Macroeconomic default modelling and stress testing

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Abstract

This paper applies a macroeconomic-based model for estimating default probabilities on Dutch data. The first part of the paper focuses on the relation between macroeconomic variables and the default behaviour of Dutch firms. A convincing relationship with GDP growth and oil price, and to a lesser extent, the interest and exchange rate exists. The second part assesses the default behaviour based on a stress scenario of two consecutive quarters of zero GDP growth. It can be concluded that a short recession of two quarters zero GDP growth does not influence the default rate significantly. A stress test scenario of two quarters zero GDP growth therefore underestimates the true credit risk.

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1 Introduction

Estimating default probabilities is the first step in assessing the credit exposure and potential losses faced by financial institutions. Default probabilities are also the basic inputs when evaluating systemic risk and stress testing financial systems. Therefore, predictors of credit risk are of natural interest to practitioners in the financial industry as well as to regulators, especially under the new capital adequacy framework (Basel II), which encourages the active involvement of banks in measuring the likelihood of defaults.

The finance literature has produced a variety of models that attempt to measure the probability of default. In this paper we consider the default rate in relation to macroeconomic variables. Macroeconomic-based models are motivated by the observation that default rates in the financial, corporate, and household sectors increase during recessions. This observation has led to the implementation of econometric models that attempt to explain default indicators, such as default probabilities or default rates, using macroeconomic variables.

We will explore the relationship between the default rate and the macroeconomy by developing a logit model with macroeconomic parameters. This fairly simple model has the following advantages: The model is straightforward, relatively easy to understand and has robust results. An interesting feature of the model is that it takes the correlation of default rates amongst sectors into account. Although it is often confirmed in the literature that the default is highly correlated amongst sectors, relatively little work has been done in previous research on estimating such a "correlation" factor. We select macroeconomic variables for which particular concerns of movements in unfavourable directions exist.

By means of the logit model and selected variables we will firstly assess which macroeconomic variables are related to the default behaviour of firms and secondly, we will examine the default behaviour in 2007 given an unfavourable macroeconomic scenario of two quarters zero GDP growth in quarters 2006.3 and 2006.4. We compare it to a base scenario and a 2.5% worst case scenario.

The remainder of this paper is organized as follows. Section 2 discusses related studies. Section 3 describes the construction of the dataset. Section 4 formulates the estimation model. Section 5 discusses the estimation results. Section 6 studies the stress test scenarios. Section 7 concludes.

2 The concept of macroeconomic-based models

Estimating default probabilities is a challenging subject. Types of models to assess credit risk can be broadly classified into either market-based models or fundamental-based models. Market based models build on Merton´s option pricing theory and rely on security prices. Chan-Lau [3] distinguishes four approaches within fundamental-based modelling to model default probabilities: Macroeconomic-based, Accounting-based, Rating-based and Hybrid models.

Macroeconomic models, used in this research, explain changes in the default rate out of macroeconomic conditions. These macroeconomic variables are cyclical indicators, e.g. GDP growth or interest rates, and financial market indicators, e.g. stockmarket prices and stockmarket volatility. Accounting-based models or credit scoring models generate default probabilities or credit ratings for individual firms using accounting information. Rating-based models can be used to infer default probabilities when external ratings information is available. Hybrid models generate default probabilities using as explanatory variables a combination of economic variables, financial ratios and ratings data.

Chan-Lau [3] lists some advantages and disadvantages of macroeconomic models. An advantage is that this type of model is very suitable to design stress scenarios. Also, because long data series are available for most countries, it is also possible to conduct cross-country comparative studies. Furthermore, the default rate used to estimate the model is observed historically, hence, one can avoid making assumptions.

On the other hand, a disadvantage of macroeconomic-models is that the time-span of the data needs to be longer than one business cycle, otherwise the model would not capture the impact of the business cycle on default probabilities. Furthermore, This type of models is subject to Lucas critique since the parameters or functional forms are unlikely to remain stable, i.e. it is almost impossible to capture the complex interaction between the state of the economy and the default risk. Finally, aggregate economic data are usually reported at substantial lags. This makes it difficult to estimate or forecast macro-economic models with up-to-date information.

Macroeconomic models can be classified into exogenous and endogenous models, i.e. whether the model allows feedback between financial distress and the explanatory economic variables. The first category of macroeconomic-based models assumes that the economic variables are exogenous and not affected by financial distress. The general approach to modelling this category is described by the following equation:

$$pd_t = g(x_1, x_2, ..., x_n) + \varepsilon \tag{1}$$

where pd is the probability of default, over a given period t. A general aggregate model sets pd_t equal to a function g, $X = (x_1, x_2, ..., x_n)$ a function of a set of economic variables and a random variable ε.

A problem of the exogenous approach is the fact that the relation between macroeconomic variables and the default rate is assumed to be the same during periods of economic downturn and expansion. This seems intuitively implausible.

The second category of macroeconomic-based models assumes that the economic variables are endogenous and differ in times of financial distress. The typical econometric framework used in these models is the vector autoregressive (VAR) methodology. See for example Hoggarth ea.[10]. We can write the VAR in a more general form as:

$$Z_{t+1} = \alpha_t + \sum_{j=1}^{p} \beta_j z_{t+1-j} + \varepsilon_{t+1} \tag{2}$$

where α_t is a constant vector, β_j are (lagged) coefficients matrices, $\varepsilon_{t+1}$is a vector of residuals/shocks, and z is the vector of endogenous variables, which includes both default probabilities and aggregate economic variables associated to the state of the business cycle. In principle, inference in VAR models is sensitive to the choice of lags. If a large number of lags is included, degrees of freedom are lost. If the lag length is too small, important lag dependencies may be omitted.

Literature about which macroeconomic variables are related to the default rate can be divided into traded and non-traded firms. Appendix A lists a set of papers on macroeconomic default modelling with a short description. These papers mainly confirm the significance of GDP growth. Relations with stock market variables have been identified several times but only for traded firms. Apparently, the stock market is a better indicator for the financial healthiness of traded firms than it is for the mostly small and medium sized, nontraded firms examined in this research. Furthermore, there is some confirmation that interest and exchange rate are significant in certain sectors. Remarkably, none of the papers examines the oil price.

3 The model

3.1 Aggregate default modelling

We model the default rates at an aggregate level, which does not allow for firm specific explanatory variables. Appendix B reviews econometric theory on which parameter estimation, hypothesis testing and confidence set construction in this paper are based.

Consider a general aggregate model which can be estimated by maximum likelihood. Let pd_t be the fraction (proportion) of firms that defaults in period t. pd_t is equal to a function $g(.)$ of relevant explanatory variables $\mathbf{z}_t$, a parameter vector $\boldsymbol{\theta}$ and a disturbance υ_t. Controlling the distribution of υ_t controls the distribution of pd_t.

$$pd_t = g(\boldsymbol{\theta}, \mathbf{z}_t, \upsilon_t) \tag{3}$$

More specifically, $pd_{t,i}$ is the fraction of firms that defaults in period t within sector i, $\forall i \in \{0, 1, \ldots, s\}$, with s the total number of sectors in the economy. The economy as a whole is denoted by $i = 0$. Let $\mathbf{z}_t$ be a vector of variables including intercept, relevant for the default rate at time t, $\upsilon_{t,i}$ a disturbance and β_i a vector of parameters.

$$pd_{t,i} = \frac{\exp\left(\mathbf{z}_t'\boldsymbol{\beta}_i + \upsilon_{t,i}\right)}{1 + \exp\left(\mathbf{z}_t'\boldsymbol{\beta}_i + \upsilon_{t,i}\right)} \tag{4}$$

Taking the logit[1] of both sides, we find

$$\tilde{pd}_{t,i} : \text{logit}(pd_{t,i}) = \mathbf{z}_t'\boldsymbol{\beta}_i + \upsilon_{t,i}. \tag{5}$$

From this, we obtain two separate models, an economy model ($i = 0$) and a sector model $\forall i \in \{1, 2, \ldots, s\}$ .For the economy default rate ($i = 0$) we assume the disturbances $\upsilon_{t,0}$ are independent and identically distributed (iid). Let $\sigma^2_{\psi,0}$ =var$(\upsilon_{t,0})$, the economy model is

$$\begin{aligned} \tilde{pd}_{t,0} &= \mathbf{z}_t'\boldsymbol{\beta}_0 + \upsilon_{t,0} \\ &\quad \text{where } \upsilon_{t,0} \overset{iid}{\sim} (0, \sigma^2_{\psi,0}). \end{aligned} \tag{6}$$

For the sector default rates, $\forall i \in \{1, 2, \ldots, s\}$, the disturbances are divided into a latent systematic (ξ_t) and an idiosyncratic ($\psi_{t,i}$) part. The systematic part should capture the correlation between the sector default rates. An advantage of taking the correlation of the sector default rates into account is that the combination of such a factor with macroeconomic indicators provides a natural test of the specification of the macro-relationship. If the macro indicators are

[1] The logit transformation is given by $\text{logit}(x) = \ln\left(\frac{x}{1-x}\right)$. Since $\frac{\exp(\text{logit}(x))}{1+\exp(\text{logit}(x))} = \frac{x/(1-x)}{1+x/(1-x)} = x$, the equation $x = \frac{\exp(y)}{1+\exp(y)}$ is solved for y by y =logit(x).

indeed informative then the fluctuations explained by the factor will be relatively small.

Let $\sigma_{\xi,i}$ and $\sigma_{\psi,i}$ be nonnegative parameters $\forall i \in \{1, \ldots, s\}$. The sector model is:

$$\begin{aligned} \tilde{pd}_{t,i} &= \mathbf{z}_t' \boldsymbol{\beta}_i + \upsilon_{t,i} \\ \text{where } \upsilon_{t,i} &= \sigma_{\xi,i}\xi_t + \sigma_{\psi,i}\psi_{t,i} \\ & \quad \xi_t \overset{iid}{\sim} (0,1)\,,\ \psi_{t,i} \overset{iid}{\sim} (0,1)\,. \end{aligned} \tag{7}$$

Estimation and inference of the parameters are based on maximizing the Gaussian quasi loglikelihood.

3.2 Dynamic effects of shocks in the aggregate default model

This section assesses the effect of shocks in $\mathbf{z}_t$ on the long and short term. Therefore, we differentiate (4) with respect to $\mathbf{z}_t$ to find

$$D_{\mathbf{z}_t} pd_{t,i} = \frac{pd_{t,i}\boldsymbol{\beta}_i}{\left(1 + \exp\left(\tilde{pd}_{t,i}\right)\right)^2}. \tag{8}$$

Given that $\tilde{pd}_{t,i}$ is in general low, we may ignore the denominator. Accordingly, for small $\Delta\mathbf{z}_t$ the elements of $\boldsymbol{\beta}_i$ are approximate semi-elasticities:

$$\Delta\% pd_{t,i} \approx \boldsymbol{\beta}_i' \Delta\mathbf{z}_t. \tag{9}$$

In order to capture persistence in the default rate, we include the lagged default rate as explanatory variable. Let $\mathbf{z}_t^*$ denote explanatory variables other than the lagged default rate and the intercept and $\boldsymbol{\beta}_i^*$ the corresponding parameter vector. We may write equation (5) as

$$\tilde{pd}_{t,i} = \boldsymbol{\beta}_{i,0} + \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i} + \boldsymbol{\beta}_i^{*\prime}\mathbf{z}_{t-1}^* + \upsilon_{t,i} \tag{10}$$

or, equivalently,

$$\tilde{pd}_{t,i} = \frac{\boldsymbol{\beta}_{i,0}}{1 - \boldsymbol{\beta}_{i,1}} + \boldsymbol{\beta}_i^{*\prime} \sum_{j=0 i,1}^{\infty \boldsymbol{\beta} j} \mathbf{z}_{t-1-j}^* + \sum_{j=0 i,1}^{\infty \boldsymbol{\beta} j} \upsilon_{t-j,i}. \tag{11}$$

The inclusion of the lagged default rate makes the current default rate depend on all lags of the explanatory variables with coefficients declining at rate $\boldsymbol{\beta}_{i,1}$. In other words, a default depends not only on the previous period but on the entire history with more recent developments being more important.

In order to estimate the short and long run effect of a small shock $\Delta\mathbf{z}^*$ occurring in period t_0 and persisting indefinitely through time, equation (9) and (11) are combined. Applying (9) to (11) and considering $\sum_{j=0 i,1}^{\infty \boldsymbol{\beta} j} \mathbf{z}_{t-1-j}^{*}$ as explanatory vector, the estimated effects are:

$$\begin{aligned} \text{Short run effect} &: \qquad (12) \\ \Delta\% pd_{t_0+1,i} &\approx \boldsymbol{\beta}_i^{*\prime} \Delta \left(\sum_{j=0 i,1}^{\infty \boldsymbol{\beta} j} \mathbf{z}_{t_0-j}^{*} \right) = \boldsymbol{\beta}_i^{*\prime} \Delta \mathbf{z}^{*} \\ \text{Long run effect} &: \\ \lim_{t\to\infty} \Delta\% pd_{t+1,i} &\approx \lim_{t\to\infty} \boldsymbol{\beta}_i^{*\prime} \Delta \left(\sum_{j=0 i,1}^{\infty \boldsymbol{\beta} j} \mathbf{z}_{t-j}^{*} \right) = \frac{\boldsymbol{\beta}_i^{*\prime}}{1-\boldsymbol{\beta}_{i,1}} \Delta \mathbf{z}^{*} \qquad (13) \end{aligned}$$

Note that at time $t > t_0$ the effect on the default rate is

$$\Delta\% pd_{t,i} \approx \boldsymbol{\beta}_i^{*\prime} \Delta \left(\sum_{j=0 i,1}^{\infty \boldsymbol{\beta} j} \mathbf{z}_{t-1-j}^{*} \right) = \left(1 - \boldsymbol{\beta}_{i,1}^{t-t_0}\right) \frac{\boldsymbol{\beta}_i^{*\prime}}{1-\boldsymbol{\beta}_{i,1}} \Delta z^{*}. \qquad (14)$$

From (14) it is clear that at time t a fraction $\left(1 - \boldsymbol{\beta}_{i,1}^{t-t_0}\right)$ of the long run effect is approximately realized. For this reason $\boldsymbol{\beta}_{i,1}$ controls the speed at which the default rate reacts to shocks. For shocks persisting shorter than indefinitely, the long run effect can be interpreted as an upper bound to the maximum effect that will be attained.

4 Data description

The default rate is defined as the ratio of the number of defaulted firms and total number of firms during a period t. Let $pd_{t,0}$ be the fraction of all firms that default during quarter t and $pd_{t,i}$ the fraction of firms in sector i that default during quarter t. We will call $pd_{t,0}$ and $pd_{t,i}$ respectively the economy and sector i default rate.

$$
\begin{aligned}
pd_{t,0} &= \frac{\text{Number of defaults in all sectors during quarter } t}{\text{Average number of firms in all sectors during quarter } t} \\
pd_{t,i} &= \frac{\text{Number of defaults in sector } i \text{ during quarter } t}{\text{Average number of firms in sector } i \text{ during quarter } t}
\end{aligned}
\tag{15}
$$

The dataset consists of Dutch defaults per quarter during the period 1983.1-2006.2 (94 quarters). The average economy default rate is about .23% per quarter or just below 1% per year. The total number of firms in the economy varies between 408,665 (1983.1) and 652,367 (2006.2). Most firms in the Netherlands are rather small.

We chose macroeconomic variables for which particular concerns of movements in unfavourable directions exist so that the variables are useful to do stress testing or scenario analysis. We stress that these need not be the variables that add the most explanatory or forecasting power. The following variables are selected:

Gross domestic product GDP equals aggregate demand of an economy. Aggregate demand is related to the sales of firms. Lower GDP growth means lower growth in sales of firms. The lower GDP growth, the harder it is for firms to generate income through sales. Lower income thus increases the possibility that firms cannot meet their obligations and default.

Interest rate Firms often finance their activities partly by debt. The costs of firms are therefore positively related to interest rates. If interest rates are higher, firms have more cost and are more likely to default.

Exchange rate The exchange rate is expressed as the price of domestic currency in terms of foreign currency. Firms in sectors doing a lot of international business are expected to be affected by exchange rates. However, the sign of the relation is ambiguous. Business conditions of importing firms depend positively (increasing demand) on the exchange rate because imports become cheaper if the exchange is high. Business conditions of exporting firms depend negatively (drop in demand) on the exchange rate because exports become more expensive if the exchange rate is high.

Stock market return and volatility Merton's theory predicts that the probability of default is negatively related to stock return and positively to volatility. Stock market return and, thereby also, volatility are popular for scenario analysis. However, since the vast majority of firms examined in this research is not listed on a stock exchange, it is doubtful wether the stock market sufficiently reflects the probability of default within this research.

Oil price The oil price affects the price of most products used by firms. Therefore, the cost of firms and thus their probability of defaulting are positively related to the oil price.

4.1 Interesting properties of the default rate

In this subsection we analyse the properties of the default rate. We observe that the default rate is persistent, negatively related to the business cycle and correlated between sectors. No seasonal effects are observed.

We examine if the economy default rate is persistent by estimating an autoregressive model of order 1 (AR(1)) with Ordinary Least Squares, which results in:

$$pd_{t,0} = .00 + .85 pd_{t-1,0} + \upsilon_t \tag{16}$$

The AR(1) model captures most of the serial correlation. The high coefficient of the first lag confirms that the default rate is persistent.

Intuitively, one would expect a negative relation of the default rate with

the business cycle. The upper panel of Figure 1 plots the economy default rate and $\Delta\%GDP_{t-1}$ against time. The relation is not obvious because GDP growth fluctuates a lot while the default rate is persistent. However, adding $\Delta\%GDP_{t-1}$ to (16) and estimating again we have

$$pd_{t,0} = .00 + .82 pd_{t-1,0} - .00\Delta\%GDP_{t-1} + \upsilon_t \tag{17}$$

or, equivalently,

$$pd_{t,0} = \frac{.00}{1 - .82} - .00 \sum_{j=0}^{\infty} .82^j \Delta\%GDP_{t-1-j} + \sum_{j=0}^{\infty} .82^j \upsilon_{t-j}. \tag{18}$$

The intercept and the coefficient of GDP growth are rounded to .00. They deviate significantly from zero though. We can conclude that persistency actually implies that the default rate is related to a weighted sum of lags of GDP growth. This relation is shown in the lower panel of Figure 1, which plots the economy default rate and $\sum_{j=0}^{19} .82^j \Delta\%GDP_{t-1-j}$ against time.

All correlations of the default rates between sectors are significant at the 1% level. So, highly significant correlation exists between the sectors.

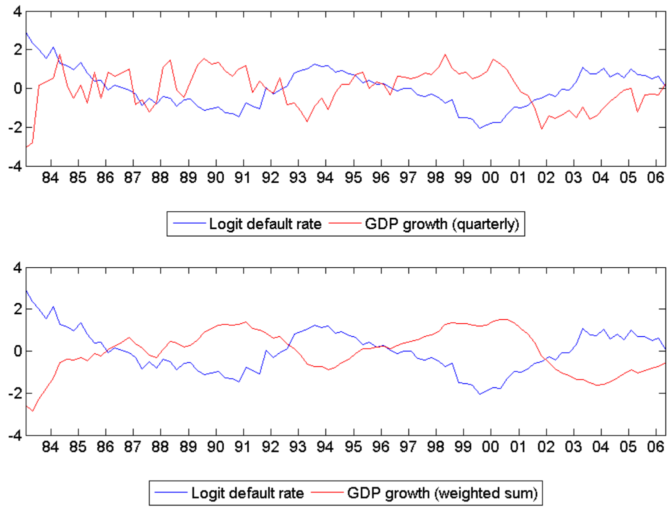

Figure 1: Economy default rate and GDP growth Explanation: The upper panel plots $pd_{t,0}$ and $\Delta\%\mathrm{GDP}_{t-1-j}$ against time. The lower panel plots $pd_{t,0}$ and $\sum_{j=0}^{19} .82^j \Delta\%\mathrm{GDP}_{t-1-j}$ against time. All series are standardized to have zero mean and unit variance.

Finally, we check for seasonal effects of the default rate. The following simple model is estimated by OLS. In this model, the function 1_A is the indicator function for the event A.

$$pd_{t,i} = \sum_{j=1}^{4} \delta_j 1_{t \in \text{ quarter } j}(t) + \upsilon_t \tag{19}$$

We test the $\delta_1 = \delta_2 = \delta_3 = \delta_4$. No indication of seasonal effects is found.

5 Estimation results

First, we substitute the selected macroeconomic variables for $\mathbf{z}_t^*$ in $\tilde{pd}_{t,i} = \boldsymbol{\beta}_{i,0} + \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i} + \boldsymbol{\beta}_i^{*\prime}\mathbf{z}_{t-1}^* + \upsilon_{t,i}$ (see equation 10) and identify which macroeconomic variables are significant. A convincing relationship with GDP growth and oil price, and to a lesser extent, the interest and exchange rate exists. Stock market return and volatility have no relation with default rate.

Gross domestic product The estimated coefficients of GDP growth have a negative sign as expected. The sectors Industry and mining, Transport, storage and communication, Financial services and Rental and corporate services are significant at the 1% level, which means that they have the strongest link with the default rate. For the overall economy the hypothesis of no relation with GDP growth is firmly rejected.

Interest rate Only the Construction sector has a significant relation between the level of the short rate and the default rate. A reason for results in this sector to be different from those in other sectors is that construction firms are substantially affected by interest rates through another channel than cost of debt; Private households find it easier to finance construction work on their homes when interest rates are low. This should especially affect small construction firms. Indeed about 86% of construction firms has less than 10 employees. The strong rejection of the hypothesis of equal coefficients supports the view that the Construction sector is an exception. Demand for construction work is negatively related to interest rates and it is interesting that the level and not the change of the interest rate is significant. Apparently, people or firms react to the level and hardly to changes in the interest rate.

We also tested the long instead of the short interest rate in $\mathbf{z}_t^*$. This yielded somewhat stronger but qualitatively similar results. Due to correlation between the short and the long rate it is not sensible to include both rates. Various term spreads can be included however. They are, with a one quarter lag, tested insignificant. Given that the term structure forecasts GDP growth, the term structure does have forecasting power when used with a lag of several years.

Exchange rate The most significant sector is Transport, storage and communication as expected, particularly in the Netherlands. For the overall economy the relation is modestly strong. No relation with the change in the exchange rate was found. Apparently, the level of exchange rate is decisive in trading. This is remarkable since one would expect firms to get used to the level and only react to changes.

Stock market return and volatility The signs of the coefficients are both negative and positive and both in case of return and volatility the coefficients do not deviate significantly from zero. We may therefore conclude that the default rate is unrelated to the stock market. A reason for this is that our dataset consists of mostly untraded firms.

Oil price Most coefficients of the level of the oil price are significant. All coefficients have the expected sign. Again, the level of the oil price is apparently more important than the change. Furthermore, remarkably, the oil price is the only significant variable for which there is no statistical reason to doubt the hypothesis of equal coefficients. This suggests that the dependence of sectors on the oil price is equal.

Second, we estimate equation (10) without the variables stock market return and volatility that were found to be insignificant. This leaves GDP growth, the short interest rate (level), the exchange rate (level) and the oil price (level) in $\mathbf{z}_t^*$. In general the behaviour of the variables is the same as noted above. Table 5.1 lists the results.

Table 1 presents short and long run effects of shocks in macroeconomic variables and their 95% confidence intervals. By applying the coefficients of the first lag into equation (14), we estimate that a year after a persistent shock, about 60% to 95% of the long run effect is realized.

Although GDP growth shocks will be examined more closely in the next section, we note already that the effects of GDP growth shocks are somewhat low. A persistent 3% decrease in GDP growth raises the long run default rate only by about 25%.

The explanatory power of the model can be assessed by comparing the variances of the macroeconomic variables, the latent systematic (ξ_t) and the idiosyncratic ($\psi_{t,i}$) disturbances. The first lag is excluded from the decomposition because (i) it explains most of the variance and (ii) it is not independent from the macroeconomic variables.

Recall that $\upsilon_{t,i} = \sigma_{\xi,i}\xi_t + \sigma_{\psi,i}\psi_{t,i}$. We can rewrite (10) into

$$\tilde{pd}_{t,i} - \boldsymbol{\beta}_{i,0} - \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i} = \boldsymbol{\beta}_i^{*\prime}\mathbf{z}_{t-1}^* + \sigma_{\xi,i}\xi_t + \sigma_{\psi,i}\psi_{t,i}.$$

This leads to the following variance decomposition, given that the systematic and idiosyncratic disturbances are independent:

$$\mathrm{var}\left(\tilde{pd}_{t,i} - \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i}\right) = \boldsymbol{\beta}_i^{*\prime}\mathrm{var}\left(\mathbf{z}_{t-1}^*\right)\boldsymbol{\beta}_i^* + \sigma_{\xi,i}^2 + \sigma_{\psi,i}^2$$

or, equivalently,

$$\frac{\boldsymbol{\beta}_i^{*\prime}\mathrm{var}\left(\mathbf{z}_{t-1}^*\right)\boldsymbol{\beta}_i^*}{\mathrm{var}\left(\tilde{pd}_{t,i} - \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i}\right)} + \frac{\sigma_{\xi,i}^2}{\mathrm{var}\left(\tilde{pd}_{t,i} - \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i}\right)} + \frac{\sigma_{\psi,i}^2}{\mathrm{var}\left(\tilde{pd}_{t,i} - \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i}\right)} = 1. \tag{20}$$

	Intercept	First lag	GDP growth	Short rate	ln(ER)	ln(Oil price)	Systematic s.d.	Idiosyncratic s.d.
Ind., min.	−3.41***	.67***	−2.75***	.97*	.28	.06	.10***	.06***
Construction	−1.77	.77***	−1.24	1.52**	.05	.03	.09***	.07***
Trade, rep. cons.	−3.39***	.78***	−1.25**	.71	.40*	.07*	.06***	.07***
Catering	−4.84**	.84***	−.50	.30	.71*	.16**	.13***	.09***
Trans., stor., com.	−10.27***	.50***	−3.22***	−.58	1.48***	.18***	.12***	.08***
Financial	−4.79**	.83***	−2.89***	−.18	.78**	.12**	.10***	.07***
Rental, corp.	−4.71***	.69***	−3.07***	1.18**	.55**	.08*	.07***	.08***
Other	−2.82	.59***	−1.15	.33	−.10	.06	.15***	.06***
Economy	−2.61**	.84***	−1.40***	.71	.32*	.05*	.08***	.00
Pooled	−3.23***	.79***	−1.77***	.78*	.32*	.07**	.10***	.07***
(P-value equal coefficients	.0003	.0000	.0421	.0011	.0001	.1305	.0000	.1153)

Table 5.1: Estimated coefficients of the parsimonious specification

Explanation: Model (10) is estimated with the macroecomic variables selected in section **??** included in z_t^*. All estimated parameters are reported. Pooled results are obtained by estimating the sector model under the restriction $\beta_{1,j} = ... = \beta_{8,j}$ or $\sigma_{j,1} = ... = \sigma_{j,8}$ for certain j while allowing the other parameters to differ per sector. These restrictions are tested and the p-values are reported. 1, 5 and 10 percent level is denoted respectively $***$, $**$ and $*$.

	Lower bound	Mean effect	Upper bound
GDP growth: +.01 (Economy)			
Short run	−2.38%	−1.40%	−.43%
Long run	−14.91%	−9.01%	−3.11%
Short rate: +.01 (Construction)			
Short run	.32%	1.52%	2.73%
Long run	.96%	6.60%	12.24%
Exchange rate: +1% (Trans.,stor.,com.)			
Short run	.75%	1.48%	2.22%
Long run	1.39%	2.95%	4.51%
Oil price: +10% (Trans.,stor.,com.)			
Short run	.52%	1.79%	3.05%
Long run	−6.46%	3.55%	13.57%

Explanation: Short and long run effects (percentage changes) on the default rate of the economy or a certain sector are computed using (12) based on estimation results from table 5.1. Upper and lower bounds of a 95% confidence interval are reported as well.

Table 1: Short and long run effects of macroeconomic shocks

We call these three fractions respectively the macroeconomic, latent systematic and idiosyncratic part, presented in Table 2. More variance is explained by the latent systematic than by the macroeconomic part. Although the macroeconomic variables were not selected because of their explanatory power, this result does illustrate the difficulty of finding relevant systematic variables. Note from Table 5.1 that the systematic standard deviations are all significantly different from zero but not from each other.

	Macroeconomic part	Latent systematic part	Idiosyncartic part
Ind., min.	22%	20%	58%
Construction	13%	35%	52%
Trade, rep. cons.	12%	51%	38%
Catering	10%	29%	60%
Trans., stor., com.	33%	23%	44%
Financial	25%	27%	48%
Rental, corp.	27%	42%	31%
Other	8%	13%	79%
Average	19%	30%	51%

Explanation: This table shows the variance decomposition (20) based on the model estimated in table 5.1.

Table 2: Variance decomposition

6 Scenario analysis of zero GDP growth

In this section results from the scenario analysis are presented. We will examine the default behaviour in 2007 given an unfavourable macroeconomic scenario of two quarters zero GDP growth in quarters 2006.3 and 2006.4. We compare it to a base scenario and a 2.5% worst case scenario.

A framework for stress testing the credit exposure to macroeconomic shocks is developed based on Virolainen (2004). In this framework, stress tests are conducted by comparing the average result of a stressed scenario, where an artificial adverse macroeconomic development is introduced, with that of the baseline scenario, where no adverse shock takes place. Estimated averages of the default rates for each sector corresponding to stressed and baseline scenarios are obtained from simulating a large number of future default rates by applying a Monte Carlo method. This is partly governed by the simulated future paths of the macroeconomic variables.

We use the macroeconomic model (10) with only GDP included in $\mathbf{z}_t$. This allows us to examine a GDP growth scenario without the need to make assumptions on the other macroeconomic variables. In formula:

$$\begin{aligned} \tilde{pd}_{t,i} &= \boldsymbol{\beta}_{i,0} + \boldsymbol{\beta}_{i,1}\tilde{pd}_{t-1,i} + \boldsymbol{\beta}_{i,2}\Delta\%\mathrm{GDP}_{t-1} + \upsilon_{t,i} \\ \upsilon_{t,i} &\overset{iid}{\sim} \left(0, \sigma^2_{\xi,i} + \sigma^2_{\psi,i}\right) \end{aligned} \tag{21}$$

Table 3 shows the estimation results. The coefficients of GDP growth are somewhat closer to zero compared to the parsimonious model. This is consistent with macroeconomic theory which states that an increase in GDP lowers the default rate but also leads to higher interest rates and an appreciating exchange rate which has an increasing effect on the default rate.

A model to forecast the behaviour of GDP growth is also required. It seems that an AR(1) model fits GDP growth quite well. Let $\boldsymbol{\gamma}$ be a parameter vector and σ_ν a nonnegative parameter.

$$\begin{aligned} \Delta\%\mathrm{GDP}_t &= \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1\Delta\%\mathrm{GDP}_{t-1} + \nu_t \\ \nu_t &\overset{iid}{\sim} \left(0, \sigma^2_\nu\right) \end{aligned} \tag{22}$$

Estimation with OLS using data over the period 1978.1-2006.2 (114 observations), leads to $\Delta\%\mathrm{GDP}_t = .01 + .61\Delta\%\mathrm{GDP}_{t-1} + .02$. All parameters are significantly different from zero at the 1% level.

In order to apply the Monte Carlo simulation, we need to draw realizations of the disturbances $\upsilon_{t,i}$ in (21) and ν_t in (22). We assume the disturbances $\upsilon_{t,i}$ and ν_t follow, after standardizing, a standardized t-distribution with df degrees

	Intercept	First lag	GDP growth	Systematic s.d.	Idiosyncratic s.d.
Ind., min.	-1.49^{***}	$.73^{***}$	-2.43^{***}	$.05^{***}$	$.10^{***}$
Construction	$-.86^{***}$	$.85^{***}$	$-.83$	$.07^{***}$	$.10^{***}$
Trade, rep. cons.	$-.92^{***}$	$.85^{***}$	-1.23^{**}	$.07^{***}$	$.07^{***}$
Catering	$-.54^{***}$	$.91^{***}$	-1.16	$.09^{***}$	$.13^{***}$
Trans., stor., com.	-1.69^{***}	$.70^{***}$	-2.93^{***}	$.08^{***}$	$.13^{***}$
Financial	$-.50^{***}$	$.89^{***}$	-2.67^{***}	$.09^{***}$	$.09^{***}$
Rental, corp.	-1.44^{***}	$.76^{***}$	-2.64^{***}	$.09^{***}$	$.06^{***}$
Other	-2.32^{***}	$.68^{***}$	-1.14	$.05^{***}$	$.16^{***}$
Economy	$-.84^{***}$	$.86^{***}$	-1.46^{***}	NA	$.08^{***}$
Pooled	$-.75^{***}$	$.86^{***}$	-1.47^{***}	$.07^{***}$	$.11^{***}$
(P-value equal coefficients	.0000	.0000	.2619	.0315	.0000)

Explanation: Model (10) is estimated with only GDP include in z_t^*. All estimated parameters are reported. Pooled results are obtained by estimating the sector model under the restriction $\beta_{1,j} = ... = \beta_{8,j}$ or $\sigma_{j,1} = ... = \sigma_{j,8}$ for certain j while allowing the other parameters to differ per sector. These restrictions are tested and the p-values are reported. 1, 5 and 10 percent level is denoted respectively $***$, $**$ and $*$.

Table 3: Estimated parameters model (10) with GDP growth only

of freedom. The pdf of a standardized t-distribution evaluated at a real number x is given by

$$\frac{\Gamma\left(\frac{df+1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)\sqrt{(df-2)\pi}}\left(1+\frac{x^2}{df-2}\right)^{\frac{df+1}{2}}$$

The standardized t-distribution with df degrees of freedom allows us to adjust the kurtosis, which is important for worst case scenarios. Moreover, the degrees of freedom is set equal to match the sample kurtoses. Table 5 shows the sample kurtoses of the disturbances and the degrees of freedom of the fitted t-distributions. The kurtoses are pooled for the sector model because the kurtoses differ quite a lot between the sectors.

$$\frac{3df-6}{df-4}$$

Table 4:

	Sample kurtosis	Degrees of freedom
Sector model	4.31	8.60
Economy model	4.12	9.38
GDP model	5.49	6.41

Explanation: The sector and economy model refer to respectively $\upsilon_{t,i}\ \forall i \in \{1,\ldots,8\}$ and $\upsilon_{t,0}$ in (21); the GDP model refers to ν_t in (22). The sample kurtoses and the degrees of freedom of the fitted t-distributions are shown.

Table 5: Sample kurtosis and degrees of freedom of disturbances in (21) and (22).

To analyse the scenarios we generate 200000 paths of the logit default rate using (21) and (22) given certain starting values for $\tilde{pd}_{t,i}$ and $\Delta\%\mathrm{GDP}_t$. Disturbances $\upsilon_{t,i}$ and ν_t are generated by multiplying draws from the t-distributions by their respective standard deviations $\sqrt{\sigma^2_{\xi,i}+\sigma^2_{\psi,i}}$ and σ_ν. Finally, we invert (5) to find the default rate:

$$pd_{t,i}=\frac{\exp(\tilde{pd}_{t,i})}{1+\exp(\tilde{pd}_{t,i})}.$$

Based on the average of the 200000 simulations, the average 2007 default rate is defined as:

$$\bar{pd}_{2007,i}=\frac{1}{4}\sum_{t=1}^{4}pd_{2007.t,i}.$$

Base scenario The expected average 2007 default rate is computed without making assumptions on what happens after 2006.2. We do so by setting the starting values for $\tilde{pd}_{t,i}$ and $\Delta\%\mathrm{GDP}_t$ equal to the known values from 2006.2, generating the logit default rates and computing $\bar{pd}_{2007,i}$.The results in Table 6 serve as a benchmark for the stress scenario. The average generated economy default rate is plotted against time in figure 2. It remains approximately constant because both the default rate and GDP growth were in 2006.2 already close to their long run averages.

	Default rate
Ind., min.	.31%
Construction	.26%
Trade, rep. cons.	.22%
Catering	.31%
Trans., stor., com.	.29%
Financial	.83%
Rental, corp.	.20%
Other	.06%
Economy	.22%

Explanation: This table shows the average generated $\bar{pd}_{2007,i}$ given the base scenario.

Table 6: Base scenario

2.5% worst case Table 7 compares the 2.5% worst cases of $\bar{pd}_{2007,i}$ of the base scenario (the .975th percentile) to the average of the base scenario. The percentage difference and its 95% confidence interval are reported. The 2.5% worst cases of the base scenario is a 31%-62% rise of the default rate depending on the sector.

Zero GDP growth The zero GDP growth scenario assumes GDP growth to be zero in quarters 2006.3 and 2006.4. After 2006.4, GDP growth evolves according to (22). Logit default rates evolve according to (21) with the $\tilde{pd}_{t,i}$ from quarter 2006.2 as starting value. We generate the logit default rates and compute $\bar{pd}_{2007,i}$. Table 8 compares the average generated $\bar{pd}_{2007,i}$ of the zero GDP growth scenario to the base scenario. The percentage difference and its 95% confidence interval are reported. The confidence interval captures uncertainty in the percentage difference of the expected effect caused by uncertainty in the estimated parameters of models (21) and (22). The effect of the GDP growth scenario in 2007 is a 4%-15% rise of the default rate depending on the sector. The effects are surprisingly small even if we look at the upper bounds. Note that, in accordance with the estimation results from table 3, the sectors

Industry and mining, Transport storage and communication, Financial services and Rental and corporate services are affected most by the zero GDP growth scenario.

The question rises if this small effect is realistic compared to historical results. During the period 1983-1991 there were about three brief sharp drops in GDP growth (Figure 1). In these cases the default rate did not visibly react. However, during the more lengthy GDP growth slowdowns of 1991-1993 and 2000-2003 the default rate approximately doubled. It appears that the default rate only reacts substantially to long lasting GDP growth developments.

The average of the 2.5% worst case scenarios are a lot worse than the average of the zero GDP growth scenarios. For most sectors the 2.5% worst case scenario is three to four times as bad as the zero GDP growth scenario. For the sectors Construction, Catering and Other, which are relatively insensitive to GDP growth, the worst case scenario is even over nine times as bad. Figure 2 plots the average generated economy default rates. The figure shows clearly that the 2.5% worst case scenario is a lot worse than then the zero GDP growth scenario.

	Lower bound	% Difference default rate	Upper bound
Ind., min.	25%	33%	41%
Construction	31%	39%	48%
Trade, rep. cons.	23%	31%	39%
Catering	54%	62%	70%
Trans., stor., com.	35%	43%	51%
Financial	43%	51%	59%
Rental, corp.	27%	35%	43%
Other	32%	40%	48%
Economy	20%	28%	36%

Explanation: This table shows the percentage difference between the .975th percentile of all generated $\bar{pd}_{2007,i}$ and the average generated $\bar{pd}_{2007,i}$ given the base scenario. The percentage difference is an estimate because of uncertainty in the estimated parameters of models (21) and (22). Upper and lower bounds of a 95% confidence interval for the percentage difference are reported.

Table 7: 2.5% worst case scenario

	Lower bound	% Difference default rate	Upper bound
Ind., min.	5%	10%	16%
Construction	−1%	4%	10%
Trade, rep. cons.	1%	6%	12%
Catering	1%	7%	12%
Trans., stor., com.	7%	12%	18%
Financial	10%	15%	21%
Rental, corp.	6%	12%	18%
Other	−1%	4%	10%
Economy	2%	8%	13%

Explanation: This table shows the percentage difference between the average generated $\bar{pd}_{2007,i}$ given the zero GDP growth scenario and the base scenario. The percentage difference is an estimate because of uncertainty in the estimated parameters of models (21) and (22). Upper and lower bounds of a 95% confidence interval for the percentage difference are reported.

Table 8: Zero GDP growth scenario

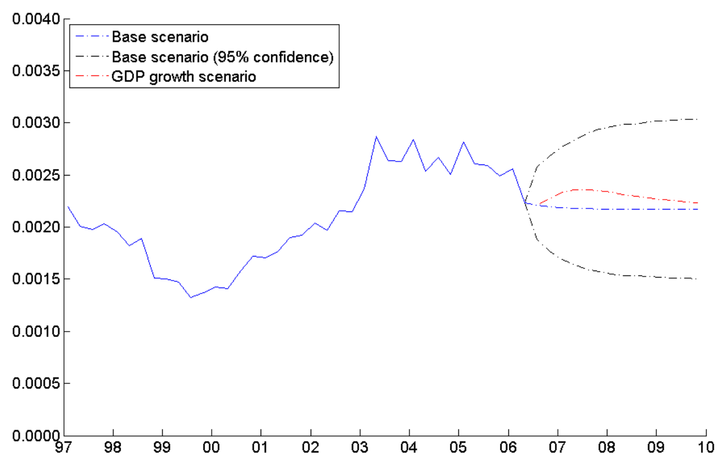

Figure 2: Forecasting the economy default rate Explanation: The figure shows (1) the average and (2) the .025th and .975th percentiles of the generated economy default rates given the base scenario as well as (3) the average generated economy default rate given the zero GDP growth scenario plotted against time.

7 Discussion and Conclusion

The focus of this paper is (i) to assess which macroeconomic variables are related to the Dutch default behaviour of firms and (ii) to assess the default behaviour given a two quarter zero GDP growth in quarters 2006.3 and 2006.4 .

(i) Assess which macroeconomic variables are related to the Dutch default behaviour of firms.

We studied GDP growth, interest rate, exchange rate, stock market return and volatility and oil price. A convincing negative relation with the default rate and GDP growth was found. The relation with the oil price is also significant in several sectors. Furthermore, there is some indication of a positive relation with the (short) interest rate for the sector Construction and with the (logarithm of the real) exchange rate for the sectors Transport, storage and communication, Financial services and Rental and corporate services. No relation with stock market return and volatility was found. Remarkably, for the interest rate, exchange rate and oil price not the change but the level of the variables turned out to be significant.

For the overall economy, the relations with the default rate and the macroeconomic variables are stable through time. The macroeconomic relations with the sector default rates are mostly unstable except for the oil price. A reason for the instability is that results amongst sectors can differ according to the growth opportunities of the sector of economic activity to which firms belong, the sector's degree of internationalization and its dependence on other sectors.

The first lag of the logit default rate has a highly significant coefficient. This implies that the effect of persistent macroeconomic shocks gradually increases over time. Without the lagged default rate taken into account, the macroeconomic variables explain on average about a fifth of the variance of the default rate. A latent factor affecting all sectors explains about thirty percent and the rest is explained by sector specific disturbances. Other literature mainly confirms the results on GDP growth and, to a limited extent, interest and exchange rate. Furthermore, the stock market is often found to be related but always for firms listed on a stock exchange.

(ii) Assess the Dutch default behaviour given a two quarter zero GDP growth in quarters 2006.3 and 2006.4 and a 2.5% worst case scenario.

The effect of the GDP growth scenario in 2007 is a 4%-15% rise of the default rate depending on the sector. Historic recessions of similar short duration are in accordance with these small numbers: the default rate does not visibly react to short recessions. However, historic recessions lead to higher long run effects of more persistent recessions. It can be concluded that a short recession of two quarters does not influence the default rate significantly. However, a longer recession would influence the default rate.

The fact that a two quarter zero GDP growth scenario underestimates the true risk is also being supported by the 2.5% worst cases of the base scenario. These lead to a 31%-62% rise of the default rate depending on the sector. The difference between the 2.5% worst case and the GDP growth scenario suggests that assessing default risk by examining a two quarter zero GDP growth scenario underestimates the true risk.

Concluding, a stress test scenario of two quarters zero GDP growth as required by Basel II, might underestimate the true risk. We would advise to do the stress test with a more severe scenario in order to get a better estimate of the true risk.

References

[1] Allen, L. and A. Saunders (2002). "Credit risk measurement: new approaches to value at risk and other paradigms. "John Wiley & Sons, second edition.

[2] Carling, K., T. Jacobson, J. Linde and K. Roszbach, (2002). "Capital charges under Basel II: Corporate credit risk modeling and the macro economy. " Working paper, Sveriges Riksbank, No. 142.

[3] Chan-Lau, J.A., (2006). Fundamentals-based estimation of default probabilities: A survey ." IMF Working paper, WP/06/149.

[4] Couderc, F. and O. Renault (2005): "Times-to-Default: Life Cycles, Global and Industry Cycle Impacts." FAME Research series, No. 142.

[5] Durbin, J. and S. Koopman (2001). "Time series analysis by state space methods." Oxford university press, first edition.

[6] Fiori, R, A Foglia and S Ianotti (2006). "Estimating macroeconomic credit risk and sectoral default rate correlations for the Italian economy." Working paper, Banca d'Italia.

[7] Jakubik, P., (2006). "Does credit risk vary with economic cycles? The case of Finland." IES Working paper, 11/2006.

[8] Johnston, J. and J. Dinardo (1997). "Econometric methods." McGraw-Hill, fourth edition.

[9] Hamerle, A., T. Liebig and H. Scheule, (2004). "Forecasting Credit Portfolio Risk ." Discussion Paper Series 2: Banking and Financial Supervision, No 01, Deutsche Bundesbank.

[10] Hoggarth, G., A. Logan and L. Zicchino, (2005). "Macro stress tests of UK banks." BIS Papers, No. 22 Part 20.

[11] Kavvathas, D. (2001). "Estimating credit rating transition probabilities for corporate bonds." AFA 2001 New Orleans Meetings.

[12] Kiefer, N. (1988). "Economic duration data and hazard functions." Journal of Economic literature 26, pp. 646-79 .

[13] Koopman, S. and A. Lucas, (2005). "Business and Default Cycles for Credit Risk ." Journal of Applied Econometrics, 20, pp. 311-323.

[14] Lucas, A., A. Monteiro, S. Koopman and R. Kraussl, (2006). "Credit cycles and macro fundamentals." Working paper 03-062/2, Tinbergen Institute.

[15] Virolainen, K. (2004). "Macro stress testing with macroeconomic credit risk model for Finland." Bank of Finland Discussion Papers 18.

[16] Vlieghe, G. (2001). "Indicators of fragility in the UK corporate sector." Working Paper no.146, Bank of England.

A Literature

- Couderc and Renault [4] estimate the default rate of firms listed in the S&P index over the period 1981 - 2003 by means of a continuous time model. They also investigate lags of variables. They show that past economic conditions are of prime importance in explaining probability changes: current shocks and long term trends jointly determine default probabilities. Significant variables are stock market return and volatility, term and credit spread and GDP growth.

- Carling, Jacobson, ea. [2] estimate a duration model to explain the survival time to default for borrowers in the business loan portfolio of a major Swedish bank over the period 1994-2000. The model takes both firm-specific characteristics and the prevailing macroeconomic conditions into account. The output gap, the yield curve and consumers' expectations of future economic development have significant explanatory power for the default risk of firms.

- Koopman and Lucas [13] estimate the default rate of US firms over the period 1933 - 1997 for a general class of periodic unobserved components time series models with stochastic trend, seasonal and cycle components. They take into account the correlation between stochastic cycle-effects. GDP growth is found to be significant.

- Fiori, Foglia and Iannotti [6] find that the explanatory power of macro factors for defaults is relatively limited, but that residual cross-section correlation of default rates suggests the presence of contagion effects from the impacts of sector-specific risk on the default rates of other sectors.

- Jakubík [7] estimates the default rate of Finish firms over the period 1988 - 2003 by means of a linear vector autoregressive model. Jakubik found GDP growth to be significant, interest rates to be somewhat insignificant and the exchange rate to be significant for the trading sector.

- Hamerle, Liebig and Scheule [9] estimate the default rate of German firms over the period 1991 - 2000 by means of a discrete time model, also including firm specific variables. They show that systematic variables make a latent factor insignificant. They find that the inclusion of variables which are correlated with the business cycle improves the forecasts of default probabilities. Asset and default correlations depend on the factors used to model default probabilities. They conclude that correlations and default probabilities should always be estimated simultaneously. GDP growth is found to be significant.

- Koopman, Kraussl, ea. [14] study the relation between the credit cycle and macro economic fundamentals using rating transition and default data of U.S. corporates from Standard and Poor's over the period 1980-2005. They conclude that many of the variables thought to explain the credit cycle, turn out to be insignificant. The main exceptions are GDP growth, and to some extent stock returns and stock return volatilities. Their economic significance appears low, however.

- Kavvathas [11] assesses the potential of conditioning on economywide state variables in improving the forecasting of the Credit Rating Transition Probability Matrix over the period 1981-1998. He finds that an increase in nominal short and long and real rates, a lower equity return and a higher equity return volatility are associated with higher relative downgrade intensities.

- Vlieghe [16], using UK data over the period 1975-1999 suggests that the substantial rise in number of defaults during the recession in the early 1990s mainly reflected deteriorating company finances, including a marked build-up of indebtedness. In the subsequent recovery, however, rising GDP relative to trend and other macroeconomic factors seem to have had greater explanatory power than changes in company finances in accounting for the fall in the corporate liquidations rate to its currently low level.

- Virolainen [15], using Finish data over the period 1986-2003 finds a significant relationship between corporate sector default rates and macroeconomic factors including GDP, interest rates and corporate indebtedness.

B Econometric theory

This appendix reviews some basic econometric theory on which parameter estimation, hypothesis testing and confidence set construction in this research are based. Parameter estimation by maximizing the likelihood is discussed in apendix B.1 and appendix B.2 explains hypothesis testing and confidence set construction.

B.1 Maximum likelihood theory

Let $\mathbf{y}_n$ be the nth observation of a vector random variables. Define $Y_n = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ and let $\boldsymbol{\theta}$ be a parameter vector. Let the probability density function (pdf) of $_n$ conditional on Y_{n-1} be given by

$$p_n(\mathbf{y}_n; \theta | Y_{n-1}).$$

(The pdf $p_n(.)$ may depend on time so we implicitly allow the pdf to be conditioned on explanatory variables.)

Suppose we have N observations. Asymptotic results in this and the following subsection are with respect to N going to infinity. Let $l_n(\boldsymbol{\theta}) = \ln p_n(\mathbf{y}_n; \boldsymbol{\theta} | Y_{n-1})$. The loglikelihood is given by

$$l(\boldsymbol{\theta}) = \sum_{n=1}^{N} l_n(\boldsymbol{\theta}).$$

We will call $l_n(\theta)$ the observation n loglikelihood. The parameter vector $\boldsymbol{\theta}$ is estimated consistently by maximizing the loglikelihood. Denote this maximum likelihood estimator by $\hat{\boldsymbol{\theta}}$. We have the following result.

$$\begin{aligned} \sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \quad &\xrightarrow{d} \quad \mathcal{N}\left(\mathbf{0}, \mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\right) \qquad (23) \\ \text{where} \quad \mathcal{I} &= \operatorname{plim} \frac{1}{N} \sum_{n=1}^{N} \left(D_{\boldsymbol{\theta}} l_n(\boldsymbol{\theta})\right) \left(D_{\boldsymbol{\theta}} l_n(\boldsymbol{\theta})\right)' \\ \mathcal{J} &= \operatorname{plim} \frac{1}{N} \sum_{n=1}^{N} D^2_{\boldsymbol{\theta}\boldsymbol{\theta}'} l_n(\boldsymbol{\theta}) \end{aligned}$$

Let $\hat{\mathcal{I}}$ and $\hat{\mathcal{J}}$ be the finite sample counterparts of respectively $\mathcal{I}$ and $\mathcal{J}$ evaluated in $\hat{\boldsymbol{\theta}}$. Because $\hat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}$ we have from the Mann and Wald theorem[2] that $\mathcal{I}$ and $\mathcal{J}$ can be estimated consistently by respectively $\hat{\mathcal{I}}$ and $\hat{\mathcal{J}}$. Result (23) implies that for large N

$$\operatorname{var}\left(\hat{\boldsymbol{\theta}}\right) \approx \frac{1}{N} \hat{\mathcal{J}}^{-1} \hat{\mathcal{I}} \hat{\mathcal{J}}^{-1}.$$

[2] Let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be sequences of random variables, X a scalar and Y a random variable and $f(X_n, Y_n))$ a function. The Mann and Wald theorem states that if $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{d} Y$ then, under mild conditions, $f(X_n, Y_n) \xrightarrow{d} f(X, Y)$. (See result 14 from Bekker[**?**].)

To derive result (23) apply a first order Taylor expansion to the first order condition for maximization of the loglikelihood. A vector $\hat{\boldsymbol{\theta}} \in (\boldsymbol{\theta}, \hat{\boldsymbol{\theta}})$ exists such that

$$D_{\boldsymbol{\theta}} l(\hat{\boldsymbol{\theta}}) = D_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) + D^2_{\boldsymbol{\theta}\boldsymbol{\theta}'} l(\hat{\boldsymbol{\theta}})\ (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = \mathbf{0}.$$

Rearranging and multiplying by $\sqrt{N}$ yields

$$\begin{aligned} \sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) &= -\sqrt{N} \left(D^2_{\boldsymbol{\theta}\boldsymbol{\theta}'} l(\hat{\boldsymbol{\theta}}) \right)^{-1} D_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) \\ &= \left(\frac{1}{N} \sum_{n=1}^{N} D^2_{\boldsymbol{\theta}\boldsymbol{\theta}'} l_n(\hat{\boldsymbol{\theta}}) \right)^{-1} \left(\frac{-1}{\sqrt{N}} \sum_{n=1}^{N} D_{\boldsymbol{\theta}} l_n(\boldsymbol{\theta}) \right). \end{aligned}$$

Because $\hat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}$ and therefore also $\hat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}$ we have $\frac{1}{N} \sum_{n=1}^{N} D^2_{\boldsymbol{\theta}\boldsymbol{\theta}'} l_n(\hat{\boldsymbol{\theta}}) \xrightarrow{p} \mathcal{J}$. Furthermore, noting that

$$\begin{aligned} E\left(D_{\boldsymbol{\theta}} l_n(\boldsymbol{\theta})\right) &= \int_{\mathbf{y}_n} D_{\boldsymbol{\theta}} l_n(\boldsymbol{\theta})\ p(\mathbf{y}_n; \boldsymbol{\theta} | Y_{n-1}) d\mathbf{y}_n \\ &= \int_{\mathbf{y}_n} D_{\boldsymbol{\theta}} p(\mathbf{y}_n; \boldsymbol{\theta} | Y_{n-1})\ d\mathbf{y}_n \\ &= D_{\boldsymbol{\theta}}[1] \ = \ \mathbf{0}. \end{aligned}$$

and applying a central limit theorem yields $\frac{-1}{\sqrt{N}} \sum_{n=1}^{N} D_{\boldsymbol{\theta}} l_n(\boldsymbol{\theta}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathcal{I})$. Now, using the Mann and Wald theorem, result (23) follows.

Results in this subsection implicitly assume some mild conditions on the pdf $p_n(\mathbf{y}_n; \boldsymbol{\theta} | Y_{n-1})$ are satisfied.

B.2 Hypothesis testing and confidence set construction

Let $R(\boldsymbol{\theta})$ be a vector function with output in $\mathcal{R}^q$. We use this function to explain both the testing of hypotheses and the construction of confidence sets. First, we derive the asymptotic distribution of $\sqrt{N}\left(R(\hat{\boldsymbol{\theta}}) - R(\boldsymbol{\theta})\right)$. Let $\Sigma = \mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}$ and $\hat{\Sigma} = \hat{\mathcal{J}}^{-1}\hat{\mathcal{I}}\hat{\mathcal{J}}^{-1}$. We use a first order Taylor expansion, result (23) and the Mann and Wald theorem. A vector $\hat{\boldsymbol{\theta}} \in (\boldsymbol{\theta}, \hat{\boldsymbol{\theta}})$ exists such that

$$\begin{aligned} \sqrt{N}\left(R(\hat{\boldsymbol{\theta}}) - R(\boldsymbol{\theta})\right) &= D_{\boldsymbol{\theta}'} R(\hat{\boldsymbol{\theta}})\ \sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \\ &\xrightarrow{d} \mathcal{N}\left(\mathbf{0}, D_{\boldsymbol{\theta}'} R(\boldsymbol{\theta})\ \Sigma\ D_{\boldsymbol{\theta}'} R(\boldsymbol{\theta})\right)'. \end{aligned} \tag{24}$$

Suppose we want to test the hypothesis

$$R(\boldsymbol{\theta}) = \mathbf{0}. \tag{25}$$

If this hypothesis is true we may rewrite (24) into

$$\left(D_{\boldsymbol{\theta}'} R(\boldsymbol{\theta})\ \frac{1}{N} \Sigma\ D_{\boldsymbol{\theta}'} R(\boldsymbol{\theta})' \right)^{-1/2} R(\hat{\boldsymbol{\theta}}) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, I_q\right)$$

which implies

$$R(\hat{\boldsymbol{\theta}})'\left(D_{\boldsymbol{\theta}'}R(\boldsymbol{\theta})\ \frac{1}{N}\Sigma\ D_{\boldsymbol{\theta}'}R(\boldsymbol{\theta})'\right)^{-1} R(\hat{\boldsymbol{\theta}}) \xrightarrow{d} \chi^2_q. \tag{26}$$

We use this result to test the hypothesis $R(\boldsymbol{\theta}) = \mathbf{0}$. Note that $R(\hat{\boldsymbol{\theta}})'\left(D_{\boldsymbol{\theta}'}R(\boldsymbol{\theta})\ \Sigma\ D_{\boldsymbol{\theta}'}R(\boldsymbol{\theta})'\right)^{-1} R(\hat{\boldsymbol{\theta}})$ (which equals the test statistic in (26) without the factor $\frac{1}{N}$ in the denominator) converges in probability to the nonnegative constant $R(\boldsymbol{\theta})'\left(D_{\boldsymbol{\theta}'}R(\boldsymbol{\theta})\ \Sigma\ D_{\boldsymbol{\theta}'}R(\boldsymbol{\theta})'\right)^{-1} R(\boldsymbol{\theta})$ so if $R(\boldsymbol{\theta}) \neq \mathbf{0}$ the test statistic goes to infinity. Therefore, we reject the hypothesis for high values of (26). From the Mann and Wald theorem the result remains valid if we replace $\boldsymbol{\theta}$ and Σ by consistent estimators so in practice we substitute $\hat{\boldsymbol{\theta}}$ and $\hat{\Sigma}$ for respectively $\boldsymbol{\theta}$ and Σ. In case we want to test the hypothesis that the ith element of $\boldsymbol{\theta}$ is zero, we often look at the t-statistic. This is the square root of the test statistic times the sign of $\hat{\boldsymbol{\theta}}_i$.

Let $\alpha \in (0, .5)$. Now suppose we want a $100(1-\alpha)\%$ confidence set $\hat{S}$ for $R(\boldsymbol{\theta})$ which is a set $\hat{S}$ such that

$$P\left(R(\boldsymbol{\theta}) \in \hat{S}\right) = 1 - \alpha. \tag{27}$$

(Remark that $R(\boldsymbol{\theta})$ is a constant vector and $\hat{S}$ a random set.) In this report we only consider confidence sets in $\mathcal{R}^1$ (which we call confidence intervals) so, hereafter, $R(.)$ is a scalar. From (24) we have for large N approximately

$$\frac{R(\boldsymbol{\theta}) - R(\hat{\boldsymbol{\theta}})}{\sqrt{D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})\ \frac{1}{N}\hat{\Sigma}\ D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})'}} \sim \mathcal{N}(0,1). \tag{28}$$

Let $\Phi(.)$ be the cdf of a standardnormal distribution. Define:

$$\begin{aligned}
\hat{S}_{\text{lower}} &= R(\hat{\boldsymbol{\theta}}) + \Phi^{-1}\left(\frac{\alpha}{2}\right)\sqrt{D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})\ \frac{1}{N}\hat{\Sigma}\ D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})'} \\
\hat{S}_{\text{upper}} &= R(\hat{\boldsymbol{\theta}}) + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})\ \frac{1}{N}\hat{\Sigma}\ D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})'}.
\end{aligned}$$

Now note that if (28) is correct we have

$$\begin{aligned}
P\left(R(\boldsymbol{\theta}) \in \left(\hat{S}_{\text{lower}}, \hat{S}_{\text{upper}}\right)\right) &= P\left(\hat{S}_{\text{lower}} < R(\boldsymbol{\theta}) < \hat{S}_{\text{upper}}\right) \\
&= P\left(\Phi^{-1}\left(\frac{\alpha}{2}\right) < \frac{R(\boldsymbol{\theta}) - R(\hat{\boldsymbol{\theta}})}{\sqrt{D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})\ \frac{1}{N}\hat{\Sigma}\ D_{\boldsymbol{\theta}'}R(\hat{\boldsymbol{\theta}})'}} < \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) \\
&= \Phi\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - \Phi\left(\Phi^{-1}\left(\frac{\alpha}{2}\right)\right) \\
&= 1 - \alpha.
\end{aligned}$$

Therefore, we set

$$\hat{S} = \left(S^*_{\text{lower}}, S^*_{\text{upper}}\right). \tag{29}$$

Results in this subsection implicitly assume some mild conditions on the function $R(.)$ are satisfied.