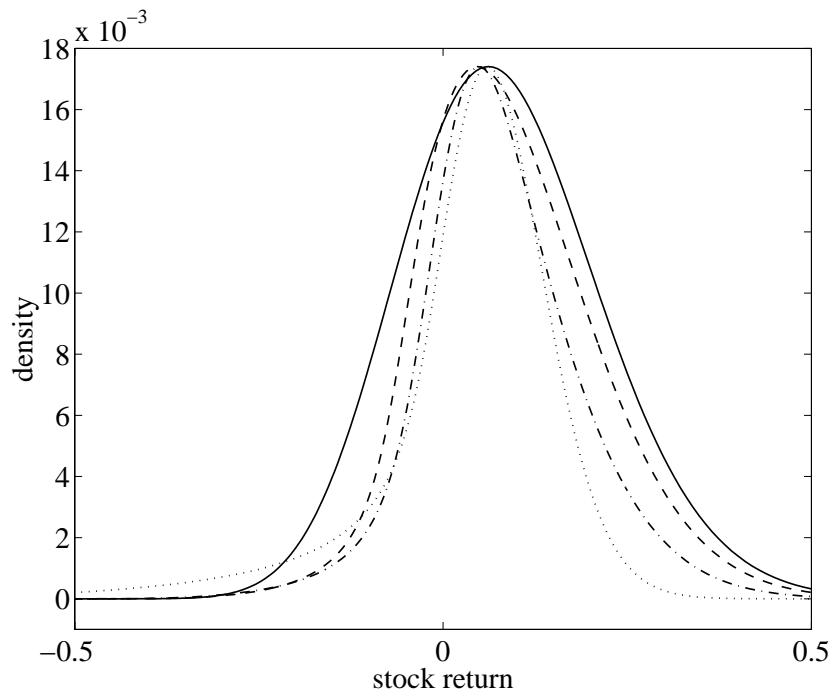
**Stock return distribution in equilibrium**



This figure shows the return distribution of stocks in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers with  $W = 0.90\%$  (dashed line), the economy with 50% VaR risk managers with  $W = 0.95\%$  (dashed-dotted line), and the economy with 50% VaR risk managers with  $W = 0.99\%$  (dotted line).

Figure 9

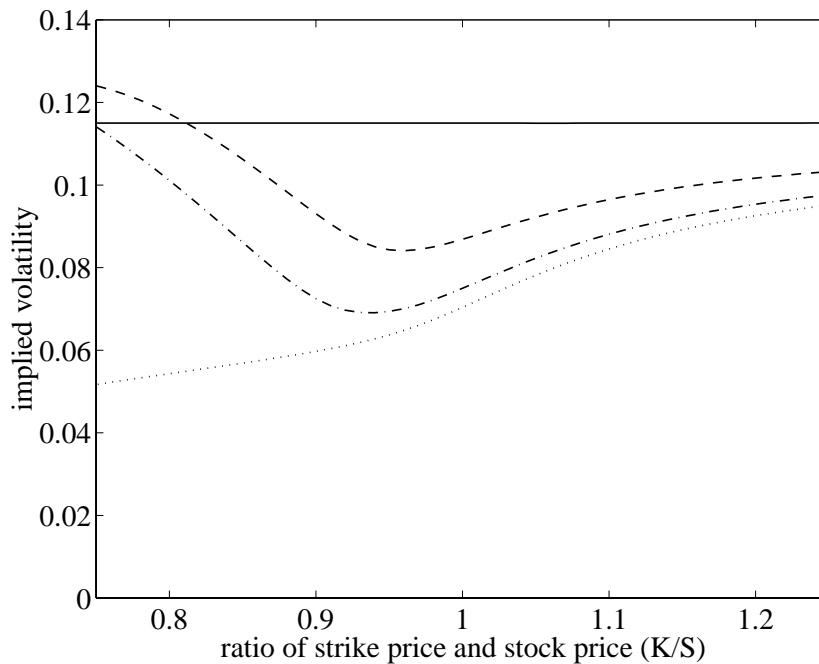
**Stock return distribution in equilibrium**



This figure shows the return distribution of stocks in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers with  $\rho_1 = \rho_2 = 20$  (dashed line), the economy with 50% VaR risk managers with  $\rho_1 = \rho_2 = 10$  (dashed-dotted line), and the economy with 50% VaR risk managers with  $\rho_1 = \rho_2 = 1$  (dotted line).

Figure 10

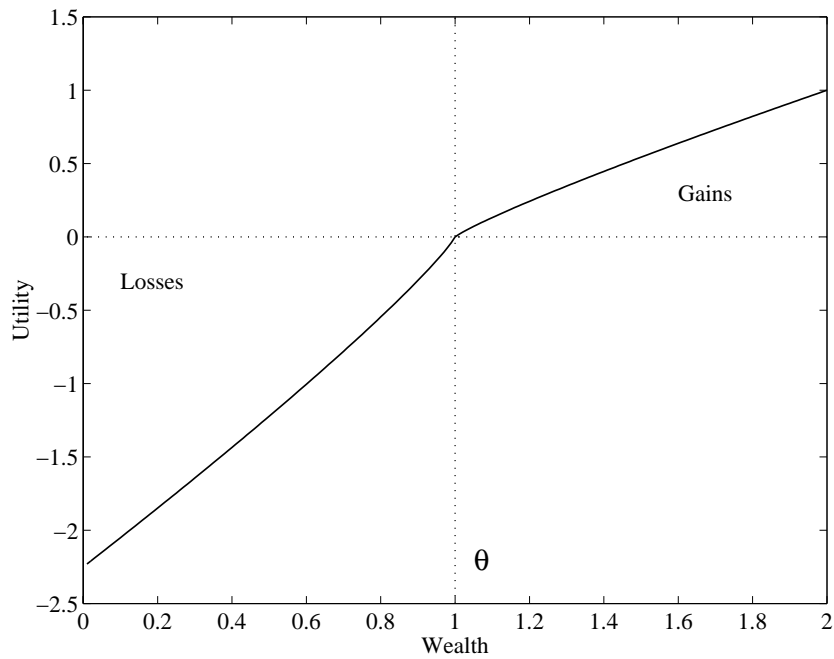
**Implied volatility of equilibrium option prices**



This figure shows the implied volatility of option prices at time  $t = 0$ . The call and put options are of the European type, with maturity at time  $t = 0.75$ . The figure displays equilibrium implied volatilities in the economy with unregulated agents (solid line), the economy with 50% VaR risk managers with  $\alpha = 5\%$  (dashed line), the economy with 50% VaR risk managers with  $\alpha = 1\%$  (dashed-dotted line), and the economy with 50% portfolio insurers (dotted line).

Figure 11

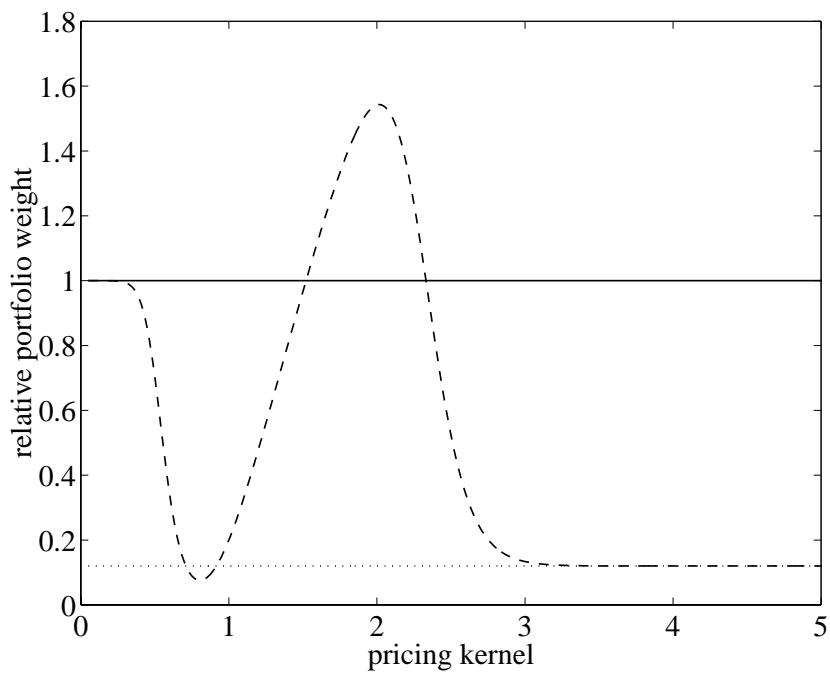
Utility function of prospect theory



This figure shows the utility function of prospect theory, with parameter values  $\gamma_1 = \gamma_2 = 0.88$ ,  $A = 2.25$ ,  $B = 1.0$  and  $\theta = 1.0$ .

Figure 12

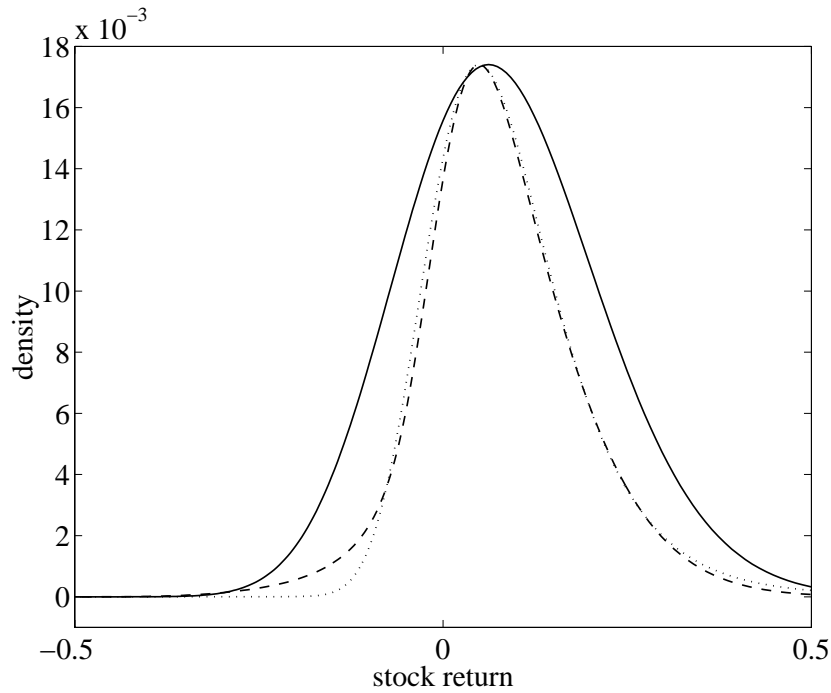
Relative portfolio weight of stocks



This figure shows the portfolio weight of stocks relative to an unregulated agent with  $\gamma = 0.88$  at time  $t = 0.75$  (solid line), for a loss-averse agent with  $\rho_{LA} = 100$  (dashed line) and for an unregulated agent with  $\gamma = 0$  (dotted line).

Figure 13

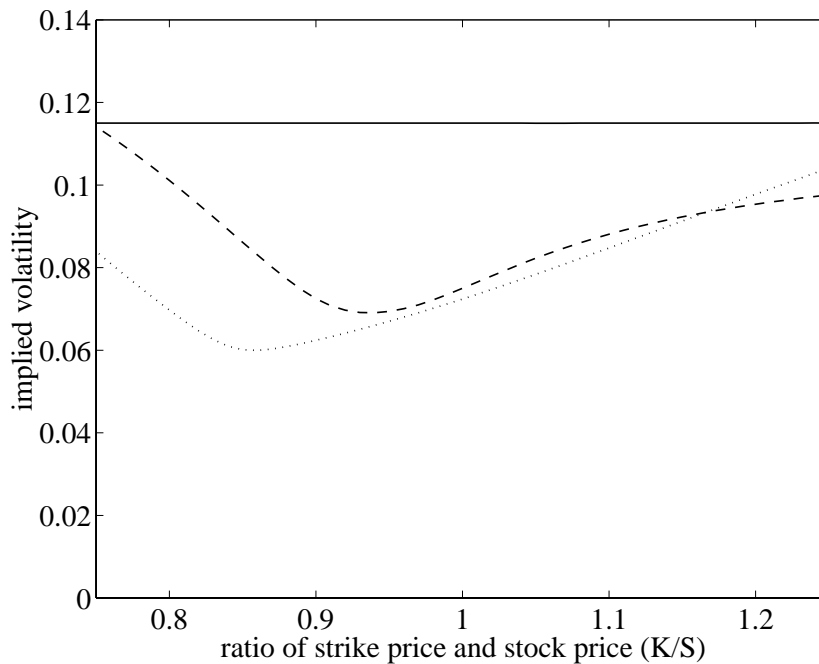
**Stock return distribution in equilibrium**



This figure shows the return distribution of stocks in equilibrium at time  $t = 0.75$ , for the economy with unregulated agents (solid line), the economy with 50% VaR risk managers with  $\alpha = 1\%$  (dashed line), and the economy with 50% loss-averse agents with  $\rho_{LA} = 100$  (dotted line).

Figure 14

**Implied volatility of equilibrium option prices**



This figure shows the implied volatility of option prices at time  $t = 0$ . The call and put options are of the European type, with maturity at time 0.75. The figure displays equilibrium implied volatilities in the economy with unregulated agents (solid line), the economy with 50% VaR risk managers with  $\alpha = 1\%$  (dashed line), and the economy with 50% loss-averse agents with  $\rho_{LA} = 100$  (dotted line).

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